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The Levi problem for the product space of a Stein space and a compact Riemann surface

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Introduction.

Since Oka [9] solved the Levi problem for unramified domains over C^n, many mathematicians extended Oka's theorem (cf. Andreotti-Narasimhan [1], Narasimhan [8]). On the other hand, recently Nakano [7] obtained the vanishing theorems for weakly 1-complete manifolds. The aim of the present paper is to give a solution of the following Levi problem for the product space of a Stein space and a compact Riemann surface.

THEOREM. *Let S be a Stein space, R be a compact Riemann surface and X be the product space of S and R. $\pi_1: X\to S$ denotes the projection of X onto S. Let D be a domain of X. Then the following assertions* (1), (2) *and* (3) *are equivalent:*

(1) *D is weakly* 1*-complete.*

(2) *D is holomorphically convex.*

(3) *Either D is a Stein space or $D=\pi_1(D)\times R$, $\pi_1(D)$ being a Stein space.*

This theorem is a generalization of the previous paper [12] and the result of Matsugu [6].

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§ 1. The Levi problem for relatively compact domains on a weakly 1-complete space.

All complex analytic spaces considered in this paper are supposed countable at infinity.

DEFINITION [7]. Let X be a complex analytic space and ϕ be a C^∞ function on X. We say that X is complete with the exhausting function ϕ if and only if

$$X_c := \{x\in X;\ \phi(x)<c\}$$

is relatively compact for every $c\in \boldsymbol{R}$. Moreover if ϕ is plurisubharmonic on

X, we say that X is weakly 1-complete.

We remark that the proof of the following proposition is based on Nakano's vanishing theorem for weakly 1-complete spaces, which is pointed out by Hironaka and is proved by Fujiki [2]. It is mentioned as follows:

THEOREM. *Let X be a weakly 1-complete space, B a positive line bundle on X and $\mathcal{S}$ be a coherent analytic sheaf on X. Then for every $c \in \boldsymbol{R}$, there exists a natural number m_0 such that*

$$H^q(X_c, \mathcal{S} \otimes \mathcal{O}(B^m)) = 0$$

for $q \geqq 1$, $m \geqq m_0$.

The following lemma is used in the next proposition.

LEMMA 1. *Let S be an n-dimensional analytic space, R be a compact Riemann surface and X be the product space of S and R. $\pi_2 : X \to R$ denotes the projection of X onto R. Suppose that D is a complete domain with the exhausting function ϕ. Then for a point q of R there exists a set A_q of measure zero such that if $c \in \boldsymbol{R} - A_q$, $D_c \cap \pi_2^{-1}(q)$ consists of finitely many connected components.*

PROOF. Let $\pi : \tilde{S} \to S$ be a resolution of singularities of an analytic space S (established by Hironaka [4]). If we put $\Pi : = \pi \times i : \tilde{S} \times R \to S \times R$ and $\tilde{D} : = \Pi^{-1}(D)$, D is complete with exausting function $\phi^* : = \phi \circ \Pi$ because Π is proper, where $i : R \to R$ is the identity map. We take a point q of R. Since ϕ^* is a C^∞ function on $\tilde{D} \cap \pi_2^{-1}(q)$, the set A_q of the critical values of ϕ^* is of measure zero by Sard's theorem (for example see [11]). Therefore for $c \in \boldsymbol{R} - A_q$, $d\phi^* \neq 0$ on $\partial \tilde{D}_c \cap \pi_2^{-1}(q)$, where $\partial \tilde{D}_c : = \{x \in \tilde{D} ; \phi^*(x) = c\}$. Now we take a point x of $\partial \tilde{D}_c \cap \pi_2^{-1}(q)$ for $c \in \boldsymbol{R} - A_q$. If we denote a local coordinate of q of R by (y_1, y_2) with $q = (0, 0)$, we can take a coordinate neighbourhood U of x whose local coordinates are denoted by $(x_1, \cdots, x_{2n-1}, \phi^* - c, y_1, y_2)$ with $x = (0, \cdots, 0)$ as a differentiable manifold. We remark that $\partial \tilde{D}_c \cap \pi_2^{-1}(q) \cap U = \{(x_1, \cdots, x_{2n-1}, \phi^* - c, y_1, y_2) \in U ; \phi^* = c, y_1 = y_2 = 0\}$ consists of only one connected component in U. $\partial \tilde{D}_c$ can be covered by such U. Since $\partial \tilde{D}_c$ is compact, we see that $\partial \tilde{D}_c \cap \pi_2^{-1}(q)$ consists of finitely many connected components. Therefore for a point q of R, there exists a set A_q of measure zero such that for $c \in \boldsymbol{R} - A_q$, $\partial D_c \cap \pi_2^{-1}(q)$ consists of finitely many connected components because Π is continuous. Q.E.D.

PROPOSITION. *Let S be a Stein space, R be a compact Riemann surface and X be the product space of S and R. $\pi_1 : X \to S$ and $\pi_2 : X \to R$ denote the projections of X onto S and R respectively. Let D be a domain of X with the following conditions:*

1) *$D \cap \pi_1^{-1}(q_1) \subsetneqq \{q_1\} \times R$ for every $q = (q_1, q_2) \in D$.*

2) *D is weakly 1-complete with the exhausting function ϕ.*

Then for every $c \in \boldsymbol{R}$, $D_c : = \{x \in D ; \phi(x) < c\}$ is a Stein space.

PROOF. D_c is weakly 1-complete with the exhausting function $(1 - e^{\phi}/e^c)^{-1}$.

So we have only to show that D_c is K-complete because of Andreotti-Narasimhan [1]. Since S is a Stein space, for any point $q=(q_1, q_2)\in D_c$, it suffices to make a holomorphic function $G(x)$ on D_c which is not constant on a neighbourhood of q in $D\cap\pi_1^{-1}(q_1)$. If $\pi_2(D_c)\subsetneqq R$, D_c is an open set of the Stein space $S\times\pi_2(D_c)$. So D_c is K-complete. Therefore we may assume that $\pi_2(D_c)=R$.

Since R is a compact Riemann surface, there exists a positive holomorphic line bundle F on R which is determined by a divisor Γ:

$$F=[\Gamma], \qquad \Gamma=\sum_{i=1}^{l} n_i P_i \quad (n_i\in \boldsymbol{Z},\ P_i\in R).$$

We take an open covering $\{U_i\,;\, i=0, 1, \cdots, l\}$ of R such that U_i is a neighbourhood of P_i with $U_i\cap U_j=\emptyset$ $(i\neq j)$ and $U_0:=R-\{P_1, \cdots, P_l\}$. Then the system of transition functions $\{f_{ij}\}$ with respect to $\{U_i\}$ of R which defines F is as follows:

(1) $$f_{0i}(z):=z_i^{n_i} \qquad \text{on} \quad U_0\cap U_i \ (i\neq 0)$$

where z_i denotes the local coordinate in U_i $(i\neq 0)$. Since S is a Stein space, the pull-back bundle π_2^*F of F by the projection $\pi_2: X\to R$ is positive on X.

We take an arbitrary but fixed point $q=(q_1, q_2)\in D_c$. We can assume that $D_c\cap\pi_2^{-1}(P_i)$ consists of finitely many connected components $\{\Delta_{ij}\,;\, j=1, 2, \cdots, k_i\}$ for $i=1, \cdots, l$ by Lemma 1. If $\Delta_{ij}\cap\pi_1^{-1}(q_1)\neq\emptyset$, we pick up a point Q_{ij} of $\Delta_{ij}\cap\pi_1^{-1}(q_1)$. If $\Delta_{ij}\cap\pi_1^{-1}(q_1)=\emptyset$, we take a point Q_{ij} of Δ_{ij}. We put

$$A:=D\cap(\pi_1^{-1}(q_1)\cup \bigcup_{\substack{1\leqq i\leqq l\\ 1\leqq j\leqq k_i}} \pi_1^{-1}\pi_1(Q_{ij}))\,.$$

Let $\mathcal{J}(A)$ be the sheaf of ideals of A in the structure sheaf $\mathcal{O}_D$ of D. There exists the exact sequence

(2) $$0\longrightarrow \mathcal{J}(A)\longrightarrow \mathcal{O}_D\longrightarrow \mathcal{O}_D/\mathcal{J}(A)\longrightarrow 0\,.$$

Since $\mathcal{J}(A)$ is a coherent analytic sheaf on D, there exists a natural number m_0 such that

$$H^1(D_c, \mathcal{J}(A)\otimes\mathcal{O}(\pi_2^*F^m))=0$$

for $m\geqq m_0$ by Fujiki [2]. Therefore we obtain the exact sequence

(3) $$H^0(D_c, \mathcal{O}(\pi_2^*F^m))\longrightarrow H^0(D_c, \mathcal{O}_D/\mathcal{J}(A)\otimes\mathcal{O}(\pi_2^*F^m))\longrightarrow 0$$

for $m\geqq m_0$ by (2).

Since the open Riemann surface $A':=D_c\cap A$ is a Stein manifold, there exists a holomorphic function $g(z)$ on A' which vanishes at q and at each Q_{ij} up to order m_0N and which is not a constant on each connected component of q and Q_{ij} in A', where $N:=\max_{1\leqq i\leqq l}|n_i|$. We put

$$\begin{cases} \phi_0(z) := g(z) & \text{on } A' \cap U_0, \\ \phi_i(z) := \dfrac{g(x)}{z_i^{m_0 n_i}} & \text{on } A' \cap U_i \ (i=1, \cdots, l). \end{cases}$$

Since $g(z)$ vanishes at Q_{ij} up to order $m_0 N$, $\phi_i(z)$ is holomorphic on $A' \cap U_i$ ($i=1, \cdots, l$). Then we have $\phi := \{\phi_i\} \in H^0(A', \mathcal{O}(\pi_2^* F^{m_0} | A'))$. Here $\pi_2^* F^{m_0} | A'$ denotes the restriction of $\pi_2^* F^{m_0}$ to A'. Since $H^0(A', \mathcal{O}(\pi_2^* F^{m_0} | A')) = H^0(D_c, \mathcal{O}_D / \mathcal{I}(A) \otimes \mathcal{O}(\pi_2^* F^{m_0}))$, there exists a holomorphic section $\tilde{\phi} = \{\tilde{\phi}_i\} \in H^0(D_c, \mathcal{O}(\pi_2^* F^{m_0}))$ such that

$$\tilde{\phi} | A' = \phi$$

by (3). We put

$$G(x) := \begin{cases} \tilde{\phi}_0(x) & \text{for } x \in D_c \cap (S \times U_0) \\ z_i^{m_0 n_i} \tilde{\phi}_i(x) & \text{for } x \in D_c \cap (S \times U_i). \end{cases}$$

Then considering the construction of g, we can check easily that $G(x)$ is a holomorphic function on D_c with $G(q)=0$ and is not a constant function on a neighbourhood of q in A'. Q. E. D.

§ 2. The proof of the main theorem.

LEMMA 2. *Let S, R, X and π_i ($i=1, 2$) be the same as in previous Proposition. Let D be a pseudoconvex (in the sense of Lelong* [**5**]*) domain of X (i.e. a domain convex with respect to the family of plurisubharmonic functions). If $\{p_1^0\} \times R$ is contained in D for a point $p_1^0 \in \pi_1(D)$, then $D = \pi_1(D) \times R$.*

PROOF. Let E be the set of all points $p=(p_1, p_2)$ of D such that $\pi_1^{-1}(p_1) \cap D = \{p_1\} \times R$. By the assumption E is a non-empty open subset of D. We prove that E is closed. Let $\{p^{(n)} = (p_1^{(n)}, p_2^{(n)})\}$ be a sequence of points in E which converges to a point $p' = (p_1', p_2')$ in D. Then $\{p_1'\} \times R$ is contained in the hull the compact set $\{p^{(n)}; n=1, 2, \cdots\} \cup \{p'\}$ of D with respect to the family of plurisubharmonic functions in D. Since D is pseudoconvex in the sense of Lelong, $\{p_1'\} \times R$ is contained in D. Hence we have $p' \in E$. So E is closed. Since D is connected, we have $D=E$. Therefore we have $D = \pi_1(D) \times R$.

Q. E. D.

We now prove the main theorem which is stated in the introduction.

THEOREM. *Let S be a Stein space, R be a compact Riemann surface and X be the product space of S and R. $\pi_1 : X \to S$ denotes the projection of X onto S. Let D be a domain of X. Then the following assertions* (1), (2) *and* (3) *are equivalent:*

(1) *D is weakly* 1*-complete.*

(2) *D is holomorphically convex.*

(3) *Either D is a Stein space or $D=\pi_1(D)\times R$, $\pi_1(D)$ being a Stein space.*

PROOF. (3)→(1) follows from Narasimhan [8]. (3)→(2) is valid by definition.

(2)→(1). Since D is holomorphically convex, it has the Remmert reduction $D\overset{\tau}{\rightarrow}Y$ with proper modification τ and Y is a Stein space (for instance, see Grauert [3]). Since Y is weakly 1-complete by Narasimhan [8], D is weakly 1-complete.

(1)→(3). If D is weakly 1-complete, D is pseudoconvex in the sense of Lelong. So if $\{p_1\}\times R$ is contained in D for a point $p_1\in\pi_1(D)$, we have $D=\pi_1(D)\times R$ by Lemma 2. Moreover we see that $\pi_1(D)$ is a Stein space by Andreotti-Narasimhan [1]. Hence we may assume that $\pi_1^{-1}(p_1)\cap D\subsetneqq\{p_1\}\times R$ for every $p=(p_1, p_2)\in D$. Since D is weakly 1-complete, there exists a C^∞ plurisubharmonic function ϕ on D such that

$$D_c:=\{p\in D\,;\ \phi(p)<c\}\Subset D$$

for every $c\in\boldsymbol{R}$. Then for every $c\in\boldsymbol{R}$, D_c is a Stein space by Proposition. We have $D=\bigcup_{k=1}^{\infty}D_k$. Moreover by Narasimhan [8] we see that D_k is a Runge domain in D_{k+1}. Therefore D is a Stein space by Stein [10]. Q. E. D.

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