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Unbounded representations of symmetric $*$-algebras

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§ 1. Introduction.

In [**16**], R. T. Powers began studying a representation of a $*$-algebra as an algebra of unbounded operators on a Hilbert space. A class of symmetric unbounded operator algebras (called symmetric $\sharp$-algebras, $EC^{\#}$-algebras, $EW^{\#}$-algebras, EC^{*}-algebras and EW^{*}-algebras) have been studied by P. G. Dixon [**3, 4**], the author [**9, 10, 11, 12**] and others.

In this paper we shall study unbounded representations of symmetric $*$-algebras. Let A be a symmetric $*$-algebra and let π be a representation of A on a Hilbert space $\mathfrak{H}$. Then we divide π into the following three types. If $\pi(x)$ is a bounded operator for all $x \in A$, then π is called a bounded representation. If π is unitarily equivalent to the direct sum of bounded representations of A, then π is called a weakly unbounded representation of A. If π has not any bounded subrepresentation of π, then π is called a strictly unbounded representaton. In § 3 we obtain the following theorems.

THEOREM 3.11. *If π is closed, then it is unitarily equivalent to the direct sum of strongly cyclic closed representations.*

THEOREM 3.13. *If π is closed, then there are a weakly unbounded closed representation π_1 of A and a strictly unbounded closed representation π_2 of A such that π is unitarily equivalent to the direct sum of π_1 and π_2.*

In § 4, we shall consider the relation of positive linear functionals and representations. Let f be a positive linear functional on A. By Gelfand-Segal construction there is a strongly cyclic closed representation π_f of A on a Hilbert space $\mathfrak{H}_f$ with a strongly cyclic vector ξ_f such that $f(x)=(\pi_f(x)\xi_f \mid \xi_f)$ for all $x \in A$. We divide f into the following three types. An f is said to be relatively bounded if π_f is bounded. An f is said to be approximately relatively bounded if an f is contained in the weak closure of $\{g; f \geqq g \geqq 0$ and g is relatively bounded$\}$. An f is said to be strictly relatively unbounded if there is not any non-zero positive linear functional g such that $f \geqq g$ and g is relatively bounded. The primary purpose of this section is to show the following two theorems.

THEOREM 4.4. *There exists a decomposition of f such that $f=f_1+f_2$, f_1 is* *approximately relatively bounded and f_2 is strictly relatively unbounded.*

THEOREM 4.5. *f is relatively bounded (resp. approximately relatively bounded, strictly relatively unbounded) if and only if π_f is bounded (resp. weakly unbounded, strictly unbounded).*

The author would like to thank Prof. Dr. R. T. Powers for giving him the basic ideas in [16].

§ 2. Preliminaries.

We begin with some basic terminology.

A $*$-algebra is an algebra A over the complex field $\mathfrak{C}$ with an involution $*$ satisfying the usual axioms;

$$(1)\qquad (\lambda x+\mu y)^*=\bar{\lambda}x^*+\bar{\mu}y^* \qquad (x, y\in A\,;\ \lambda, \mu\in\mathfrak{C}),$$

$$(2)\qquad (xy)^*=y^*x^* \qquad (x, y\in A),$$

$$(3)\qquad x^{**}=x \qquad (x\in A).$$

An element x of A is called hermitian if $x^*=x$. The set of all hermitian elements of A is denoted by A_h.

DEFINITION 2.1. Let A be a $*$-algebra with identity e. If, for every $x\in A$, $(e+x^*x)^{-1}$ exists in A, then A is said to be symmetric.

If S and T are linear operators on a Hilbert space $\mathfrak{H}$ with domains $\mathfrak{D}(S)$ and $\mathfrak{D}(T)$ we say S is an extension of T, denoted by $S\supset T$, if $\mathfrak{D}(S)\supset\mathfrak{D}(T)$ and $S\xi=T\xi$ for all $\xi\in\mathfrak{D}(T)$. If S is a closable operator we denote by $\bar{S}$ the smallest closed extension of S. Let $\mathfrak{A}$ be a set of closable operators on $\mathfrak{H}$. Then we set

$$\bar{\mathfrak{A}}=\{\bar{S}\,;\, S\in\mathfrak{A}\}\,.$$

If S is a linear operator with dense domain $\mathfrak{D}(S)\subset\mathfrak{H}$ we denote by S^* the hermitian adjoint of S. S^* is always a closed operator. However, the domain $\mathfrak{D}(S^*)$ may not be dense in $\mathfrak{H}$. In fact, S is closable if and only if $\mathfrak{D}(S^*)$ is dense in $\mathfrak{H}$ and if $\mathfrak{D}(S^*)$ is dense in $\mathfrak{H}$ then $\bar{S}=S^{**}$. Let S, T be closed operators on $\mathfrak{H}$. If $S+T$ is closable, then $\overline{S+T}$ is called the strong sum of S and T, and is denoted $S+T$. The strong product is likewise defined to be $\overline{ST}$ if it exists, and is denoted $S\cdot T$. The strong scalar multiplication $\lambda\in\mathfrak{C}$ and S is defined by $\lambda\cdot S=\lambda S$ if $\lambda\neq 0$, and $\lambda\cdot S=0$ if $\lambda=0$.

Let $\mathfrak{D}$ be a pre-Hilbert space with an inner product $(\ |\ )$ and let $\mathfrak{H}$ be the completion of $\mathfrak{D}$. We denote by $\mathfrak{L}(\mathfrak{D})$ the set of all linear operators on $\mathfrak{D}$. A subalgebra $\mathfrak{A}$ of $\mathfrak{L}(\mathfrak{D})$ is called a $\#$-algebra on $\mathfrak{D}$ if there exists an involution $S\to S^{\#}$ on $\mathfrak{A}$ satisfying

$$(S\xi\,|\,\eta)=(\xi\,|\,S^{\#}\eta)$$

for all $S\in\mathfrak{A}$ and $\xi, \eta\in\mathfrak{D}$. Let $\mathfrak{A}$ be a $\#$-algebra on $\mathfrak{D}$ and let $\mathfrak{B}(\mathfrak{H})$ be the set of all bounded linear operators on $\mathfrak{H}$. We set

$$\mathfrak{A}_b = \{S \in \mathfrak{A} ; \bar{S} \in \mathfrak{B}(\mathfrak{H})\} .$$

DEFINITION 2.2. Let $\mathfrak{A}$ be a $\#$-algebra on $\mathfrak{D}$ with an identity operator I. If $(I+S^{\#}S)^{-1}$ exists and lies in $\mathfrak{A}_b$ for all $S \in \mathfrak{A}$, then $\mathfrak{A}$ is called a symmetric $\#$-algebra on $\mathfrak{D}$. Let $\mathfrak{A}$ be a symmetric $\#$-algebra on $\mathfrak{D}$. If $\mathfrak{A}_b$ is a C^*-algebra (resp. W^*-algebra), then $\mathfrak{A}$ is said to be an $EC^{\#}$-algebra (resp. $EW^{\#}$-algebra). In particular, a symmetric $\#$-algebra (resp. $EC^{\#}$-algebra, $EW^{\#}$-algebra) $\mathfrak{A}$ is said to be pure if $\mathfrak{A} \neq \mathfrak{A}_b$.

These algebras are examples of symmetric $*$-algebras and are of frequent occurrence in functional analysis. In fact, in [**11**] we have showed that if a maximal Hilbert algebra is not a Hilbert space then there necessarily exist pure $EW^{\#}$-algebras.

For a more complete discussion of the basic properties of unbounded operator algebras the reader is referred to [**9, 10, 11, 12**].

§ 3. Representations of symmetric $*$-algebras.

In this paper let A be a symmetric $*$-algebra with identity e.

DEFINITION 3.1. We call π a representation of A on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(\pi)$ if $\mathfrak{D}(\pi)$ is a dense subspace of $\mathfrak{H}$ and π is a homomorphism of A onto a $\#$-algebra on $\mathfrak{D}(\pi)$. That is,

(1) $$\pi(A) \subset \mathfrak{L}(\mathfrak{D}(\pi)) ,$$

(2) $$\pi(\lambda x + \mu y) = \lambda\pi(x) + \mu\pi(y) \qquad (x, y \in A ; \ \lambda, \mu \in \mathfrak{C}) ,$$

(3) $$\pi(xy) = \pi(x)\pi(y) \qquad (x, y \in A) ,$$

(4) $$\pi(x^*) = \pi(x)^{\#} \qquad (x \in A) .$$

LEMMA 3.2. *Let π be a representation of A on a Hilbert space $\mathfrak{H}$. Then $\pi(A)$ is a symmetric $\#$-algebra on $\mathfrak{D}(\pi)$.*

PROOF. For every $x \in A$ we have $I+\pi(x)^{\#}\pi(x) = \pi(e+x^*x)$ and since A is symmetric, $(I+\pi(x)^{\#}\pi(x))^{-1}$ exists and equals $\pi((e+x^*x)^{-1})$. Hence we have only to show $\overline{(I+\pi(x)^{\#}\pi(x))^{-1}} \in \mathfrak{B}(\mathfrak{H})$. In fact, $\overline{\pi(x)}$ is a closed operator, and so $(\bar{I}+\overline{\pi(x)}^*\overline{\pi(x)})^{-1} \in \mathfrak{B}(\mathfrak{H})$. It is easy to show

$$(\bar{I}+\overline{\pi(x)}^*\overline{\pi(x)})^{-1}/\mathfrak{D}(\pi) = (I+\pi(x)^{\#}\pi(x))^{-1} ,$$

where $S/\mathfrak{D}(\pi)$ denotes the restriction of an operator S onto $\mathfrak{D}(\pi)$. Therefore we have $\overline{(I+\pi(x)^{\#}\pi(x))^{-1}} = (I+\overline{\pi(x)}^*\overline{\pi(x)})^{-1}$.

LEMMA 3.3. *Let π be a representation of A on a Hilbert space $\mathfrak{H}$. Then*

$$\overline{\overline{\pi(x)}+\overline{\pi(y)}} = \overline{\pi(x+y)} , \qquad \overline{\overline{\pi(x)}\cdot\overline{\pi(y)}} = \overline{\pi(xy)} ,$$

$$\lambda\cdot\overline{\pi(x)}=\overline{\pi(\lambda x)}\,, \qquad \overline{\pi(x)}^*=\overline{\pi(x^*)}\,,$$

for all $x, y\in A$ *and* $\lambda\in\mathfrak{C}$. *Therefore* $\overline{\pi(A)}$ *is a* $*$*-algebra of closed operators under the operations of strong sum, strong product, adjoint and strong scalar multiplication. Furthermore,* $(\bar{I}+\overline{\pi(x)}^*\overline{\pi(x)})^{-1}$ *exists and lies in* $\overline{\pi(A)}_b$ *for every* $x\in A$.

PROOF. This follows from Lemma 3.2 and ([**9**] Theorem 2.3).

Let π be a representation of A on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(\pi)$. Then there is a natural induced topology τ_0 on $\mathfrak{D}(\pi)$. This topology is defined as follows. Suppose S is a finite subset of A. We define the seminorm $\|\ \|_S$ on $\mathfrak{D}(\pi)$ as

$$\|\xi\|_S=\sum_{x\in S}\|\pi(x)\xi\|\,,$$

where $\|\xi\|$ is the Hilbert space norm of ξ. We define the induced topology on $\mathfrak{D}(\pi)$ as the topology generated by the seminorms $\{\|\ \|_S\,;\ S$ is a finite subset of $A\}$.

DEFINITION 3.4. Let π be a representation of A on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(\pi)$. If $(\mathfrak{D}(\pi)\,;\tau_0)$ is complete, then π is said to be closed.

LEMMA 3.5. *If* π *is a closed representation of* A *on a Hilbert space* $\mathfrak{H}$, *then we have*

$$\mathfrak{D}(\pi)=\bigcap_{x\in A}\mathfrak{D}(\overline{\pi(x)})=\bigcap_{x\in A}\mathfrak{D}(\pi(x)^*)\,.$$

PROOF. This follows from Lemma 3.3 and ([**16**] Lemma 2.6).

In this section let π be a closed representation of A on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(\pi)$.

DEFINITION 3.6. The commutant of $\pi(A)$, denoted by $\pi(A)'$, consists of all bounded operators C on $\mathfrak{H}$ such that

$$(C\pi(x)\xi\,|\,\eta)=(C\xi\,|\,\pi(x^*)\eta)$$

for all ξ, $\eta\in\mathfrak{D}(\pi)$ and $x\in A$.

LEMMA 3.7. *The commutant* $\pi(A)'$ *is a von Neumann algebra. Furthermore, for each* $C\in\pi(A)'$ *we have*

$$C\mathfrak{D}(\pi)\subset\mathfrak{D}(\pi) \quad \textit{and} \quad C\pi(x)\xi=\pi(x)C\xi$$

for all $x\in A$ *and* $\xi\in\mathfrak{D}(\pi)$.

PROOF. This follows from Lemma 3.5 and ([**16**] Lemma 4.6).

DEFINITION 3.8. A vector $\xi\in\mathfrak{D}(\pi)$ is said to be strongly cyclic if $\{\pi(A)\xi\}$ is dense in $(\mathfrak{D}(\pi)\,;\tau_0)$. If π has a strongly cyclic vector, then it is said to be strongly cyclic.

Let $\mathfrak{M}$ be a linear subspace of $\mathfrak{D}(\pi)$. If $\pi(x)\mathfrak{M}\subset\mathfrak{M}$ for all $x\in A$, then $\mathfrak{M}$ is said to be π-invariant. We denote by $\bar{\mathfrak{M}}$ (resp. $\mathfrak{M}^-$) the closure of $\mathfrak{M}$ under the Hilbert space norm (resp. the induced topology τ_0). If $\mathfrak{M}=\bar{\mathfrak{M}}$ (resp. $\mathfrak{M}=\mathfrak{M}^-$), then $\mathfrak{M}$ is said to be closed (resp. τ_0-closed). We denote by $\pi/\mathfrak{M}$ the representation π restricted to $\mathfrak{M}$. If $\mathfrak{M}$ is a τ_0-closed π-invariant subspace of $\mathfrak{D}(\pi)$, then $\pi/\mathfrak{M}$ is a closed representation of A on $\bar{\mathfrak{M}}$.

After a slight modification of Powers' ([**16**] Theorem 4.7), we have the following theorem.

THEOREM 3.9. *Let $\pi(A)'_p$ denote the set of all projections in $\pi(A)'$.*

(1) *Suppose $E \in \pi(A)'_p$. Then $\mathfrak{M} \underset{\text{def.}}{=} E\mathfrak{D}(\pi)$ is a π-invariant τ_0-closed subspace of $\mathfrak{D}(\pi)$.*

(2) *Conversely suppose that $\mathfrak{M}$ is a π-invariant τ_0-closed subspace of $\mathfrak{D}(\pi)$. Then the projection $E_{\mathfrak{M}}$ onto $\bar{\mathfrak{M}}$ is in $\pi(A)'$.*

Hence there is a one-to-one correspondence between projections in $\pi(A)'$ and π-invariant τ_0-closed subspaces of $\mathfrak{D}(\pi)$.

DEFINITION 3.10. We call π_1 a subrepresentation of π if there is a π-invariant τ_0-closed subspace $\mathfrak{M}$ of $\mathfrak{D}(\pi)$ such that $\pi_1 = \pi/\mathfrak{M}$, and is denoted by $\pi_{\mathfrak{M}}$ or $\pi_{E_{\bar{\mathfrak{M}}}}$.

It is easily showed that $\pi_{\mathfrak{M}}$ is a closed representation of A on $\bar{\mathfrak{M}}$ with domain $\mathfrak{M}$.

We define the direct sum of representations of A. Suppose that $\{\pi_\alpha; \alpha \in \Lambda\}$ is a collection of closed representations π_α of A on Hilbert spaces $\mathfrak{H}_\alpha$. We denote the direct sum of these representations by $\rho = \bigoplus_{\alpha \in \Lambda} \pi_\alpha$ and define ρ as follows. Let $\mathfrak{H} = \bigoplus_{\alpha \in \Lambda} \mathfrak{H}_\alpha$ be the direct sum of $\mathfrak{H}_\alpha$ and let

$$\mathfrak{D}(\rho) = \{\xi = \{\xi_\alpha\} \in \mathfrak{H}\,;\, \xi_\alpha \in \mathfrak{D}(\pi_\alpha) \quad \text{for all } \alpha \in \Lambda \\ \text{and } \sum_{\alpha \in \Lambda} \|\pi_\alpha(x)\xi_\alpha\|^2 < \infty \quad \text{for all } x \in A\}\,.$$

We define $\rho(x)\xi = \rho(x)\{\xi_\alpha\} = \{\pi_\alpha(x)\xi_\alpha\}$ for all $\xi = \{\xi_\alpha\} \in \mathfrak{D}(\rho)$ and $x \in A$. It is easily seen that ρ is a closed representation of A on $\mathfrak{H}$ with domain $\mathfrak{D}(\rho)$.

Let π' be a representation of A on a Hilbert space $\mathfrak{H}'$. If there exists a unitary transform U of $\mathfrak{H}'$ onto $\mathfrak{H}$ such that $U\mathfrak{D}(\pi') = \mathfrak{D}(\pi)$ and $\pi(x)U\xi = U\pi'(x)\xi$ for all $x \in A$ and $\xi \in \mathfrak{D}(\pi')$, then π and π' are said to be unitarily equivalent, and are denoted by $\pi \cong \pi'$.

THEOREM 3.11. *The π is unitarily equivalent to the direct sum of strongly cyclic closed representation.*

PROOF. Let ξ_0 be a non-zero vector in $\mathfrak{D}(\pi)$ and $\mathfrak{M}_0 = \{\pi(A)\xi_0\}$. Then $\mathfrak{M}_0^{-}$ is a π-invariant τ_0-closed subspace of $\mathfrak{D}(\pi)$. From Theorem 3.9, $E_0 \underset{\text{def.}}{=} E_{\bar{\mathfrak{M}}_0} \in \pi(A)'$ and $\mathfrak{M}_0^{-} = E_0\mathfrak{D}(\pi)$. If $E_0 = I$, then π is strongly cyclic. Suppose $E_0 \neq I$. Since $(I - E_0) \in \pi(A)'_p$ and Lemma 3.7, $(I - E_0)\mathfrak{D}(\pi) \subset \mathfrak{D}(\pi)$. From the density of $\mathfrak{D}(\pi)$ in $\mathfrak{H}$, there exists a non-zero vector $\xi_1 \in \mathfrak{D}(\pi)$ such that $\xi_1 \in \bar{\mathfrak{M}}^{0\perp} \underset{\text{def.}}{=} \mathfrak{H} - \bar{\mathfrak{M}}_0$.

Now we consider $\mathfrak{M}_1 \underset{\text{def.}}{=} \{\pi(A)\xi_1\}$. Since $(\pi(x)\xi_0|\pi(y)\xi_1)=(\pi(y^*x)\xi_0|\xi_1)=0$ for all $x, y\in A$, we have $\bar{\mathfrak{M}}_0 \perp \bar{\mathfrak{M}}_1$. Thus, by Zorn's lemma, there is a maximal family $\{\mathfrak{M}_\alpha\}_{\alpha\in\Lambda}$ ($\mathfrak{M}_\alpha=\{\pi(A)\xi_\alpha\}$, $\xi_\alpha\in\mathfrak{D}(\pi)$) such that $\bar{\mathfrak{M}}_\alpha \perp \bar{\mathfrak{M}}_\beta$ for $\alpha\neq\beta$. Putting $E_\alpha=E_{\bar{\mathfrak{M}}_\alpha}$ and $\pi_\alpha=\pi_{E_\alpha}$, π_α is a strongly cyclic closed representation of A on $\bar{\mathfrak{M}}_\alpha$ with $\mathfrak{D}(\pi_\alpha)=\mathfrak{M}_\alpha^-$. We set

$$\mathfrak{H}'=\bigoplus_{\alpha\in\Lambda}\bar{\mathfrak{M}}_\alpha \quad\text{and}\quad \pi'=\bigoplus_{\alpha\in\Lambda}\pi_\alpha .$$

Since the subspaces $\bar{\mathfrak{M}}_\alpha$ are pairwise orthogonal in $\mathfrak{H}$, the series $\sum_\alpha \zeta_\alpha$ converges to an element of $\mathfrak{H}$ for each $\{\zeta_\alpha\}\in\mathfrak{H}'$. Therefore, $\{\zeta_\alpha\}\to\sum_\alpha\zeta_\alpha$ is a unitary transform U of $\mathfrak{H}'$ into $\mathfrak{H}$. Furthermore, we have $U\mathfrak{D}(\pi')=\mathfrak{D}(\pi)$. In fact, from the definition of $\mathfrak{D}(\pi')$ we can easily show $U\mathfrak{D}(\pi')\subset\mathfrak{D}(\pi)$. On the other hand, by the maximality of $\{\mathfrak{M}_\alpha\}_{\alpha\in\Lambda}$ we have $\sum_{\alpha\in\Lambda}E_\alpha=I$. For each $\xi\in\mathfrak{D}(\pi)$ we have $\xi=\sum_\alpha E_\alpha\xi$, $E_\alpha\xi\in\mathfrak{M}_\alpha^-=\mathfrak{D}(\pi_\alpha)$ and $\sum_{\alpha\in\Lambda}\|\pi_\alpha(x)E_\alpha\xi\|^2=\sum_{\alpha\in\Lambda}\|\pi(x)E_\alpha\xi\|^2=\|\pi(x)\xi\|^2$. Therefore, $\{E_\alpha\xi\}\in\mathfrak{D}(\pi')$ and $\xi=\sum_\alpha E_\alpha\xi=U\{E_\alpha\xi\}$. Hence, $\mathfrak{D}(\pi)\subset U\mathfrak{D}(\pi')$. Thus we have $\mathfrak{D}(\pi)=U\mathfrak{D}(\pi')$. Since $\mathfrak{D}(\pi)$ and $\mathfrak{D}(\pi')$ are dense in $\mathfrak{H}$ and $\mathfrak{H}'$ respectively, U is extended to a unitary transform of $\mathfrak{H}'$ onto $\mathfrak{H}$. Furthermore, we have

$$U\pi'(x)\{\zeta_\alpha\}=U\{\pi_\alpha(x)\zeta_\alpha\}=\sum_\alpha\pi_\alpha(x)\zeta_\alpha$$

and

$$\pi(x)U\{\zeta_\alpha\}=\pi(x)\sum_\alpha\zeta_\alpha=\sum_\alpha\pi_\alpha(x)\zeta_\alpha$$

for all $x\in A$ and $\{\zeta_\alpha\}\in\mathfrak{D}(\pi')$. Hence π and π' are unitarily equivalent.

DEFINITION 3.12. If $\pi(x)$ is a bounded operator for all $x\in A$, then π is said to be bounded. If π is unitarily equivalent to the direct sum of bounded representations of A, then π is said to be weakly unbounded. If π has not any bounded subrepresentation of π, then π is said to be strictly unbounded.

If π is a bounded representation of A, then $\pi(A)$ is a Banach $*$-algebra under the uniform topology. If π is a weakly unbounded representation of A, then $\pi(A)$ is an LMC $*$-algebra defined by E. A. Michael [**14**]. (He defined an LMC $*$-algebra to be a $*$-algebra with a locally convex topology given by a family of seminorms $\{P_\lambda\}_{\lambda\in\Lambda}$ satisfying the conditions; $P_\lambda(xy)\leqq P_\lambda(x)P_\lambda(y)$ and $P_\lambda(x^*)=P_\lambda(x)$.) In fact, let $\pi=\bigoplus_{\alpha\in\Lambda}\pi_\alpha$, where π_α is a bounded representation of A on $\mathfrak{H}_\alpha$ for every $\alpha\in\Lambda$. We set

$$\|\pi(x)\|_\alpha=\|\pi_\alpha(x)\|, \qquad x\in A,$$

where $\|\pi_\alpha(x)\|$ denotes the operator norm of $\pi_\alpha(x)$. Then $\|\ \|_\alpha$ is a seminorm on $\pi(A)$. It is not difficult to show that $(\pi(A);\{\|\ \|_\alpha\}_{\alpha\in\Lambda})$ is an LMC $*$-algebra.

THEOREM 3.13. *The π is unitarily equivalent to the direct sum of a weakly unbounded representation of A and a strictly unbounded representation of A.*

PROOF. Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a maximal family of non-zero mutually orthogonal projections in $\pi(A)'$ such that π_{E_α} is a bounded representation for all $\alpha\in\Lambda$. We set

$$E_1=\sum_{\alpha\in\Lambda}E_\alpha,\ E_2=I-E_1,\ \pi_1=\pi_{E_1}\quad\text{and}\quad\pi_2=\pi_{E_2}.$$

Then we have $\pi\cong\pi_1\oplus\pi_2$ and $\pi_1\cong\bigoplus_{\alpha\in\Lambda}\pi_\alpha$. Therefore π_1 is weakly unbounded. If $E_2\neq 0$, then π_2 is strictly unbounded. In fact, suppose that π_2 is not strictly unbounded. Then there is a non-zero projection E_0 in $\pi_2(A)'$ such that $(\pi_2)_{E_0}$ is a bounded subrepresentation of π_2. Clearly we can regard E_0 as an element of $\pi(A)'_p$. Hence π_{E_0} is a bounded subrepresentation of π and $0\neq E_0\leqq E_2=I-E_1$. This contradicts the maximality of $\{E_\alpha\}_{\alpha\in\Lambda}$. Therefore π_2 is strictly unbounded.

§ 4. Positive linear functionals and representations.

A linear functional f on A is said to be positive if $f(x^*x)\geqq 0$ for every $x\in A$. If f is a positive linear functional on A, then $f(x^*)=\overline{f(x)}$, $x\in A$ and $|f(y^*x)|^2\leqq f(y^*y)f(x^*x)$, $x, y\in A$ (the Cauchy-Schwartz inequality for positive functionals). Let f, g be linear functionals on A. We write $f\leqq g$ for $g-f\geqq 0$. Let $A^*(+)$ denote the set of all positive linear functionals on A.

PROPOSITION 4.1. *Define a positive linear functional f on A by*

$$f(x)=(\pi(x)\xi\,|\,\xi),\qquad \xi\in\mathfrak{D}(\pi).$$

Then the following facts are satisfied.

(1) *If $T\in\pi(A)'$ with $0\leqq T\leqq I$, then the functional*

$$x\longrightarrow(\pi(x)T\xi\,|\,\xi)$$

on A is a positive linear functional f_T and $f_T\leqq f$.

(2) *If ξ is cyclic for π, then $T\to f_T$ is injective.*

(3) *Let $f'\in A^*(+)$. Then $f'\leqq f$ if and only if there is a $T\in\pi(A)'$ such that $0\leqq T\leqq I$ and $f'=f_T$.*

PROOF. (1), (2); Obvious.

(3); Suppose $f'\leqq f$. Define

$$\langle\pi(x)\xi,\ \pi(y)\xi\rangle=f'(y^*x)$$

for all $x, y\in A$. Then $\langle\ ,\ \rangle$ is a bilinear functional on $\mathfrak{M}=\{\pi(A)\xi\}$. Since

$$|\langle\pi(x)\xi,\ \pi(y)\xi\rangle|^2=|f'(y^*x)|^2\leqq f'(y^*y)f'(x^*x)$$
$$\leqq f(y^*y)f(x^*x)=\|\pi(x)\xi\|^2\|\pi(y)\xi\|^2,$$

the bilinear functional $\langle\ ,\ \rangle$ on $\mathfrak{M}$ is uniquely extended to the bounded bilinear functional on $\bar{\mathfrak{M}}$ and

$$|\langle \eta, \zeta \rangle| \leqq \|\eta\| \|\zeta\|$$

for all $\eta, \zeta \in \bar{\mathfrak{M}}$. Therefore there exists a $T_0 \in \mathfrak{B}(\bar{\mathfrak{M}})$ (the set of all bounded linear operators on $\bar{\mathfrak{M}}$) such that $0 \leqq T_0 \leqq I$ and

$$f(y^*x) = (\pi(x)\xi \mid T_0\pi(y)\xi)$$

for all $x, y \in A$. Since $\mathfrak{M}$ is a π-invariant subspace of $\mathfrak{D}(\pi)$, we have $E_{\bar{\mathfrak{M}}} \in \pi(A)'$. Define $T = T_0 E_{\bar{\mathfrak{M}}}$. Clearly we have $T \in \mathfrak{B}(\mathfrak{H})$ and $0 \leqq T \leqq I$. We shall show $T \in \pi(A)'$. That is,

$$(T\pi(x)\eta \mid \zeta) = (T\eta \mid \pi(x^*)\zeta)$$

for all $x \in A$ and $\eta, \zeta \in \mathfrak{D}(\pi)$. Since $E_{\bar{\mathfrak{M}}}\mathfrak{D}(\pi) = \mathfrak{M}^-$, we have only to show

$$(T_0\pi(x)\pi(y)\xi \mid \pi(z)\xi) = (T_0\pi(y)\xi \mid \pi(x^*)\pi(z)\xi)$$

for all x, y and z in A. We have

$$\begin{aligned} (T_0\pi(x)\pi(y)\xi \mid \pi(z)\xi) &= f'(z^*xy) = f'((x^*z)^*y) \\ &= (\pi(y)\xi \mid T_0\pi(x^*z)\xi) \\ &= (T_0\pi(y)\xi \mid \pi(x^*)\pi(z)\xi) . \end{aligned}$$

Therefore we have $T \in \pi(A)'$. Furthermore, for each $x \in A$ we have

$$\begin{aligned} f'(x) &= \langle \pi(x)\xi, \pi(e)\xi \rangle = (\pi(x)\xi \mid T\xi) \\ &= (T\pi(x)\xi \mid \xi) = (\pi(x)T\xi \mid \xi) \\ &= f_T(x) . \end{aligned}$$

By Gelfand-Segal construction, there is a strongly cyclic closed representation π_f of A on a Hilbert space $\mathfrak{H}_f$ with a strongly cyclic vector ξ_f such that $f(x) = (\pi_f(x)\xi_f \mid \xi_f)$ for all $x \in A$.

DEFINITION 4.2. Let $f \in A^*(+)$. An f is said to be relatively bounded if π_f is a bounded representation of A on $\mathfrak{H}_f$. That is, there exists a constant M_x such that $f(a^*x^*xa) \leqq M_x f(a^*a)$ for all $a \in A$. An f is said to be approximately relatively bounded if an f is contained in the weak closure of $\{g \in A^*(+) ; f \geqq g \geqq 0$ and g is relatively bounded$\}$. An f is said to be strictly relatively unbounded if there is not a non-zero element g of $A^*(+)$ such that $f \geqq g \geqq 0$ and g is relatively bounded.

THEOREM 4.3. *If f_1 and f_2 are relatively bounded (resp. approximately relatively bounded, strictly relatively unbounded), then $f_1 + f_2$ is relatively bounded (resp. approximately relatively bounded, strictly relatively unbounded).*

PROOF. Let f_1 and f_2 be relatively bounded (resp. approximately relatively bounded). Then it is easy to show that $f_1 + f_2$ is relatively bounded (resp. approximately relatively bounded).

Suppose that f_1 and f_2 are strictly relatively unbounded and there is a non-zero element g of $A^*(+)$ such that $f \underset{\text{def.}}{=} f_1+f_2 \geqq g$ and g is relatively bounded. From Proposition 4.1 there are elements T, T_1 and T_2 of $\pi_f(A)'$ such that $0 \leqq T \leqq I$, $0 \leqq T_i \leqq I$ $(i=1, 2)$ and for all $x \in A$

$$g(x)=(\pi_f(x)T\xi_f \,|\, \xi_f), \qquad f_i(x)=(\pi_f(x)T_i\xi_f \,|\, \xi_f) \qquad (i=1, 2).$$

Since g is relatively bounded, for all $x, a \in A$ we have

$$\begin{aligned}\|\pi_f(a)T^{1/2}\pi_f(x)\xi_f\|^2 &= g(x^*a^*ax) \\ &\leqq r_a g(x^*x) \qquad (r_a\,;\ \text{constant}) \\ &= r_a\|T^{1/2}\pi_f(x)\xi_f\|^2 \leqq r_a\|T^{1/2}\|^2\|\pi_f(x)\xi_f\|^2 .\end{aligned}$$

Hence $\overline{\pi_f(a)T^{1/2}}$ is a bounded operator on $\mathfrak{H}_f$ for all $a \in A$. Since $f=f_1+f_2$, we have $T_1+T_2=I$. Let

$$T=\int_0^1 \lambda dE(\lambda), \qquad T_1=\int_0^1 \lambda dE_1(\lambda),$$

where $E(\lambda)$ (resp. $E_1(\lambda)$) is the spectral resolution of T (resp. T_1). Since $T_1+T_2=I$, we have

$$T_2=\int_0^1 (1-\lambda)dE_1(\lambda).$$

(1); Suppose that there exists a λ_0 such that $0<\lambda_0<1$,

$$0<E_1(\lambda_0)<I \quad \text{and} \quad E_1(\lambda_0)TE_1(\lambda_0) \neq 0 .$$

Then we have

$$T_2 \geqq \int_0^{\lambda_0} (1-\lambda)dE_1(\lambda) \geqq \lambda_0 E_1(\lambda_0) \neq 0 .$$

From Proposition 4.1 we have $f_2=f_{T_2} \geqq (f)_{\lambda_0 E_1(\lambda_0)} \neq 0$. Since $E_1(\lambda_0)TE_1(\lambda_0) \neq 0$ and $T \neq 0$, there are the following two cases.

① There is a μ_0 such that $0<\mu_0<1$, $0<E(\mu_0)<I$ and $E_1(\lambda_0)E(\mu_0)E_1(\lambda_0) \neq 0$.

② For each $\mu \in (0, 1)$ with $0<E(\mu)<I$ we have $E_1(\lambda_0)E(\mu)E_1(\lambda_0)=0$.

①; For each $\mu \in (0, 1)$ with $0<E(\mu)<I$ we have

$$T \geqq \int_\mu^1 \lambda dE(\lambda) \geqq \mu E(1-\mu),$$

and so we get, for all $x \in A$,

$$\pi_f(x^*x)T \geqq \mu\pi_f(x^*x)E(1-\mu).$$

Then, since $\overline{\pi_f(x^*x)T}$ is bounded, we have

$$\begin{aligned}\|\pi_f(x)E(1-\mu)\xi\|^2 &= (\pi_f(x^*x)E(1-\mu)\xi \,|\, \xi) \\ &\leqq \frac{1}{\mu}(\pi_f(x^*x)T\xi \,|\, \xi)\end{aligned}$$

$$\leqq \frac{1}{\mu}\|\overline{\pi_f(x^*x)T}\|\|\xi\|^2$$

for all $\xi\in\mathfrak{D}(\pi_f)$. Therefore $\overline{\pi_f(x)E(1-\mu)}$ is bounded for all $x\in A$. In particular, $\overline{\pi_f(x)E(\mu_0)}$ is bounded for all $x\in A$. Since

$$\lambda_0 E_1(\lambda_0)\geqq \lambda_0 E_1(\lambda_0)E(\mu_0)E_1(\lambda_0)\neq 0\,,$$

we have

$$f_2\geqq (f)_{\lambda_0 E_1(\lambda_0)}\geqq g' \underset{\text{def.}}{=} (f)_{\lambda_0 E_1(\lambda_0)E(\mu_0)E_1(\lambda_0)}\neq 0\,.$$

We shall show that g' is relatively bounded. In fact, for all $x, a\in A$ we have

$$\begin{aligned}g'(x^*a^*ax) &= (\pi_f(x^*a^*ax)\lambda_0 E_1(\lambda_0)E(\mu_0)E_1(\lambda_0)\xi_f\,|\,\xi_f)\\ &= \lambda_0\|\pi_f(a)E(\mu_0)E_1(\lambda_0)\pi_f(x)\xi_f\|^2\\ &\leqq \|\overline{\pi_f(a)E(\mu_0)}\|^2\lambda_0\|E(\mu_0)E_1(\lambda_0)\pi_f(x)\xi_f\|^2\\ &= \|\overline{\pi_f(a)E(\mu_0)}\|^2 g'(x^*x)\,.\end{aligned}$$

This contradicts that f_2 is strictly relatively unbounded. Therefore f is strictly relatively unbounded.

②; It is easy to show $E_1(\lambda_0)TE_1(\lambda_0)=\|T\|E_1(\lambda_0)$. We define

$$g'=(f)_{\lambda_0\|T\|E_1(\lambda_0)}\,.$$

Since $0<\|T\|\leqq 1$, we have $f_2\geqq (f)_{\lambda_0 E_1(\lambda_0)}\geqq g'\neq 0$. We shall show that g' is relatively bounded. For each $x, a\in A$ we have

$$\begin{aligned}g'(x^*a^*ax) &= (\pi_f(x^*a^*ax)\lambda_0 E_1(\lambda_0)TE_1(\lambda_0)\xi_f\,|\,\xi_f)\\ &= \lambda_0\|\pi_f(a)T^{1/2}E_1(\lambda_0)\pi_f(x)\xi_f\|^2\\ &\leqq \|\overline{\pi_f(a)T^{1/2}}\|^2\lambda_0\|E_1(\lambda_0)\pi_f(x)\xi_f\|^2\\ &= \frac{1}{\|T\|}\|\overline{\pi_f(a)T^{1/2}}\|^2 g'(x^*x)\,.\end{aligned}$$

This contradicts that f_2 is strictly relatively unbounded. Therefore f is strictly relatively unbounded.

(2); Suppose that there is a λ_0 such that $0<\lambda_0<1$,

$$0<E_1(\lambda_0)<I \quad\text{and}\quad E_1(1-\lambda_0)TE_1(1-\lambda_0)\neq 0\,.$$

After a slight modification of (1), we can show that f is strictly relatively unbounded.

(3); Suppose that for each $\lambda\in(0, 1)$ with $0<E_1(\lambda)<I$, we have

$$E_1(\lambda)TE_1(\lambda)=0 \quad\text{and}\quad (I-E_1(\lambda))T(I-E_1(\lambda))=0\,.$$

Then we have $(I-E_1(\lambda))T=TE_1(\lambda)$, i.e., if $E_1(\lambda)\xi=\xi$ (resp. $E_1(1-\lambda)\xi=\xi$), then $T\xi\in E_1(1-\lambda)\mathfrak{H}_f$ (resp. $T\xi\in E_1(\lambda)\mathfrak{H}_f$). Therefore we have $E_1(\lambda)T^2E_1(\lambda)=T^2E_1(\lambda)$ and $E_1(1-\lambda)T^2E_1(1-\lambda)=T^2E_1(1-\lambda)$. Since $T\neq 0$, we have

$$T^2E_1(\lambda)=E_1(\lambda)T^2E_1(\lambda)\neq 0 \quad \text{or} \quad T^2E_1(1-\lambda)=E_1(1-\lambda)T^2E_1(1-\lambda)\neq 0\,.$$

Therefore, after a slight modification of (1) we can show that f is strictly relatively unbounded.

THEOREM 4.4. *Let $f\in A^*(+)$. Then there exists a decomposition of f such that $f=f_1+f_2$, f_1 is an approximately relatively bounded positive linear functional on A and f_2 is a strictly relatively unbounded positive linear functional on A.*

PROOF. Let $B(f)$ (resp. $B^a(f)$) be the set of all relatively bounded (resp. approximately relatively bounded) positive linear functionals on A. If $B(f)=\{0\}$, then f is strictly relatively unbounded. Suppose $B(f)\neq\{0\}$. $B^a(f)$ is clearly a partially ordered set by the relation $\leqq$. Let B be each totally ordered subset of $B^a(f)$. For each $g\in B$, from Proposition 4.1, there exists a $T_g\in\pi_f(A)'$ such that $0\leqq T_g\leqq I$ and $g(x)=(\pi_f(x)T_g\xi_f\,|\,\xi_f)$ for all $x\in A$. We can easily show that $g_1\leqq g_2$ if and only if $T_{g_1}\leqq T_{g_2}$. Hence there exists an element T of $\pi_f(A)'$ such that $\{T_g\,;g\in B\}$ converges weakly to T and $0\leqq T_g\leqq T$ for all $g\in B$. Then we can easily show that $f_T\in B^a(f)$ and $g\leqq f_T$ for all $g\in B$. Therefore B has an upper bounded element f_T in $B^a(f)$. By Zorn's lemma $B^a(f)$ contains a maximal element f_1. We set

$$f_2=f-f_1\,.$$

Then we shall show that f_2 is strictly relatively unbounded. If not, then there exists a non-zero element g of $A^*(+)$ such that $f_2\geqq g$ and g is relatively bounded. Therefore we have $g\in B(f)$, and so we have $f_1+g\in B^a(f)$ from Theorem 4.3 and $f\geqq f_1+g>f_1$. This contradicts that f_1 is maximal. Therefore f_2 is strictly relatively unbounded.

THEOREM 4.5. *Let $f\in A^*(+)$. Then the following facts are satisfied.*

(1) f is relatively bounded if and only if π_f is bounded.

(2) f is approximately relatively bounded if and only if π_f is weakly unbounded.

(3) f is strictly relatively unbounded if and only if π_f is strictly unbounded.

PROOF. (1); This follows from Definition 4.2.

(3); Suppose that f is strictly relatively unbounded and π_f is not strictly unbounded. Then there is a non-zero element E of $(\pi_f(A)')_p$ such that $(\pi_f)_E$ is bounded. Since

$$f_E(x)=(\pi_f(x)E\xi_f\,|\,\xi_f)$$

for all $x\in A$, we have

$$0<f_E\leqq f \quad \text{and} \quad f_E(x^*a^*ax)\leqq\|\overline{\pi_f(a)E}\|^2 f_E(x^*x)$$

for all $x, a\in A$. This contradicts that f is strictly relatively unbounded.

Conversely suppose that π_f is strictly unbounded. Let g be each non-zero element of $A^*(+)$ with $f\geqq g$. From Proposition 4.1 there is a $T\in\pi_f(A)'$ such that $0<T\leqq I$ and $g(x)=(\pi_f(x)T\xi_f|\xi_f)$ for all $x\in A$. Suppose that g is relatively bounded. Then we have $f\neq g$, and so $T\neq I$. Since g is relatively bounded, $\pi_f(x)T$ is bounded for all $x\in A$. Let

$$T=\int_0^1 \lambda dE(\lambda),$$

where $E(\lambda)$ is the spectral resolution of T. Since $0<T<I$, there is a λ_0 such that $0<\lambda_0<1$ and $0<E(\lambda_0)<I$. We set

$$g_1(x)=f_{\lambda_0 E(1-\lambda_0)}(x)=(\pi_f(x)\lambda_0 E(1-\lambda_0)\xi_f|\xi_f).$$

Since $T\geqq\lambda_0 E(1-\lambda_0)$, we have $g\geqq g_1$. Clearly g_1 is relatively bounded. Therefore $(\pi_f)_{E(1-\lambda_0)}$ is a non-zero bounded subrepresentation of π_f. This contradicts that π_f is strictly unbounded. Therefore g is not relatively bounded, and so f is strictly relatively unbounded.

(2); Suppose that π_f is weakly unbounded. Then π_f is unitarily equivalent to $\bigoplus_{\alpha\in\Lambda}\pi_\alpha$ such that π_α is a bounded representation of A on a Hilbert space $\mathfrak{H}_\alpha$. Let E_α denote the projection onto $\mathfrak{H}_\alpha$. Clearly $E_\alpha\in(\pi_f(A)')_p$. From Theorem 4.4 there are $f_1, f_2\in A^*(+)$ such that f_1 is approximately relatively bounded, f_2 is strictly relatively unbounded and $f=f_1+f_2$. Since

$$f_{E_\alpha}(x)=(\pi_f(x)E_\alpha\xi_f|\xi_f)=(\pi_\alpha(x)E_\alpha\xi_f|E_\alpha\xi_f),$$

we have f_{E_α} is relatively bounded, and so $f_1\neq 0$. Suppose $f_2\neq 0$. Then $f>f_2\neq 0$, and so there is a $T_2\in\pi_f(A)'$ such that $0<T_2<I$ and $f_2(x)=(\pi_f(x)T_2\xi_f|\xi_f)$ for all $x\in A$. Let

$$T_2=\int_0^1 \lambda dE(\lambda),$$

where $E(\lambda)$ is the spectral resolution of T_2. Then there is a λ_0 such that $0<\lambda_0<1$ and $0<E(\lambda_0)<I$. We set

$$g=f_{\lambda_0 E(1-\lambda_0)}.$$

Since

$$T_2\geqq\int_{\lambda_0}^1 \lambda dE(\lambda)\geqq\lambda_0 E(1-\lambda_0)>0,$$

we have $f_2\geqq g\gneqq 0$. Since f_2 is strictly relatively unbounded, g is not relatively bounded, and so $(\pi_f)_{E(1-\lambda_0)}$ is unbounded. Since $\sum_{\alpha\in\Lambda}E_\alpha=I$, there is an $\alpha_0\in\Lambda$ such that $E(1-\lambda_0)E_{\alpha_0}E(1-\lambda_0)\neq 0$. We set

$$g'=f_{\lambda_0 E(1-\lambda_0)E_{\alpha_0}E(1-\lambda_0)}.$$

Then we have $f_2\geqq g\geqq g'\gneqq 0$ and since $\pi_f(x)E_{\alpha_0}$ is bounded, for each $x, a\in A$ we have

$$g'(x^*a^*ax)\leqq\|\overline{\pi_f(a)E_{\alpha_0}}\|^2 g'(x^*x).$$

and hence g' is relatively bounded. This contradicts that f_2 is strictly relatively unbounded. Therefore, $f_2=0$, and so $f=f_1$. That is, f is approximately relatively bounded.

Conversely suppose that f is approximately relatively bounded. From Theorem 3.13 there are a weakly unbounded representation π_1 of A on a Hilbert space $\mathfrak{H}_1$ and a strictly unbounded representation π_2 of A on a Hilbert space $\mathfrak{H}_2$ such that $\pi_f=\pi_1\oplus\pi_2$. Putting

$$E_1=E_{\mathfrak{H}_1} \quad \text{and} \quad E_2=E_{\mathfrak{H}_2},$$

$E_1, E_2\in\pi_f(A)'_p$ and $E_1+E_2=I$. We set

$$\xi_1=E_1\xi_f \quad \text{and} \quad \xi_2=E_2\xi_f .$$

Then it is not difficult to show that ξ_1 and ξ_2 are strongly cyclic vectors for π_1 and π_2 respectively. We define

$$f_1(x)=(\pi_1(x)\xi_1|\xi_1) \quad \text{and} \quad f_2(x)=(\pi_2(x)\xi_2|\xi_2) .$$

Then we have $f=f_1+f_2$, $\pi_1\cong\pi_{f_1}$ and $\pi_2\cong\pi_{f_2}$. Since π_1 is weakly unbounded, f_1 is approximately relatively bounded. Therefore $f-f_1$ is approximately relatively bounded. On the other hand, since π_2 is strictly unbounded, f_2 is strictly relatively unbounded. Therefore we have $f-f_1=f_2=0$. Hence, $\pi_2=0$, i. e., $\pi_f=\pi_1$. Hence π_f is weakly unbounded.

REMARKS. (1) In [**3**], P. G. Dixon has characterized a class of symmetric locally convex $*$-algebras called GB*-algebras as a certain class of closed operators on a Hilbert space. Hence, as representations of symmetric locally convex $*$-algebras it seems that we should consider unbounded representations. We note that we can obtain same results as those in this paper for unbounded representations of symmetric locally convex $*$-algebras. However, all arguments of this paper are algebraic. In order to investigate such representations in detail, it seems that we should begin by studying a class of unbounded operator algebras. In [**9, 10, 11, 12**], we have studied unbounded operator algebras.

(2) R. Godement [**6**] has obtained the integral representation for a unitary (relatively bounded in this paper) positive linear functional f of a commutative $*$-algebra. After that, A. E. Nussbaum [**15**] has extended Godement's theorem to positive linear functionals which satisfy certain growth conditions, but which are not necessarily unitary, By analogy of the Nussbaum's result, we obtain the following result:

Let A be a commutative symmetric $*$-algebra with identity e. We denote by $\hat{A}$ the set of all homomorphisms of A onto $\mathfrak{C}$. If f is a positive linear functional on A satisfying the separability condition (d); there exists a countable subset D of A such that for every $x\in A$ there exists a $y\in A$ which is a polynomial with complex coefficients in finitely many elements of D such that $f(x^*xzz^*)\leqq f(yy^*zz^*)$ for all $z\in A$, then there exists a finite positive Radon measure μ_f on a locally compact subset σ_f of $\hat{A}$ such that

(a) $\hat{x}(\varphi)=\varphi(x)$ belongs to $L^2(\mu_f)$ for every $x\in A$,

(b) $f(x)=\int_{\sigma_f}\varphi(x)d\mu_f(\varphi)$

for all $x\in A$.

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