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A remark on the continuous variation of secondary characteristic classes for foliations

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§ 1. Introduction.

Thurston [3] has constructed a family of codimension n foliations on certain $(2n+1)$ manifold with continuously varying Godbillon-Vey invariant, establishing a surjection,

$$gv : H_{2n+1}(B\Gamma_n^\infty ; \boldsymbol{Z}) \longrightarrow \boldsymbol{R} \longrightarrow 0 .$$

The purpose of this note is to show that, using his results, we can show that some secondary characteristic classes other than that of Godbillon-Vey vary also continuously. We can also show that these classes are independent. Thus we can construct a surjective homomorphism

$$H_{2n+1}(B\Gamma_n^\infty ; \boldsymbol{Z}) \longrightarrow \boldsymbol{R}^{[n+1/2]} \longrightarrow 0$$

(see § 3).

The foliations which we are going to construct to realize some characteristic classes are product foliations. The reason why these classes vary continuously is that they are in some sense "decomposable." The Godbillon-Vey class is "indecomposable" in the sense that it vanishes on any product foliation. In § 3, we shall remark that there are infinitely many series of indecomposable classes, first of which are those of Godbillon-Vey.

To evaluate the characteristic classes on product foliations, we need the "Cartan formula" for the secondary classes. This will be done in § 2.

The author would like to express his hearty thanks to Dr. Liang for many conversations and to Dr. Lazarov for helpful suggestions. Actually the formulation of the Cartan formula in § 2 is due to him.

§ 2. The Cartan formula.

Suppose we are given two foliations $(M, \mathcal{F})$ and $(N, \mathcal{G})$ of codimensions p and q respectively. Then we can construct the product foliation $(M\times N, \mathcal{F}\times\mathcal{G})$. The classifying map for this foliation factors through $B\Gamma_p^\infty\times B\Gamma_q^\infty$;

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$$M\times N \longrightarrow B\Gamma_p^\infty \times B\Gamma_q^\infty \xrightarrow{\mu} B\Gamma_n^\infty$$

where $n=p+q$ and μ is induced from the natural inclusion $\Gamma_p^\infty\times\Gamma_q^\infty\subset\Gamma_{p+q}^\infty$.

To calculate the characteristic classes of the product foliation $(M\times N, \mathcal{F}\times\mathcal{G})$, it is necessary to identify the image of the classes in $H^*(B\Gamma_n^\infty;\boldsymbol{R})$ under the homomorphism μ^*. To do this, let us define characteristic classes for codimension n foliations in a slightly different way from that of Bott [2]*. Recall that Bott used the class c_i, which is the degree $2i$ part of $\det\left(I+\frac{1}{2\pi}\Omega\right)$ where Ω is the curvature matrix of Bott's connection on the normal bundle of foliation. We simply replace c_i by $\Sigma_i=\left(\frac{1}{2\pi}\right)^i \mathrm{Tr}\,\Omega^i$ and let γ_{2i+1} be the form corresponding to h_{2i+1} in the Bott's definition. Let WO'_n be a differential graded algebra defined by

$$WO'_n=\hat{\boldsymbol{R}}[\Sigma_1,\cdots,\Sigma_n]\otimes E(\gamma_1,\gamma_3,\cdots,\gamma_{2k+1})$$

$$\deg\Sigma_i=2i,\ \deg\gamma_{2i+1}=4i+1,\quad\text{and}\quad d\Sigma_i=0,\ d\gamma_{2i+1}=\Sigma_{2i+1}.$$

($2k+1$ is the greatest odd integer satisfying $2k+1\leqq n$.) Then by exactly the same way as in [2], we obtain a homomorphism

$$H^*(WO'_n)\longrightarrow H^*(B\Gamma_n^\infty;\boldsymbol{R}).$$

Now let us define a homomorphism of differential graded algebras

$$\alpha: WO'_n\longrightarrow WO'_p\otimes WO'_q$$

by

$$\alpha(\Sigma_i)=\tilde{\Sigma}_i\otimes 1+1\otimes\tilde{\Sigma}_i$$

$$\alpha(\gamma_{2i+1})=\tilde{\gamma}_{2i+1}\otimes 1+1\otimes\tilde{\gamma}_{2i+1}$$

where $\tilde{\Sigma}_i$ and $\tilde{\gamma}_{2i+1}$ are defined as follows

$$\tilde{\Sigma}_i\otimes 1=\begin{cases}\Sigma_i\otimes 1 & i\leqq p\\ 0 & i>p\end{cases}$$

$$1\otimes\tilde{\Sigma}_i=\begin{cases}1\otimes\Sigma_i & i\leqq q\\ 0 & i>q.\end{cases}$$

If $2i+1>r=$size of a matrix A, then as is well-known, $\mathrm{Tr}\left(\frac{1}{2\pi}A\right)^{2i+1}$ can be uniquely expressed as a polynomial of $\mathrm{Tr}\left(\frac{1}{2\pi}A\right),\cdots,\mathrm{Tr}\left(\frac{1}{2\pi}A\right)^r$;

$$\mathrm{Tr}\left(\frac{1}{2\pi}A\right)^{2i+1}=\sum a^r_{\substack{j_1\cdots j_s\\ k_1\cdots k_s}}\left(\mathrm{Tr}\left(\frac{1}{2\pi}A\right)^{j_1}\right)^{k_1}\cdots\left(\mathrm{Tr}\left(\frac{1}{2\pi}A\right)^{j_s}\right)^{k_s}$$

* This idea is due to Lazarov.

where we may assume that j_1 is odd. Then we define

$$\tilde{\gamma}_{2i+1}\otimes 1=\begin{cases}\gamma_{2i+1}\otimes 1 & 2i+1\leqq p\\ \sum\limits_{k_1\cdots k_s} a^q_{j_1\cdots j_s}\gamma_{j_1}\,\Sigma_{j_1}{}^{k_1-1}\cdots\Sigma_{j_s}{}^{k_s} & 2i+1>p\,.\end{cases}$$

$1\otimes\tilde{\gamma}_{2i+1}$ is defined similarly. Then we have

PROPOSITION 1 (Cartan formula). *The following diagram is commutative.*

$$\begin{array}{ccc} H^*(B\Gamma_n^\infty;\mathbf{R}) & \xrightarrow{\mu^*} & H^*(B\Gamma_p^\infty\times B\Gamma_q^\infty;\mathbf{R}) \\ \uparrow & & \uparrow\ \text{cross product} \\ & & H^*(B\Gamma_p^\infty;\mathbf{R})\otimes H^*(B\Gamma_q^\infty;\mathbf{R}) \\ \uparrow & & \uparrow \\ H^*(WO'_n) & \xrightarrow{\alpha_*} & H^*(WO'_p)\otimes H^*(WO'_q) \end{array}$$

PROOF. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two foliations of codimensions p and q on smooth manifolds M_1 and M_2 respectively and we consider the product foliation $\mathcal{F}=\mathcal{F}_1\times\mathcal{F}_2$ of codimension $n=p+q$ on $M=M_1\times M_2$. The normal bundle $\nu(\mathcal{F})$ is canonically isomorphic to the exterior direct sum of $\nu(\mathcal{F}_1)$ and $\nu(\mathcal{F}_2)$;

$$\nu(\mathcal{F})=\nu(\mathcal{F}_1)\widehat{\oplus}\nu(\mathcal{F}_2)\,.$$

Moreover exterior direct sum of Riemannian (resp. Bott) connections of $\nu(\mathcal{F}_1)$ and $\nu(\mathcal{F}_2)$ defines those of $\nu(\mathcal{F})$. Therefore

$$\Sigma_i(\mathcal{F})=\pi_1^*\Sigma_i(\mathcal{F}_1)+\pi_2^*\Sigma_i(\mathcal{F}_2)$$

where $\Sigma_i(\mathcal{F})$ is the form $\mathrm{Tr}\left(\frac{1}{2\pi}\Omega^1_{\mathcal{F}}\right)^i$ ($\Omega^1_{\mathcal{F}}$ is the curvature matrix associated to the Bott connection of $\nu(\mathcal{F})$) and $\pi_i: M_1\times M_2\to M_i$ $(i=1, 2)$ is the natural projection.

By the Bott's vanishing theorem [2], we have

$$\Sigma_i(\mathcal{F})=\pi_1^*\tilde{\Sigma}_i(\mathcal{F}_1)+\pi_2^*(\tilde{\Sigma}_i(\mathcal{F}_2)\,.$$

This justifies the definition of α on Σ_i's. Next we consider γ_{2i+1}. Let $\Omega_{\mathcal{F}}$ be the curvature matrix associated to the canonical connection $\tilde{\nabla}_{\mathcal{F}}$ of $\nu(\mathcal{F})\times I$ which connects the Riemannian connection $\nabla^0_{\mathcal{F}}$ and the Bott connection $\nabla^1_{\mathcal{F}}$ of $\nu(\mathcal{F})$ (cf. [2]). Then γ_{2i+1} is defined by

$$\gamma_{2i+1}(\mathcal{F})=\pi_*\mathrm{Tr}\left(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}}\right)^{2i+1}$$

where π_* is the integration along the fibre I. Then it is easy to see that

$$\gamma_{2i+1}(\mathcal{F})=\pi_1^*\pi_*\mathrm{Tr}\left(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\right)^{2i+1}+\pi_2^*\pi_*\mathrm{Tr}\left(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_2}\right)^{2i+1}.$$

Now we prove the following lemma.

LEMMA 2. *If* $2i+1>p$, *then*

$$\pi_*\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{2i+1}-\sum a^{p}_{\substack{j_1\cdots j_s\\ k_1\cdots k_s}}\gamma_{j_1}(\mathcal{F}_1)(\Sigma_{j_1}(\mathcal{F}_1))^{k_1-1}\cdots(\Sigma_{j_s}(\mathcal{F}_1))^{k_s}$$

is an exact form.

PROOF OF THE LEMMA. Since

$$\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{2i+1}=\sum a^{p}_{\substack{j_1\cdots j_s\\ k_1\cdots k_s}}\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\Big)^{k_1}\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\Omega_{\mathcal{F}_1}\Big)^{j_s}\Big)^{k_s}$$

it suffices to prove the following

$$\omega=\pi_*\Big[\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\Big)^{k_1}\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_s}\Big)^{k_s}\Big]$$
$$-\Big[\pi_*\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\Big)\Big]\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\Omega^1_{\mathcal{F}_1}\Big)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\big(\Omega^1_{\mathcal{F}_1}\big)^{j_s}\Big)^{k_s}$$

is an exact form.

Now let us consider the bundle $\nu(\mathcal{F}_1)\times I^2$, which is a vector bundle over $M_1\times I^2$. We define a connection $\bar{\nabla}$ on this bundle as follows. Over $M_1\times I\times\{1\}$, $\bar{\nabla}$ is equal to the connection $\tilde{\nabla}_{\mathcal{F}_1}$ and over $M_1\times I\times\{0\}$, it is equal to $\pi^*(\nabla_{\mathcal{F}_1})$, where $\pi: M_1\times I\to M_1$ is the natural projection. Then we define $\bar{\nabla}$ to be the canonical connection which connects $\tilde{\nabla}_{\mathcal{F}_1}$ and $\pi^*(\nabla^1_{\mathcal{F}_1})$. Let us write $\bar{\Omega}$ for the curvature matrix of this connection. Let $\pi': M_1\times I\times I\to M_1\times I$ be the projection onto the first two factors and let π'_* be the integration along the fibre I. Then we have

$$\omega=\pi_*\Big[\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\Big\{\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_s}\Big)^{k_s}$$
$$-\pi^*\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\Omega_{\mathcal{F}_1}\Big)^{j_1}\Big)^{k_1-1}\Big)\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\Omega^1_{\mathcal{F}_1}\Big)^{j_s}\Big)^{k_s}\Big\}\Big]$$
$$=\pi_*\Big[\mathrm{Tr}\Big(\frac{1}{2\pi}\Omega_{\mathcal{F}_1}\Big)^{j_1}d\pi'_*\Big\{\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_s}\Big)^{k_s}\Big\}\Big]$$
$$=-\pi_*d\Big[\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\pi'_*\Big\{\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_s}\Big)^{k_s}\Big\}\Big]$$
$$=(d\pi_*+i_0^*-i_1^*)\Big[\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\pi'_*\Big\{\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_s}\Big)^{k_s}\Big\}\Big]$$

where $i_\varepsilon: M_1\to M_1\times I$ ($\varepsilon=0, 1$) is the inclusion; $i_\varepsilon(x)=(x, \varepsilon)$.

Now

$$i_0^*\Big[\mathrm{Tr}\Big(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\Big)^{j_1}\pi'_*\Big\{\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\Big(\frac{1}{2\pi}\bar{\Omega}\Big)^{j_s}\Big)^{k_s}\Big\}\Big]$$

$$=-\mathrm{Tr}\left(\frac{1}{2\pi}\Omega^0_{\mathcal{F}_1}\right)^{j_1}\pi_*\Big[\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\right)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\right)^{j_s}\Big)^{k_s}\Big]$$

$$=0$$

because, $\mathrm{Tr}\left(\frac{1}{2\pi}\Omega^0_{\mathcal{F}_1}\right)^{j_1}=0$ (recall that j_1 is odd and $\Omega^0_{\mathcal{F}_1}$ is the curvature matrix associated to the Riemannian connection of $\nu(\mathcal{F}_1)$).

Next we have

$$i_1^*\Big[\mathrm{Tr}\left(\frac{1}{2\pi}\tilde{\Omega}_{\mathcal{F}_1}\right)^{j_1}\pi'_*\Big\{\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\bar{\Omega}\right)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\bar{\Omega}\right)^{j_s}\Big)^{k_s}\Big\}\Big]$$

$$=\mathrm{Tr}\left(\frac{1}{2\pi}\Omega^1_{\mathcal{F}_1}\right)^{j_1}i_1^*\pi'_*\Big\{\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\bar{\Omega}\right)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\bar{\Omega}\right)^{j_s}\Big)^{k_s}\Big\}.$$

But clearly we have

$$i_1^*\pi'_*\Big\{\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\bar{\Omega}\right)^{j_1}\Big)^{k_1-1}\cdots\Big(\mathrm{Tr}\left(\frac{1}{2\pi}\bar{\Omega}\right)^{j_s}\Big)^{k_s}\Big\}=0.$$

Thus we have shown that ω is an exact form and this proves our lemma.

By the above lemma, we can conclude that

$$(*)\qquad \gamma_{2i+1}(\mathcal{F})-(\pi_1^*\tilde{\gamma}_{2i+1}(\mathcal{F}_1)+\pi_2^*\tilde{\gamma}_{2i+1}(\mathcal{F}_2))$$

is an exact form.

Now let $x=\gamma_{i_1}\cdots\gamma_{i_l}\Sigma_{j_1}\cdots\Sigma_{j_m}$ be a cocycle in WO'_n (namely $i_k+j_1+\cdots+j_m>n$ for all $k=1,\cdots,l$). Then using (*), it is easy to show that $x(\mathcal{F})-\alpha(x)(\mathcal{F}_1\times\mathcal{F}_2)$ is an exact form. However according to Vey, cocycles of this type can be chosen as a basis for $H^*(WO'_n)$ (note that WO'_n and WO_n are mutually isomorphic differential graded algebras). This shows that our definition of α is true for this particular product foliation case. Then the general case follows from the usual argument (cf. [**2**]).

§ 3. Main theorem.

Let S be a complex analytic surface constructed by Kodaira (cf. [**1**]), having the following properties.

(i) S is the total space of a fibre bundle over a curve with fibre another curve.

(ii) $\mathrm{sign}(S)\neq 0$.

Let $\xi\subset\tau(S)$ be the tangent bundle along the fibres. From (ii) we conclude that

$$P_1(\xi)=3\,\mathrm{sign}\,S\neq 0.$$

According to Thurston [**5**], $B\Gamma_2$ is 3-connected. Therefore again by Thurston [**4**], ξ is homotopic to the normal bundle of a codimension 2 foliation $\mathcal{F}$. We have

$$P_1(\nu(\mathcal{F}))=3\,\mathrm{sign}\,S\neq 0.$$

Now let $(M^{2n+1}, \mathcal{G}_t^n)$ be the family of codimension n foliations on a manifold M^{2n+1} constructed by Thurston [3], such that

$$\langle gv(\mathcal{G}_t^n), [M^{2n+1}]\rangle = t \in \boldsymbol{R}$$

where t ranges over some open set of $\boldsymbol{R}$. We consider the product foliation

$$\underbrace{(S, \mathcal{F})\times \cdots \times(S, \mathcal{F})}_{i}\times(M^{2n+1-4i}\mathcal{G}t).$$

These are a family of codimension n foliations on $(2n+1)$ manifold $(S)^i\times M^{2n+1-4i}$. We claim that some characteristic class in $H^{2n+1}(B\Gamma_n^\infty; \boldsymbol{R})$ varies continuously on this family.

More precisely we have the following. Let $r(n)$ be the greatest integer satisfying the inequality $2n+1-4r(n)\geqq 3$. Thus $r(n)+1=\left[\frac{n+1}{2}\right]$. Then we have

THEOREM. There is a surjective homomorphism

$$H_{2n+1}(B\Gamma_n^\infty; \boldsymbol{Z}) \longrightarrow \boldsymbol{R}^{[n+1/2]} \longrightarrow 0$$

for any $n\geqq 1$.

PROOF. We consider the following family of foliations.

$$(N^i, \mathcal{H}_t^i)=((S)^i\times M^{2n+1-4i}, \mathcal{F}^i\times\mathcal{G}t) \qquad i=0, 1, \cdots, r(n).$$

Next we consider characteristic classes $\gamma_1\Sigma_1^n, \gamma_1\Sigma_1^{n-2}\Sigma_2, \cdots, \gamma_1\Sigma_1^{n-2r(n)}\Sigma_2^{r(n)}$. We claim that

$$(1) \qquad \langle\gamma_1\, \Sigma_1^{n-2i}\, \Sigma_2^i\,(\mathcal{H}^i t), [N^i]\rangle = i\,!\cdot(-6 \operatorname{sign} S)^i\cdot t \qquad i=0, \cdots, r(n).$$

$$(2) \qquad \langle\gamma_1\, \Sigma_1^{n-2i}\, \Sigma_2^i\,(\mathcal{H}^j t), [N^i]\rangle = 0 \qquad \text{for } j>i.$$

Clearly our theorem follows from these two statements.

Now we first verify (1). We have a map

$$\alpha: WO'_n \longrightarrow WO'_2\otimes\cdots\otimes WO'_2\otimes WO'_{n-2i},$$

which is an iteration of the maps of type $WO'_n\to WO'_p\otimes WO'_q$ considered in § 2. Let $x\in H^*(WO'_2\otimes\cdots\otimes WO'_2\otimes WO'_{n-2i})$ be a cohomology class, and let $(a_1, a_2, \cdots, a_{i+1})$ be an $(i+1)$-tuple of non-negative integers. We define $x_{(a_1,a_2,\cdots,a_{i+1})}$ to be the multi-degree $(a_1, a_2, \cdots, a_{i+1})$ part of x. Then we have, by Proposition 1,

$$\begin{aligned}&\alpha_*([\gamma_1\, \Sigma_1^{n-2i}\, \Sigma_2^i])_{(4,4,\cdots,4,2n+1-4i)}\\ &\quad = i\,![\Sigma_2]\otimes[\Sigma_2]\otimes\cdots\otimes[\Sigma_2]\otimes[\gamma_1\Sigma_1^{n-2i}].\end{aligned}$$

Therefore

$$\langle[\gamma_1\Sigma_1^{n-2i}\Sigma_2^i](\mathcal{H}^i t),\ [N^i]\rangle = i!(-6\,\mathrm{sign}\,S)^i\cdot t.$$

Next we prove (2).

The same calculation as above using proposition 1 yields

$$\alpha_*([\gamma_1\Sigma_1^{n-2i}\Sigma_2^i])_{(4,4,\cdots,4,2n+1-4j)} = 0 \qquad \text{for } j>i.$$

This proves (2) and hence our Theorem.

REMARK 3. Let us call an element $x\in H^{2n+1}(WO'_n)$ "indecomposable" if $\alpha_*(x)=0$ for any factorization $\alpha: WO'_n \to WO'_{i_1}\otimes\cdots\otimes WO'_{i_k}$ $(i_1+\cdots+i_k=n)$. Let x be such an element. Then obviously,

$$x(\text{product foliation}) = 0.$$

It is easy to show that the classes

$$\gamma_{2i+1}\Sigma_{j_1}^{k_1}\cdots\Sigma_{j_s}^{k_1} \in H^{2n+4i+1}(WO'_{n+2i}) \qquad (n=j_1k_1+\cdots+j_sk_s)$$

are indecomposable if j_l is odd for every $l=1,\cdots,s$. Thus we have infinitely many series of indecomposable elements. For example,

$$\gamma_1\Sigma_1,\ \gamma_1\Sigma_1^2,\ \gamma_1\Sigma_1^3,\ \cdots$$

$$\gamma_1\Sigma_3,\ \gamma_1\Sigma_1\Sigma_3,\ \gamma_1\Sigma_1^2\Sigma_3,\ \cdots$$

$$\gamma_1\Sigma_5,\ \gamma_1\Sigma_1\Sigma_5,\ \gamma_1\Sigma_1^2\Sigma_5,\ \cdots.$$

We also have another type of indecomposable elements, e.g. $\gamma_1\Sigma_2\in H^5(WO'_2)$.

By the argument in this note, the problem of continuous variation of characteristic classes in $H^{2n+1}(WO'_n)$ splits into the following two problems.

(1) To construct a family of codimension n foliation on some M^{2n+1}'s on which indecomposable elements take values continuously and independently.

(2) To show that the natural map $\pi^*: H^{4k}(BGL_{2k}\boldsymbol{R};\boldsymbol{R})\to H^{4k}(B\Gamma_{2k}^\infty;\boldsymbol{R})$ is injective. (This is the case for $k=1$ by Thurston [**5**].)

REMARK 4. Finally we remark the relationship between Bott's definition and our definition of secondary classes. Let us define a homomorphism

$$\beta: WO'_n \longrightarrow WO_n$$

as follows.

As is well-known, Σ_i can be expressed as a polynomial of c_i's (and vice versa). For example $\Sigma_1=c_1$, $\Sigma_2=c_1^2-2c_2$ and so on;

$$\Sigma_i = \sum_{k_1\cdots k_s} b^i_{j_1\cdots j_s} c_{j_1}^{k_1}\cdots c_{j_s}^{k_s}.$$

Then we define

$$\beta(\Sigma_i) = \sum_{k_1\cdots k_s} b^i_{j_1\cdots j_s} c_{j_1}^{k_1}\cdots c_{j_s}^{k_s}.$$

If i is odd, then at least one j_l $(l=1, \cdots, s)$ is odd. We assume that j_1 is odd. Then we define

$$\beta(\gamma_{2i+1}) = \sum_{k_1\cdots k_s} b^{2i+1}_{j_1\cdots j_s} h_{j_1} c_{j_1}^{k_1-1} \cdots c_{j_s}^{k_s}.$$

We have

PROPOSITION 5. *The following diagram is commutative.*

$$\begin{array}{ccc} H^*(WO'_n) & \searrow & \\ \downarrow \beta_* & & H^*(B\Gamma_n^\infty; \boldsymbol{R}). \\ H^*(WO_n) & \nearrow & \end{array}$$

The proof of this proposition is very similar to that of Proposition 1 and omitted.

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