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On the signature invariants of a non-singular complex sesqui-linear form

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The purpose of this note is to make clear the relationship between two types of signatures defined for a non-singular real bilinear or complex sesqui-linear form, and then, to get a result in the algebraic topology.

Let $l: V\times V\rightarrow \boldsymbol{C}$ be a complex sesqui-linear form of finite dimension; a matrix representation $x^*\Gamma y$ is used and a symbol "$*$" stands for the transpose of the conjugate of the matrix or the vector. Let t be an indeterminant which may be thought either as an automorphism or as a variable ranging over the complex numbers. We call $\Gamma(t)=\Gamma-\Gamma^* t$ an Alexander matrix and $\det\Gamma(t)$ the Alexander polynomial. The first series of signatures consists of the signature τ_ω of the hermitian form $l_\omega=x^*\Gamma_\omega y$ with $\Gamma_\omega=(1/2)\{(1-\bar{\omega})\Gamma+(1-\omega)\Gamma^*\}$. Since $\tau_\xi=\mathrm{sign}(1-\mathrm{Re}\,\xi)\tau_\omega$ with $\omega=-(1-\xi)/(1-\bar{\xi})$, the only interesting case is when ω is on the unit circle, where Γ_ω reduces to $\Gamma_\omega=(1/2)(1-\bar{\omega})\Gamma(\omega)$.

A hermitian form $l_+=x^*Ay$ where $A=(1/2)(\Gamma+\Gamma^*)$ and a skew-hermitian form $l_-=x^*(-Q)y$ where $Q=(1/2)(\Gamma^*-\Gamma)$ are considered; then $\Gamma=A-Q$ and of course $2A=\Gamma_{-1}$. If the form l is non-singular, then the matrix $P=(\Gamma^*)^{-1}\Gamma$ gives an automorphism t of l, i.e., $P^*\Gamma P=\Gamma$, and hence of l_ω, l_+ and l_-. The eigen-values α of the automorphism t associate another series of signatures $\sigma_{(\alpha)}$ which are defined by the hermitian form l_+; where l_+ is restricted to the α-root subspaces $V_\alpha=\{x\in V; (t-\alpha)^k x=0 \text{ for some } k\}$. Note that $\dim V_\alpha>0$ if and only if α is a root of the Alexander polynomial and we have a generalized Cayley transformation $Q(I+P)=A(I-P)$. Moreover, we can remark that, if $\alpha\neq\pm 1$, $\sigma_{(\alpha)}=\mathrm{sign}(V_\alpha; l_+)$ is equal to $\mathrm{sign}(\mathrm{Im}\,\alpha)\,\mathrm{sign}(V_\alpha; il_-)$. (Cf. § 1, case (b).) We define $\sigma_{(-1\pm 0i)}$ by $\pm\mathrm{sign}(V_{-1}; il_-)$.

THEOREM 1 (Complex case). *For $\omega=\exp(i\varphi)$ and $\alpha=\exp(i\theta)$ with $-\pi<\varphi<\pi$ and $-\pi<\theta<\pi$,*

$$(*)\qquad \tau_\omega=\mathrm{sign}(\mathrm{Im}\,\omega)\{\sum_{|\alpha|=1,\alpha\neq -1}\mathrm{sign}(\varphi-\theta)\sigma_{(\alpha)}+\sigma_{(-1+0i)}\}$$

holds, provided either the automorphism t is semi-simple, or ω is not a root of the Alexander polynomial.

REMARK. If $\omega=-1$, $(*)$ is replaced by $(*')$ $\mathrm{sign}(l_+)=\sum\sigma_{(\alpha)}(|\alpha|=1, \alpha\neq -1)$. The formula, $(*)$ or $(*')$, does not always hold. The excluded cases will be studied in the § 3.

If l is a real non-singular bilinear form, then we shall deduce the following theorem with the more appropriate notation: $\sigma_0=\sigma_{(1)}$ and for $0<\theta<\pi$, $\sigma_\theta=\sigma_{(\alpha)}+\sigma_{(\bar{\alpha})}$ where $\alpha=\exp(i\theta)$.

THEOREM 2 (Real case). *For* $\omega=\exp(\pm i\varphi)$ *with* $0<\varphi\leqq\pi$,

$$(**)\qquad \tau_\omega=\sum_{0\leqq\theta<\varphi}\sigma_\theta+\frac{1}{2}\sigma_\varphi$$

holds, provided either the automorphism t is semi-simple, or ω *is not a root of the Alexander polynomial.*

The study on the classification of sesqui-linear forms is summarized in [**4**]. And the reader can find a definition of σ_θ for a knot in [**2**], which can be seen to be equal to σ_θ for the non-singular Seifert matrix. The hermitian form l_ω is defined and used by Levine [**1**] and Tristram [**3**] in the algebraic topology of knots and links. In the last section we are concerned with the calculation of σ_θ for some algebraic links and we generalize the Brieskorn criterion [**5**]. Finally we mention a totally elementary proof of the result of Rokhlin [**8**] in an interesting special case.

§ 1. Proof of Theorem 1.

Since $l_\omega(f(t)x, y)=l_\omega(x, \overline{f(t^{-1})}y)$ for any complex polynomial $f(t)$, V_α is orthogonal to V_β with respect to the hermitian form l_ω unless $\bar{\alpha}\beta=1$. It follows that the only contributions to the signature arise from V_α with $|\alpha|=1$.

On the other hand by the generalized Cayley transformation $Q(I+P)=A(I-P)$, we know that if $\det(I+P)\neq 0$ then $\Gamma_\omega=A(1-\bar{\omega})(P-\omega)(I+P)^{-1}$ and if $\det(I-P)\neq 0$ then $\Gamma_\omega=Q(1-\bar{\omega})(P-\omega)(I-P)^{-1}$.

(a) *The case when t is semi-simple, that is,* $V_\alpha=\{x\in V;\ (t-\alpha)x=0\}$: If $x, y\in V_{-1}$, then $l_\omega(x, y)=(\omega-\bar{\omega})l_-(x, y)$. Hence $\operatorname{sign}(V_{-1}; l_\omega)=\operatorname{sign}(\operatorname{Im}\omega)\operatorname{sign}(V_{-1}; il_-)$.

If $|\alpha|=1$ and $\alpha\neq -1$, we have $l_\omega(x, y)=(1-\bar{\omega})(1-\bar{\alpha}\omega)(1+\bar{\alpha})^{-1}l_+(x, y)$, provided $x, y\in V_\alpha$. Noting that $(1+\bar{\alpha})(1+\alpha)=2+(\alpha+\bar{\alpha})>0$, we have only to study the sign of the following function f.

$$f=(1-\bar{\omega})(1-\bar{\alpha}\omega)(1+\alpha)=-8\sin(-\varphi/2)\sin((\varphi-\theta)/2)\cos(\theta/2).$$

We get $\operatorname{sign} f=\operatorname{sign}(\operatorname{Im}\omega)\operatorname{sign}(\varphi-\theta)$, provided $-\pi<\theta$, $\varphi<\pi$.

(b) *The case when* $l(x, y)$ *is a general non-singular sesqui-linear form*: We restrict Γ to V_α with $|\alpha|=1$, and then perturb it. Assuming $\alpha\neq -1$, we have $Q=A(I-P)(I+P)^{-1}$ and another skew-hermitian matrix ${}_0Q=A(1-\alpha)(1+\alpha)^{-1}$. A family of skew-hermitian matrices ${}_sQ=sQ+(1-s){}_0Q$, $0\leqq s\leqq 1$, is considered and we get a family of sesqui-linear forms ${}_sl=x^*{}_s\Gamma y$, $0\leqq s\leqq 1$ by defining

${}_s\Gamma=A-{}_sQ$. It follows that ${}_s\Gamma=2A((1-s)\alpha+(s+\alpha)P)(I+P)^{-1}(1+\alpha)^{-1}$ is non-singular and ${}_sP-\alpha=(I+s\alpha+(1-s)P)^{-1}s(1+\alpha)(P-\alpha)$ is nilpotent for the automorphism ${}_sP=(A+{}_sQ)^{-1}(A-{}_sQ)$. Hence, for any s with $0\leqq s\leqq 1$, the Alexander polynomial ${}_s\Gamma(t)$ associated to ${}_s\Gamma$ does not vanish except $t=\alpha$, that is, the hermitian form ${}_sl_\omega=x^*{}_s\Gamma_\omega y$ is non-degenerate unless $\omega=\alpha$. Therefore, if $|\alpha|=1$, $\alpha\neq-1$ and $\omega\neq\alpha$, then $\operatorname{sign}(V_\alpha;{}_0l_\omega)=\operatorname{sign}(V_\alpha;l_\omega)$. This follows from the perturbation invariance of the signature of non-degenerate hermitian forms. Note also that ${}_0l_+=l_+$. As a consequence, if $\omega=\exp(i\varphi)$ and $\alpha=\exp(i\theta)$ with $-\pi<\varphi\neq\theta<\pi$, then $\operatorname{sign}(V_\alpha;l_\omega)-\operatorname{sign}(\operatorname{Im}\omega)\operatorname{sign}(\varphi-\theta)\operatorname{sign}(V_\alpha;l_+)=\operatorname{sign}(V_\alpha;{}_0l_\omega)-\operatorname{sign}(\operatorname{Im}\omega)\operatorname{sign}(\varphi-\theta)\operatorname{sign}(V_\alpha;{}_0l_+)$; the latter vanishes, because ${}_0P=\alpha I$. If ω is not a root of the Alexander polynomial, then $V_\alpha=0$ and this completes the proof for $\alpha\neq-1$. Remark also that ${}_sQ$ are non-degenerate for $0\leqq s\leqq 1$, then we get $\operatorname{sign}(V_\alpha;l_+)=\operatorname{sign}(\operatorname{Im}\alpha)\operatorname{sign}(V_\alpha;i(-{}_0Q))=\operatorname{sign}(\operatorname{Im}\alpha)\operatorname{sign}(V_\alpha;-iQ)$.

If $\alpha=-1$, we use the inverse Cayley transformation $A=Q(I+P)(I-P)^{-1}$ and put ${}_sA=sA$. Then, ${}_s\Gamma={}_sA-Q$, $0\leqq s\leqq 1$, are non-singular and so are ${}_sl_\omega(x,y)$. Note that ${}_0l_\omega(x,y)=\operatorname{sign}(\operatorname{Im}\omega)il_-(x,y)$. Therefore, we get $\operatorname{sign}(V_{-1},l_\omega)=\operatorname{sign}(\operatorname{Im}\omega)\operatorname{sign}(V_{-1},il_-)$.

§ 2. Proof of Theorem 2.

In view of the theorem 1 and the remark, it is sufficient to prove $\sigma_{(\alpha)}=\sigma_{(\bar\alpha)}$ for any real bilinear form with $\alpha=\exp(i\theta)$, $0<\theta<\pi$ and $\sigma_{(-1+0)i}=0$. But this is also deduced from the theorem 1 as follows. Because Γ is a real matrix, the transpose of Γ_ω is equal to $\Gamma_{\bar\omega}$ and hence $\tau_\omega=\operatorname{sign}(\text{transpose of }\Gamma_\omega)=\tau_{\bar\omega}$. Let $\alpha_\pm$ denote $\exp(i(\theta\pm\varepsilon))$ for a small positive number ε. Then, from the theorem 1, we get

$$\sigma_{(\alpha)}=\tau_{\alpha+}-\tau_{\alpha-}=\tau_{\beta+}-\tau_{\beta-}=\sigma_{(\bar\alpha)}\,,\qquad\text{where}\quad \beta_\pm=\bar\alpha_\pm\,.$$

Therefore, $\tau_\omega=\sum\sigma_\theta+(1/2)\sigma_\varphi+\operatorname{sign}(\operatorname{Im}\omega)\sigma_{(-1+0i)}$. But $\tau_\omega=\tau_{\bar\omega}$ implies $\sigma_{(-1+0i)}=0$ from that.

§ 3. Excluded cases.

We use the notation of the § 1. By decomposing V_α into t-invariant subspaces, we may assume P is the triangular matrix of rank r: $P_{i,i}=\alpha$, $P_{i,i+1}=1$ and otherwise $P_{i,j}=0$. Then, the fact that $P^*AP=A$ and $\alpha\bar\alpha=1$ implies that A is the triangular matrix: $A_{i,j}=0$ if $i+j\leqq r$. We investigate the case $\omega=\alpha$ and $\alpha\neq-1$. (The case $\alpha=-1$ is treated in the same way by using Q instead of A). Remember the matrix Γ_α is AX with $X=(1-\alpha)(I-\bar\alpha P)(I+P)^{-1}$. The matrices X and hence AX are the strongly triangular matrices: $X_{i,j}=0$ if $i\geqq j$ and $(AX)_{i,j}=0$ if $i+j\leqq r+1$. The non-degeneracy of $\Gamma=A-Q=2AP(I+P)^{-1}$

implies that rank $A=r$ and rank $AX=r-1$. If $r=$odd, we have $\operatorname{sign}(AX)=0$ and (*). (Note: $|\operatorname{sign} A|=1$ in the case $r\geqq 3$). If $r=$even, we have $|\operatorname{sign}(AX)|=1$. So in this case (*) does not hold.

If we note that $\Gamma\oplus\bar{\Gamma}$ may be transformed to a real matrix, we understand that (**) has also counterexamples.

§ 4. Signatures of algebraic links.

We shall give a criterion to calculate σ_θ for the algebraic links of Fermat-Pham-Brieskorn type:

$$\{z_1^{a_1}+\cdots+z_n^{a_n}=0\}\cap S^{2n-1}.$$

The Seifert matrix with integral coefficients is described as $\Gamma=(-1)^{n(n+1)/2}\Gamma(a_1)\oplus\cdots\oplus\Gamma(a_n)$, where $\Gamma(a_\nu)$ denotes a triangular matrix of rank $a_\nu-1$ with $\Gamma(a_\nu)_{i,j}=\delta_{i,j}-\delta_{i+1,j}$, $1\leqq i,\ j\leqq a_\nu-1$ (cf. [7]). The intersection matrix and the monodromy matrix of the Milnor fiber are

$$-(\Gamma+(-1)^{n-1}\Gamma^*) \quad \text{and} \quad (-1)^n(\Gamma^*)^{-1}\Gamma$$

respectively. They have the same real bases (cf. [6]). It is enough to know the case when $n=$odd, because Γ becomes either Γ or $-\Gamma$ after we add the term z_{n+1}^2. Now, for $0\leqq\theta\leqq\pi$, A_θ denotes the finite set of integers,

$$A_\theta=\{(j_1,\cdots,j_n)\,;\ 1\leqq j_\nu\leqq a_\nu-1 \text{ and } \pi+2\pi\textstyle\sum(j_\nu/a_\nu)\equiv\theta \text{ or } -\theta \bmod 2\pi\}\,.$$

PROPOSITION 3. *Suppose n is odd. The partial signatures $\sigma_\theta=\sigma_\theta^+-\sigma_\theta^-$ and the nullity n of $\Gamma+\Gamma^*$ are given as follows: If $0\leqq\theta<\pi$, then*

$$\sigma_\theta^-=\textit{number of } (A_\theta\cap\{0<\textstyle\sum(j_\nu/a_\nu)<1 \bmod 2\})\,,$$

$$\sigma_\theta^+=\textit{number of } (A_\theta\cap\{1<\textstyle\sum(j_\nu/a_\nu)<2 \bmod 2\})$$

and

$$n=\textit{rank of } V_{-1}=\textit{number of } A_\pi\,.$$

The signatures τ_ω are given by the sum formula in the theorem 2, because the monodromy is semi-simple. We shall give an outline of the proof of the proposition 3.

Let $T(a)$ be the transformation matrix with $T(a)_{i,j}=1-\xi^{ij}$ and $\xi=\exp(2\pi\sqrt{-1}/a)$. (The bases must be written as $x_s=(1-\xi^s)\sum\xi^{si}\omega^i$ $(0\leqq i\leqq a-1)$ in the notation of [7] and changes to $x_s=-\xi^s\sum\xi^{si}\omega^i$ in that of [5].) Then, $T^*(a)\Gamma(a)T(a)$ is a diagonal matrix $(a(1-\xi^{-i})\delta_{i,j})$. Therefore, the transformed matrix $T^*\Gamma T$ and the transformed automorphism $T^{-1}(\Gamma^*)^{-1}\Gamma T$ by $T=T(a_1)\oplus\cdots\oplus T(a_n)$ are

$$((-1)^{n(n+1)/2}\textstyle\prod a_\nu\prod(1-\xi_\nu^{-i_\nu})\prod\delta_{i_\nu,j_\nu}) \quad \text{and} \quad ((-1)^n\prod\xi_\nu^{i_\nu}\prod\delta_{i_\nu,j_\nu})$$

respectively. Since these are diagonal matrices, it is easy to deduce the proposition by the same technique of the calculation of sign of the function f in the proof of the theorem 1 (cf. p. 12 of [5]).

As a final remark it is noticed that the result of Rokhlin [8] in the case $M=CP^2$ has an elementary proof: Apply the direct calculation in this note for the algebraic link $\{z_1^d+z_2^d=0\}\cap S^3$ to the inequality of Tristram [3] with respect to τ_ω; $\omega=-1$ if $d=$ even and $\omega=\exp(m\pi\sqrt{-1}/2m+1)$ if $2m+1$ is an odd prime power which divides d.

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