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The representability of modular forms by theta series

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§ 1. Introduction.

Let N be a natural number and $\Gamma_0(N)$ the congruence modular group of level N, i. e. $\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2, Z) \mid c\equiv 0 \pmod{N}\right\}$. Let $S_k(N)$ denote the space of cusp forms (of Haupt or principal type) on $\Gamma_0(N)$ of weight k, $k\geqq 2$. Eichler considered the problem ("the basis problem") of concretely giving a basis of $S_k(N)$ in terms of theta series attached to a rational quaternion algebra in the case N is square free (see [**2**], [**3**] and [**4**]). Hijikata and Saito ([**6**]) generalized Eichler's results to the case N has at least one simple prime factor (i. e. $N=pM$, $(p, M)=1$). Here we generalize these results to the case N is not a perfect square.

The basic idea is to consider Brandt Matrices $B(n)$ which occur in the theory of quaternion algebras and which are analogous to the Hecke Operators $T(n)$. In fact they both generate semi-simple commutative rings and the Brandt Matrices give a representation of the Hecke Operators on a space generated by theta series. In [**8**] we defined more general Brandt Matrices than considered in [**4**] or [**6**] and also computed their traces. Theorem 4 below gives a relation between the trace of the Brandt Matrix and the traces of Hecke Operators. From this theorem we obtain several results on the representability of modular forms by theta series. In particular Corollary 7 gives an explicit procedure for obtaining (in a concrete manner) all new forms on $\Gamma_0(N)$ if N is not a perfect square and Corollary 11 gives a version of Eichler's main Theorem in Chapter IV of [**4**].

§ 2. Brandt matrices.

Let $q_1=p_1^{s_1}\cdots p_f^{s_f}$ where the p_i are distinct primes and the $f, s_1, \cdots, s_f$ are all *odd* positive integers. Let q_2 be any positive integer with $(q_1, q_2)=1$. Let $\mathfrak{A}$ be the (unique) quaternion algebra over Q ramified precisely at the primes $\{p_1, \cdots, p_f, \infty\}$. Let $\mathfrak{O}$ be an order of level q_1q_2 in $\mathfrak{A}$ (see Definition 22 in [**8**]). Fixing a set of representatives of the (left) $\mathfrak{O}$-ideal classes, we define (gener-

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alized) Brandt Matrices $B(n)=B_l(n\,;q_1q_2)$ in exactly the same manner as Eichler (see [4], equations 15 and 15a on page 105). Here l and n are both non-negative integers. As in Theorem 2, Chapter II of [4], the $B(n)$ (for fixed l, q_1, q_2) with $(n, q_1q_2)=1$ generate a semi-simple commutative ring and satisfy the same identities as do the Hecke Operators $T(n)$, $(n, q_1q_2)=1$ acting on $S_{l+2}(q_1q_2)$.

Consider the Brandt Matrix Series

$$(1)\qquad (\theta_{ij}(\tau))=\sum_{n=0}^{\infty} B_l(n\,;q_1, q_2)e^{2\pi i n\tau}.$$

Then just as in Theorem 1, Chapter II of [4], the $\theta_{ij}(\tau)$ in (1) are modular forms of weight $l+2$ on $\Gamma_0(N)$ where $N=q_1q_2$. If $l>0$, they are cusp forms. If $l=0$, then we have a homomorphism from the ring generated by the Brandt Matrices $B_0(n\,;q_1, q_2)$, $(n, q_1q_2)=1$ to Z which sends the Brandt Matrix $B_0(n)$ to $b(n)$ where $b(n)$ denotes the number of left (or right) integral $\mathfrak{O}$-ideals of norm n. This is analogous to the representation $T(n)\to\deg T(n)$ of Hecke Operators. Thus we can simultaneously reduce the $B_0(n)=B_0(n\,;q_1, q_2)$, $(n, q_1q_2)=1$ to block form $\begin{pmatrix} B_0'(n) & 0 \\ 0 & b(n)\end{pmatrix}$ and then the entries $\theta_{ij}'(\tau)$ of the matrix series $(\theta_{ij}'(\tau))=\sum_{n=0}^{\infty} B_0'(n\,;q_1, q_2)e^{2\pi i n\tau}$ are cusp forms.

Finally, just as in the Proposition on page 138 of [4], the action of the Hecke Operators on the theta series $\theta_{ij}(\tau)$ is given by Brandt Matrices. Specifically, let $T_k(n)$ denote the Hecke Operator $T(n)$ acting on $S_k(q_1q_2)$. Then for $k>2$, the action of $T_k(n)$ for $(n, q_1q_2)=1$ on the $\theta_{ij}(\tau)$ is given (formally) by the Brandt Matrix $B_{k-2}(n)$ and for $k=2$, the action of $T_2(n)$ for $(n, q_1q_2)=1$ on the $\theta_{ij}'(\tau)$ is given (formally) by the $B_0'(n)$.

§ 3. Traces of Hecke operators.

Let $\mathrm{tr}_N T_k(n)$ denote the trace of the Hecke Operator $T_k(n)$ acting on $S_k(N)$. Hijikata has computed $\mathrm{tr}_N T_k(n)$ and it is given by (see (2) in [6])

THEOREM 1.

$$\mathrm{tr}_N T_k(n)=-\sum_s a_k(s)\sum_f b(s, f)\prod_{p\mid N} c'(s, f, p)$$
$$+\delta(\sqrt{n})\frac{k-1}{12}\cdot N\cdot\prod_{p\mid N}\left(1+\frac{1}{p}\right)+\delta(k)\deg T_k(n)$$

where the notation is exactly the same as in [6].

For the convenience of the reader, we copy Theorem 26 of [8] as

THEOREM 2. *The trace of the Brandt Matrix is given by*

$$\operatorname{tr} B_{k-2}(n\,;\,q_1,\,q_2)=\sum_s a_k(s)\sum_f b(s,\,f)\prod_{p\mid q_1q_2} c(s,\,f,\,p)$$

$$+\delta(\sqrt{n})\frac{k-1}{12}q_1q_2\prod_{p\mid q_1}\left(1-\frac{1}{p}\right)\prod_{p\mid q_2}\left(1+\frac{1}{p}\right)$$

where the notation is exactly the same as in Theorem 26 *of* [8].

REMARK. The notation used in Theorem 1 is exactly the same as in Theorem 2. In fact the similarity of both formulas is striking. The only major difference is that where $c'(s, f, p)$ appears in Theorem 1 we have $c(s, f, p)$ in Theorem 2. Let us recall the definitions of the $c(s, f, p)$ and the $c'(s, f, p)$. Let $\mathfrak{O}$ be an order of level q_1q_2 in a (definite) quaternion algebra over Q (see Definition 22 in [8]). Let $\mathfrak{O}'$ be the order $\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\middle| a, b, c, d\in Z,\ c\equiv 0 \pmod N\right\}$ in the 2×2 matrix algebra $M_2(Q)$. Let $\mathfrak{o}$ be the order of discriminant s^2-4n/f^2 (here n is the "n" of $B(n)$ or $T(n)$) in $Q[x]/(x^2-sx+n)$. Then $c(s, f, p)$ (resp. $c'(s, f, p)$) is the number of inequivalent $\operatorname{mod} U(\mathfrak{O}_p)$ (resp. $\operatorname{mod} U(\mathfrak{O}'_p)$) optimal embeddings of $\mathfrak{o}_p=\mathfrak{o}\otimes Z_p$ into $\mathfrak{O}_p$ (resp. $\mathfrak{O}'_p$).

LEMMA 3. *Let $N=q_1q_2$. Then $c(s, f, p)=c'(s, f, p)$ for $p\nmid q_1$ where $c(s, f, p)$ appears in the trace of $B_{k-2}(n\,;\,q_1, q_2)$ (Theorem 2 above) and $c'(s, f, p)$ appears in the trace of $T_k(n)$ (Theorem 1 above).*

PROOF. This is obvious as $\mathfrak{O}_p\cong\mathfrak{O}'_p$ for $p\nmid q_1$ in the notation of the above remark.

§ 4. The trace formula.

Fix a prime p and let $q_1=p^{2r+1}$ with $r\geqq 0$ and let $q_2=M$ with M any positive integer prime to p. Denote by $\operatorname{tr}_N T_k(n)$ the trace of the Hecke Operator $T_k(n)$ acting on $S_k(N)$. Then we have the following

THEOREM 4. *Let M be any positive integer prime to p. Then for all positive integers n with $(n, pM)=1$ and for all $r\geqq 0$ we have*

$$\operatorname{tr}_{p^{2r+1}M}T_k(n)+2\sum_{i=0}^{r-1}\operatorname{tr}_{p^{2i+1}M}T_k(n)+\begin{cases}\deg T_k(n) & \textit{for}\quad k=2\\ 0 & \textit{for}\quad k>2\end{cases} \tag{2}$$

$$=\operatorname{tr} B_{k-2}(n\,;\,p^{2r+1},\,M)+2\sum_{i=0}^{r}\operatorname{tr}_{p^{2i}M}T_k(n)\,.$$

REMARK. If $r=0$ and M is square free this is essentially equation (5) on page 140 of Eichler [4] and if $r=0$, it is Lemma 1 of Hijikata and Saito [6].

PROOF. The formulas for $\operatorname{tr}_{p^sM}T_k(n)$ (Theorem 1) and for $\operatorname{tr} B_{k-2}(n\,;\,p^{2r+1}, M)$ (Theorem 2) are very similar. In fact they are both sums over the same index set and we shall show the equality holds in (2) term by term. First it is obvious that the $\deg T_k(n)$ term causes no problem (it appears the same number of times on both the left and right hand sides of (2). Ignoring the $\deg T_k(n)$ terms, by induction we need only show that

$$\operatorname{tr}_{p^{2r+1}M}T_k(n)+\operatorname{tr}_{p^{2r-1}M}T_k(n)+\operatorname{tr} B_{k-2}(n\,;p^{2r-1},M) = \operatorname{tr} B_{k-2}(n\,;p^{2r+1},M)+2\operatorname{tr}_{p^{2r}M}T_k(n) \tag{3}$$

where if $r=0$, the expressions in (3) containing p^{-1} do not occur. First we check that equality holds in (3) for the mass terms, i.e.

$$p^{2r+1}M\prod_{q\mid pM}\Big(1+\frac{1}{q}\Big)+p^{2r-1}M\prod_{q\mid pM}\Big(1+\frac{1}{q}\Big)+p^{2r-1}M\Big(1-\frac{1}{p}\Big)\prod_{q\mid M}\Big(1+\frac{1}{q}\Big)$$
$$=p^{2r+1}M\Big(1-\frac{1}{p}\Big)\prod_{q\mid M}\Big(1+\frac{1}{q}\Big)+2p^{2r}M\prod_{q\mid p^rM}\Big(1+\frac{1}{q}\Big)$$

where of course the 2nd and 3rd terms of the left hand side do not occur if $r=0$. It is easy to check that this equality is valid. By Lemma 3, we have $c(s,f,q)=c'(s,f,q)$ for all $q\neq p$. Thus fixing n, s and f (hence $\Delta=s^2-4n/f^2$), to complete the proof we need only show that

$$-c'(s,f,p)_{p^{2r+1}}-c'(s,f,p)_{p^{2r-1}}+c(s,f,p)_{p^{2r-1}}=c(s,f,p)_{p^{2r+1}}-2c'(s,f,p)_{p^{2r}} \tag{4}$$

where we have used the subscript to denote the level of the corresponding order, i.e. for example $c'(s,f,p)_{p^{2t}}$ denotes the number of inequivalent optimal embeddings of an order $\mathfrak{O}$ of discriminant $\Delta=s^2-4n/f^2$ into $\begin{pmatrix} Z_p & Z_p \\ p^{2t}Z_p & Z_p\end{pmatrix}$. Hijikata has given a method of calculating the $c'(s,f,p)$ in Theorem 2.3 of [5]. The $c(s,f,p)$ are given by Theorem 14 and 15 of [8]. We tabulate the values of $c(s,f,p)$ and $c'(s,f,p)$ below.

Let p be a prime $\neq 2$. Let u be a quadratic non-residue mod p. Set $\Delta=s^2-4n/f^2 \pmod{U(Z_p)^2}$. Then the values of the $c(s,f,p)$ and $c'(s,f,p)$ are given by the tables below. Note in all tables read $p^{-1}=0$.

$\Delta=p^{2m}$

	$t<m$	$t=m$	$t>m$
$c(s,f,p)_{p^{2t+1}}$	0	0	0
$c'(s,f,p)_{p^{2t+1}}$	$2p^t$	$2p^m+2p^{m-1}$	$2p^m+2p^{m-1}$
$c'(s,f,p)_{p^{2t}}$	p^t+p^{t-1}	p^m+2p^{m-1}	$2p^m+2p^{m-1}$

$\Delta=p^{2m}u$

	$t<m$	$t=m$	$t>m$
$c(s,f,p)_{p^{2t+1}}$	0	$2p^m-2p^{m-1}$	$2p^m-2p^{m-1}$
$c'(s,f,p)_{p^{2t+1}}$	$2p^t$	0	0
$c'(s,f,p)_{p^{2t}}$	p^t+p^{t-1}	p^m	0

$\Delta=p^{2m+1}a$ where $a=1$ or u

	$t<m$	$t=m$	$t>m$
$c(s, f, p)_{p^{2t+1}}$	0	p^m	0
$c'(s, f, p)_{p^{2t+1}}$	$2p^t$	p^m	0
$c'(s, f, p)_{p^{2t}}$	p^t+p^{t-1}	p^m+p^{m-1}	0

Let $p=2$ and set $\Delta=s^2-4n/f^2 \pmod{U(Z_p)^2}$. Then the values of the $c(s, f, 2)$ and $c'(s, f, 2)$ are given by the tables below. Note in all tables read $p^{-1}=0$.

$\Delta=1$

	$t=0$	$t>0$
$c(s, f, 2)_{2^{2t+1}}$	0	0
$c'(s, f, 2)_{2^{2t+1}}$	2	2
$c'(s, f, 2)_{2^{2t}}$	1	2

$\Delta=5$

	$t=0$	$t>0$
$c(s, f, 2)_{2^{2t+1}}$	2	2
$c'(s, f, 2)_{2^{2t+1}}$	0	0
$c'(s, f, 2)_{2^{2t}}$	1	0

$\Delta=2^{2m+2}$

	$t<m$	$t=m$	$t=m+1$	$t>m+1$
$c(s, f, 2)_{2^{2t+1}}$	0	0	0	0
$c'(s, f, 2)_{2^{2t+1}}$	2^{t+1}	2^{m+1}	$3\cdot 2^{m+1}$	$3\cdot 2^{m+1}$
$c'(s, f, 2)_{2^{2t}}$	2^t+2^{t-1}	2^m+2^{m-1}	2^{m+2}	$3\cdot 2^{m+1}$

$\Delta=2^{2m+2}\cdot 5$

	$t<m$	$t=m$	$t=m+1$	$t>m+1$
$c(s, f, 2)_{2^{2t+1}}$	0	0	2^{m+1}	2^{m+1}
$c'(s, f, 2)_{2^{2t+1}}$	2^{t+1}	2^{m+1}	0	0
$c'(s, f, 2)_{2^{2t}}$	2^t+2^{t-1}	2^m+2^{m-1}	2^{m+1}	0

$\Delta=2^{2m+2}\cdot a$ where $a=3, 7, 6, 10$ or 14

	$t<m$	$t=m$	$t>m$
$c(s, f, 2)_{2^{2t+1}}$	0	2^m	0
$c'(s, f, 2)_{2^{2t+1}}$	2^{t+1}	2^m	0
$c'(s, f, 2)_{2^{2t}}$	2^t+2^{t-1}	2^m+2^{m-1}	0

Using the above tables one can easily check that the equality (4) holds in all cases. For example, in the case $\Delta=p^{2m}$ $(p\neq 2)$ and $r<m$, (4) becomes

$$-2p^r-2p^{r-1}+0=0-2(p^r+p^{r-1})$$

which (recalling $p^{-1}=0$) is obvious. This completes the proof of Theorem 4.

§ 5. Consequences.

Consider the Brandt Matrix series $(\theta_{ij}(\tau))=\sum_{n=0}^{\infty}B_{k-2}(n\,;p^{2r+1}, M)e^{2\pi in\tau}$ where $k\geqq 2$, $r\geqq 0$ and $(p, M)=1$. The $\theta_{ij}(\tau)$ are modular forms of weight k on $\Gamma_0(p^{2r+1}M)$. If $k>2$, they are cusp forms. If $k=2$, replacing the Brandt Matrix $B_0(n)$ by $B_0'(n)$ as in § 2, the entries $\theta'_{ij}(\tau)$ of $(\theta'_{ij}(\tau))=\sum_{n=0}^{\infty}B_0'(n\,;p^{2r+1}, M)e^{2\pi in\tau}$ are cusp forms of weight 2 on $\Gamma_0(p^{2r+1}M)$. We can simultaneously diagonalize the $B_{k-2}(n)$ (and the $B_0'(n)$) for $(n, pM)=1$. Let $\Phi_k(p^{2r+1}, M)$ denote the set of (cusp) forms appearing on the diagonal of the diagonalized Brandt Matrix Series $\sum_{n=0}^{\infty}B_{k-2}(n\,;p^{2r+1}, M)e^{2\pi in\tau}$ for $k>2$. For $k=2$, we let $\Phi_2(p^{2r+1}, M)$ denote the set of (cusp) forms appearing on the diagonal of the diagonalized series $\sum_{n=0}^{\infty}B_0'(n\,;p^{2r+1}, M)e^{2\pi in\tau}$. As the Brandt Matrices give (formally) the action of the Hecke Operators on the $\theta_{ij}(\tau)$, we have that the forms in $\Phi_k(p^{2r+1}, M)$ are eigen forms for the action of the Hecke Operators $T_k(n)$ with $(n, pM)=1$ and further the action of the $T_k(n)$, $(n, pM)=1$ is just given by the diagonalized Brandt Matrix $B_{k-2}(n\,;p^{2r+1}, M)$ for $k>2$ or $B_0'(n, p^{2r+1}, M)$ if $k=2$. We have the following fundamental

THEOREM 5. *Let $\Phi_k(p^{2r+1}, M)=\{\theta_1, \cdots, \theta_d\}$ and let $\langle\theta_i\rangle$ denote the 1-dimensional (complex) vector space generated by θ_i. Writing $+$ for $\oplus$ (sometimes), we have*

$$S_k(p^{2r+1}M)\oplus 2\sum_{s=0}^{r-1}S_k(p^{2s+1}M)\cong\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\oplus 2\sum_{s=0}^{r}S_k(p^{2s}M) \tag{5}$$

where the isomorphism is as a module for the Hecke algebra H generated by the $T_k(n)$ with $(n, pM)=1$. Here $2S_k(p^{2s}M)=S_k(p^{2s}M)\oplus S_k(p^{2s}M)$, etc.

PROOF. As H is a semi-simple ring, we need only check that the traces of the transformations induced by $T_k(n)$ on both sides of (5) are equal. But for $k>2$, this is exactly what Theorem 4 says. For $k=2$, we need to find the trace of $B_0'(n; p^{2r+1}, M)$. But

$$\begin{aligned}\operatorname{tr} B_0'(n; p^{2r+1}, M) &= \operatorname{Tr}(B_0(n; p^{2r+1}, M)) - b(n)\\ &= \operatorname{Tr}(B_0(n; p^{2r+1}, M)) - \deg T(n) \quad \text{for } (n, pM)=1\end{aligned}$$

since $b(n)=\deg T(n)$ for $(n; pM)=1$. Thus again Theorem 4 gives exactly what is needed. q. e. d.

Let $\delta(M)$ denote the number of (positive) divisors of M and let $S_k{}^0(N)$ denote the space generated by the "new forms" on $S_k(N)$ (or the "essential part" of $S_k(N)$) (see [1]).

THEOREM 6. *Let* $\Phi_k(p^{2r+1}, M)=\{\theta_1, \cdots, \theta_d\}$. *Then*

$$\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\cong\sum_{a\mid M}\delta\Big(\frac{M}{a}\Big)[S_k{}^0(pa)\oplus S_k{}^0(p^3a)\oplus S_k{}^0(p^5a)\oplus\cdots\oplus S_k{}^0(p^{2r+1}a)]$$

as $H=\langle T(n)\mid(n, pM)=1\rangle$ *modules.*

PROOF. By Theorem 5 of [1] we have $S_k(N)\cong\sum_{a\mid N}\delta\Big(\frac{N}{a}\Big)S_k{}^0(a)$ as H-modules. Thus

$$S_k(p^nM)\cong\sum_{t=1}^{n+1}t\sum_{a\mid M}\delta\Big(\frac{M}{a}\Big)S_k{}^0(p^{n+1-t}a).$$

Letting $\Omega(s)=\sum_{a\mid M}\delta\Big(\frac{M}{a}\Big)S^0(p^sa)$ we have

$$\begin{aligned}&\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\oplus 2\sum_{s=0}^{r}S(p^{2s}M)\\ &\quad\cong\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\oplus\sum_{s=0}^{r}\sum_{t=1}^{2s+1}t\Omega(2s+1-t)\\ &\quad\cong\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\oplus 2\sum_{t=2}^{2r+2}(t-1)\Omega(2r+2-t)\\ &\qquad\oplus 2\sum_{s=0}^{r-1}\sum_{t=2}^{2s+2}(t-1)\Omega(2s+2-t).\end{aligned}$$

Also

$$\begin{aligned}S(p^{2r+1}M)\oplus 2\sum_{s=0}^{r-1}S(p^{2s+1}M)&\cong\sum_{t=1}^{2r+2}t\Omega(2r+2-t)\oplus 2\sum_{s=0}^{r-1}\Omega(2s+1)\\ &\qquad\oplus 2\sum_{s=0}^{r-1}\sum_{t=2}^{2s+2}t\Omega(2s+2-t).\end{aligned}$$

Thus Theorem 5 implies

$$\begin{aligned}&\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\oplus 2\sum_{t=2}^{2r+2}(t-1)\Omega(2r+2-t)\\ &\quad\cong\sum_{t=1}^{2r+2}t\Omega(2r+2-t)\oplus 2\sum_{s=0}^{r-1}\Omega(2s+1)\oplus 2\sum_{s=0}^{r-1}\sum_{t=2}^{2s+2}\Omega(2s+2-t).\end{aligned}$$

Hence

$$\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle+\sum_{t=3}^{2r+1}(t-2)\Omega(2r+2-t)$$

$$\cong\Omega(2r+1)+\sum_{w=3}^{2r+2}2\gamma(w)\Omega(2r+2-w)$$

where $\gamma(w)$ is the number of representatives of w as $2(r-s)+t$ where $0\leqq s\leqq r-1$ and $1\leqq t\leqq 2s+2$. But for w odd, we have $\gamma(w)=\frac{w-1}{2}$ and for w even, we have $\gamma(w)=\frac{w-2}{2}$ (for $3\leqq w\leqq 2r+2$). Thus

$$\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\cong\Omega(2r+1)\oplus\sum_{s=1}^{r}\Omega(2r+2-(2s+1))$$

$$\cong\sum_{s=1}^{r}\Omega(2s+1)$$

$$\cong\sum_{s=1}^{r}\sum_{a\mid M}\delta\Big(\frac{M}{a}\Big)S^0(p^{2s+1}a). \qquad \text{q. e. d.}$$

Theorem 6 gives a concrete inductive process for constructing new forms on $\Gamma_0(N)$ if N is not a perfect square. In fact we have the

COROLLARY 7. *If $N=p^{2r+1}M$ with $(p, M)=1$, then the new forms in $S_k(N)$ are the $\theta_i\in\Phi_k(p^{2r+1}, M)$ which are not "equivalent" to any θ_i in $\Phi_k(p^{2r-1}, M)$ (i. e. which do not have the same eigen values for the $T_k(q)$ with $(q, N)=1$ as any $\theta_i\in\Phi_k(p^{2r-1}, M)$) and which are not "equivalent" (i. e. do not have the same eigen values for $T_k(q)$ with $(q, N)=1$) as any other remaining $\theta_i\in\Phi_k(p^{2r+1}, M)$.*

PROOF. We just have to find the θ_i which belong to $S_k^0(p^{2r+1}M)$ and by [1] elements of $S_k^0(p^{2r+1}M)$ which are eigen values for the Hecke Operators are determined by their eigen values under the $T_k(q)$ with $(q, N)=1$.

In the case of $\Gamma_0(p^{2r+1})$ we have

COROLLARY 8. *Let $\Phi_k(p^{2r+1}, 1)=\{\theta_1, \cdots, \theta_d\}$. Then $\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle\cong S_k^0(p)\oplus S_k^0(p^3)\oplus\cdots\oplus S_k^0(p^{2r+1})$ as H-modules.*

In the case of $\Gamma_0(p^{2r+1})$ we also have

COROLLARY 9. *Let $\Phi_k(p^{2r+1}, 1)=\{\theta_1, \cdots, \theta_d\}$. Then the $\theta_1, \cdots, \theta_d$ are linearly independent over* $\boldsymbol{C}$.

PROOF. This follows immediately from Corollary 8 since by Lemma 24 of [1], the θ_i have distinct eigen values.

One might conjecture that in general the $\theta_i\in\Phi_k(p^{2r+1}, M)$ are linearly independent. Then $\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_d\rangle$ would be naturally embedded in $S_k(N)$, $N=p^{2r+1}M$ and an obvious question arises.

(6) Is $\langle\theta_1, \cdots, \theta_d\rangle=\sum_{a\mid M}\sum_{d\mid \frac{M}{a}}(S_k^0(pa)^d\oplus S_k^0(p^3a)^d\oplus\cdots\oplus S_k^0(p^{2r+1}a)^d)$

where if $S_k^0(m)=\langle f_1(\tau), \cdots, f_t(\tau)\rangle$, then $S_k^0(m)^d=\langle f_1(d\tau), \cdots, f_t(d\tau)\rangle$? Here equality is as subspaces of $S_k(N)$. Though we can not show this, we do get the following result (which proves (6) in the case $r=0$).

Define inductively on r sets of cusp forms $\Phi'_k(p^{2r+1}, M)$ by: if $r=0$, $\Phi'_k(p, M)=\Phi_k(p, M)$. Assume $\Phi'_k(p^{2r-1}, M)$ has been defined. Let $\Phi_k(p^{2r+1}, M)=\{\theta_1, \cdots, \theta_d\}$ and among these θ_i let $\theta_{i_1}, \cdots, \theta_{i_s}$ be those which are not "equivalent" (i.e. do not have the same eigen values for the $T_k(n)$ with $(n, pM)=1$) to any $\theta_j\in\Phi'_k(p^{2r-1}, M)$. Then $\Phi'_k(p^{2r+1}, M)=\Phi'_k(p^{2r-1}, M)\cup\{\theta_{i_1}, \cdots, \theta_{i_s}\}$. Note that in general the elements in the $\Phi_k(p^{2r+1}, M)$ are linearly independent if and only if the elements in the $\Phi'_k(p^{2r+1}, M)$ are linearly independent.

THEOREM 10. *Let $\Phi'_k(p^{2r+1}, M)=\{\theta'_1, \cdots, \theta'_d\}$ and assume the $\theta'_1, \cdots, \theta'_d$ are linearly independent. Then*

$$(7)\qquad \langle\theta'_1, \cdots, \theta'_d\rangle=\sum_{a\mid M}\sum_{d\mid Ma^{-1}}(S_k{}^0(pa)^d\oplus S_k{}^0(p^3a)^d\oplus\cdots\oplus S_k{}^0(p^{2r+1}a)^d)$$

as subspaces of $S_k(N)$, $N=p^{2r+1}M$.

PROOF. The dimensions of both sides of (7) are equal by our assumption, Theorem 6, and Theorem 5 of [1]. Thus we need only show that each θ'_i is contained in the right hand side. We show this by induction on r. If $r=1$, then by Theorem 6 any $\theta'_i\in\Phi'_k(p, M)$ has the same eigen values for $T_k(q)$ with $(q, pM)=1$ as some new form (on $\Gamma_0(pa)$) $f(\tau)\in S_k{}^0(pa)$ for $a\mid M$. Hence by Corollary 2 of Theorem 2 of [7], $\theta'_i=\sum_{d\mid\frac{pM}{pa}}\lambda_d f(d\tau)=\sum_{d\mid\frac{M}{a}}\lambda_d f(d\tau)\in$ R. H. S. of (7).

Assume the theorem is true for $r-1$, i.e.

$$(8)\qquad \langle\theta'_1, \cdots, \theta'_e\rangle=\sum_{a\mid M}\sum_{d\mid Ma^{-1}}(S_k{}^0(pa)^d\oplus\cdots\oplus S_k{}^0(p^{2r-1}a)^d)$$

(where $\{\theta'_1, \cdots, \theta'_e\}=\Phi'_k(p^{2r-1}, M)$) as subspaces of $S_k(p^{2r-1}M)$. As $S_k(p^{2r-1}M)\subseteqq S_k(p^{2r+1}M)$, (8) remains true as subspaces of $S_k(p^{2r+1}M)$. Thus we need only consider the $\theta'_i\in\Phi'(p^{2r+1}, M)-\Phi'(p^{2r-1}, M)$. Such a $\theta'_i(\tau)$ must have the same eigen values for the $T_k(q)$ with $(q, pM)=1$ as $f(\tau)\in S_k{}^0(p^{2r+1}a)$ (of course this is the reason we defined $\Phi'_k(p^{2r+1}, M)$ as we did). Thus again $\theta'_i(\tau)=\sum_{d\mid\frac{p^{2r+1}M}{p^{2r+1}a}}\lambda_d f(d\tau)=\sum_{d\mid Ma^{-1}}\lambda_d f(d\tau)\in$ R. H. S. of (7) and this completes the proof.

In the case $N=p^{2r+1}M$ is square free, Theorem 10 yields a version of Eichler's Main Theorem in Chapter IV of [4] as follows.

Let $\Theta_k(p, M)$ with $(p, M)=1$ be the subspace spanned by the elements of $\Phi_k(p, M)$. Then we have the following

COROLLARY 11. *Let $N=p_1p_2\cdots p_r$ be square free. Assume the elements of $\Phi_k(p_i, p_{i+1}\cdots p_r)$ are linearly independent for $i=1, 2, \cdots, r$. Then*

$$S_k(N)=\Theta_k(p_1, p_2\cdots p_r)\oplus\Theta_k(p_2, p_3\cdots p_r)\oplus\Theta_k(p_2, p_3\cdots p_r)^{p_1}$$
$$\oplus\cdots\oplus\sum_{d\mid p_1\cdots p_{r-1}}\Theta_k(p_r, 1)^d\oplus\sum_{d\mid N}S_k(1)^d\,.$$

REMARK. This should be viewed as a version of Eichler's main theorem in Chapter IV of [4]. The elements of the various $\Phi_k(p_i, p_{i+1} \cdots p_r)$ are probably linearly independent and thus the hypotheses of the theorem are unecessary, but the proof of this fact is not totally clear.

PROOF. Note first that $\Phi_k(p_i, p_{i+1} \cdots p_r) = \Phi'_k(p_i, p_{i+1} \cdots p_r)$ so that we can use Theorem 10. By Theorem 5 of [1], $S_k(N) = \sum_{a \mid N} \sum_{d \mid Na^{-1}} S_k^0(a)^d$. We now use induction on r. If $r=1$,

$$\begin{aligned} S_k(p_1) &= S_k^0(p_1) \oplus S_k(1) \oplus S_k(1)^{p_1} \\ &= \Theta_k(p_1, 1) \oplus S_k(1) \oplus S_k(1)^{p_1} \end{aligned}$$

by Theorem 10. Now assume the Corollary holds in the case $r-1$. Then for $N = p_1 \cdots p_r$,

$$\begin{aligned} S_k(N) = &\sum_{a \mid p_2 \cdots p_r} \sum_{d \mid p_2 \cdots p_r a^{-1}} S_k^0(p_1 a)^d \\ &\oplus \sum_{a \mid p_2 \cdots p_r} \sum_{d \mid p_2 \cdots p_r a^{-1}} S_k^0(a)^d \\ &\oplus \sum_{a \mid p_2 \cdots p_r} \sum_{d \mid p_2 \cdots p_r a^{-1}} S_k^0(a)^{p_1 d} \\ = &\Theta_k(p_1, p_2 \cdots p_r) \oplus S_k(p_2 \cdots p_r) \oplus S_k(p_2 \cdots p_r)^{p_1} \end{aligned}$$

by Theorem 10 and Theorem 5 of [1]. Now by induction we are done.

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