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A construction of fundamental solution of Schrödinger's equation on the sphere

By Daisuke FUJIWARA

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§ 1. Introduction.

The aim of this note is to construct the fundamental solution of Schrödinger's equation on the unit n-sphere S^n. In § 2, following works of Birkoff [**4**], Maslov [**14**] and Leray [**12**], we shall construct a parametrix. Contrary to the case of heat equation, a local parametrix is not sufficient for us to construct a fundamental solution. Maslov's theory of canonical operators plays an essential rôle in constructing a global parametrix. However, Maslov restricted his discussions to the case where the symplectic manifold is T^*R^n. So we shall discuss, in Appendix I, a simple definition of Maslov's index of a curve in a Lagrangean submanifold of the cotangent bundle T^*M of a Riemannian manifold M. The basic fact is that the horizontal subspace of the Levi-Civita connection is a Lagrangean subspace. Our discussion makes use of the Riemannian metric on M but it is proved, in Appendix I, that Maslov's index thus defined is independent of particular choice of the Riemannian metric of M.

In § 3, we shall present two methods to construct the fundamental solution. The first is the iteration method stated in Theorem 3 that is commonly used in the case of heat equations. The second is Feynman's method stated in Theorem 4. We shall prove that "the Riemannian finite sum approximation of Feynmann's integral" converges in an operator norm to the fundamental solution. Of these two methods, Feynmann's method seems advantageous in two points. The first point is that the evolution property is very easily seen. The second is that the parametrix is needed only for very short interval of time.

The techniques employed here are essentially the same as those used in our previous work [**6**] but are much more complicated. This is mainly because two waves emanating from a point with different velocities may meet at this point or at its antipodal point after a while. The key fact with which we shall get rid of this difficulty is that two such waves are mutually almost orthogonal in the space $L^2(S^n)$. Applying the result of Appendix II, we shall show this fact in Lemma 3.6 along the same lines as the discussions in our work [3].

§ 2. A parametrix.

The equation to be treated is that of Schrödinger's equation

$$\left(i\nu\partial_t+\frac{1}{2}\Delta\right)u=0\,, \qquad \partial_t=\frac{\partial}{\partial t}\,, \tag{1}$$

on the natural unit sphere S^n of dimension n. Here Δ is the Laplace-Beltrami operator associated with the natural metric ds^2 and ν is a parameter greater than one. We shall denote by L the Lagrangean function of a free particle with unit mass on the sphere. Given a curve $\gamma=\gamma(s)$ on S^n, the action $S(\gamma)$ along γ is

$$S(\gamma)=\int_0^t Lds\,, \tag{2}$$

where the Lagrangean L is the square of the length of the tangent vector of γ at s. Since the Lagrangean function L is hyper-regular, in the terminology of Abraham [1], the Hamiltonian function $H: T^*S^n \rightarrow R$ is defined from L through the Legendre transformation

$$FL: T^*S^n \longrightarrow TS^n\,.$$

(See Abraham [1].) Here T^*S^n denotes the cotangent bundle of S^n with projection ρ. We shall denote by σ the canonical 2-form on T^*S^n. The Hamiltonian vector field $\mathcal{X}_H$ is defined by $i_{\mathcal{X}_H}\sigma=-dH$. The vector field $\mathcal{X}_H$ is complete. Integration of this gives rise to a one-parameter group $\{\chi_t\}_{t\in R}$ of global canonical transformations of T^*S^n. The graph $G(\chi_t)$ of χ_t is a $2n$ dimensional Lagrangean submanifold of $T^*S^n\times T^*S^n$ provided with the canonical form $\pi_1^*\sigma-\pi_2^*\sigma$, where π_1 and π_2 denote the projection of $T^*S^n\times T^*S^n$ onto the first and the second factor. We shall call this first factor the initial space and the second the image space. Since energy is conjugate to time, we assign the energy for each point of $\{G(\chi_t)\}_{t\in R}$ and obtain a $2n+1$ dimensional Lagrangean submanifold $G(\chi)$ in $T^*(R\times S^n\times S^n)$.

Following Maslov [14] and Leray [13], we shall construct an oscillatory function associated with this Lagrangean submanifold $G(\chi)$. In order to determine a point of $G(\chi)$ we have only to indicate the time t and the point of the initial space T^*S^n. Thus we have a diffeomorphism $G(\chi)\overset{\theta}{\longrightarrow} R\times T^*S^n$. Let v_0 be the Riemannian volume element of S^n. We shall denote by v the n-form which is the pull-back of v_0 by the following map;

$$G(\chi)\overset{\theta}{\longrightarrow} R\times T^*S^n \overset{\text{proj}}{\longrightarrow} T^*S^n \overset{\rho}{\longrightarrow} S^n\,.$$

We can define another n-form w on $G(\chi)$ that is the pull-back of v_0 by the following map;

$$G(\chi) \longrightarrow T^*R\times T^*S^n\times T^*S^n \xrightarrow{\pi_2} T^*S^n \xrightarrow{\rho} S^n,$$

where the first is the embedding map. A point of $G(\chi)$ is called focal if this n-form w vanishes there.

Now we shall introduce a coordinate system on S^n in order to describe Maslov's oscillatory functions. We shall write $U_\delta(g)=\{x\in S^n \mid \mathrm{dist}\,(x, g)<\delta\}$ for any point $g\in S^n$ and $0<\delta\leqq\pi$. We can choose $g_1, \cdots, g_m$ such that $U_\delta(g_1), \cdots, U_\delta(g_m)$ cover S^n. We may assume that there exists a constant $C_0>0$ such that min dist $(g_i, g_j)\geqq C_0\delta$. Note that $U_{\pi/2}(g_j)$, $j=1, \cdots, m$, is an open hemisphere with pole at g_j. We identify $U_{\pi/2}(g_j)$ with an open set in R^n by coordinate functions $x_{(j)}=(x_j^1, x_j^2, \cdots, x_j^n)$. The portion $\rho^{-1}U_{\pi/2}(g_j)$ of T^*S^n can be identified with $U_{\pi/2}(g_j)\times R^n$ by coordinate functions $(x_{(j)}, \xi^{(j)})=(x_j^1 \cdots x_j^n, \xi_1^j \cdots \xi_n^j)$. We shall call $x_{(j)}=(x_j^1, x_j^2, \cdots, x_j^n)$ position coordinates and $\xi^{(j)}=(\xi_1^j, \xi_2^j, \cdots, \xi_n^j)$ momentum coordinates. When there is no fear of confusion, we shall omit subscripts and write simply $x=(x^1, x^2, \cdots, x^n)$ instead of $x_{(j)}=(x_j^1, x_j^2, \cdots, x_j^n)$. Similar abbreviations will be used for momentum coordinates. The Hamiltonian function H is expressed as

$$H(x, \xi)=\frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k, \tag{3}$$

where $\sum_k g^{jk}(x)g_{kl}(x)=\delta_{jl}$ and $ds^2=\sum g_{jk}(x)dx^j dx^k$ is the metric on S^n. The canonical 2-form on T^*S^n is

$$\sigma=d\xi_1\wedge dx^1+d\xi_2\wedge dx^2+\cdots+d\xi_n\wedge dx^n$$

and the volume element on S^n is

$$v_0=\sqrt{g(x)}\,|dx^1\wedge\cdots\wedge dx^n|,$$

where $g(x)=\det\,(g_{jk}(x))$.

If $(y, \eta, x, \xi)\in G(\chi_t)$, then we have coordinate expressions:

$$v=\sqrt{g(y)}\,|dy^1\wedge\cdots\wedge dy^n|,$$

and

$$w=\sqrt{g(x)}\,|dx^1\wedge\cdots\wedge dx^n|.$$

In the correspondence $\chi_t: (y, \eta)\rightarrow(x, \xi)$, the partial Jacobian $\det\left(\frac{\partial x}{\partial y}\right)$ does not vanish if $(y, \eta, x, \xi)\in G(\chi_t)$ is not focal. Therefore, in this case, we can adopt functions (t, x, η) as local coordinates in some neighbourhood of (H, t, y, η, x, ξ) in $G(\chi)$.

Let (y, η, x, ξ) be in $G(\chi_t)$. Then there exists a curve $\tilde{\gamma}: \tilde{\gamma}(s)=\chi_s(y, \eta)=(x(s), \xi(s))$ in T^*S^n joining (y, η) to (x, ξ). This is the integral curve of $\mathcal{X}_H$ emanating from (y, η) and ending at (x, ξ). We shall denote $\gamma=\rho\tilde{\gamma}$ and we have $\gamma(s)=(x(s))$. The classical action $S(\gamma)$ along γ is a function of (t, y, η). If (H, t, y, η, x, ξ) is not focal, we can consider this as a function of (t, x, η) in some neighbourhood in $G(\chi)$ of this point. We set $S_0(t, x, \eta)=S(\gamma)$. The law of conservation of energy yields

$$S_0(t, x, \eta)=tH(y, \eta)=\frac{1}{2}\frac{r_\gamma(x, y)^2}{t} \tag{4}$$

where $r_\gamma(x, y)$ is the length of the geodesic γ joining x to y. Using this function, we shall define

$$S(t, x, \eta)=S_0(t, x, \eta)+y(t, x, \eta)\cdot\eta\,, \tag{5}$$

where $y(t, x, \eta)\cdot\eta=\sum_{j=1}^n y^j(t, x, \eta)\cdot\eta_j$ and $(y(t, x, \eta), \eta)$ is the initial point of the point $G(\chi_t)$ determined by coordinates (x, η). This function $S(t, x, \eta)$ is defined only at non-focal points in the open set of $G(\chi)$ which is the inverse image of $\rho^{-1}U_\pi(g_j)$ with some j by the map

$$G(\chi)\xrightarrow{\ \theta\ } R\times T^*S^n \xrightarrow{\ \text{proj}\ } \rho^{-1}U_\pi(g_j)\subset T^*S^n\,.$$

At these points, the oscillatory function we need is of the form

$$E_0(t, x, \eta)=e_0(t, x, \eta)e^{i\nu S(t,x,\eta)} \tag{6}$$

where

$$e_0(t, x, \eta)=\left|\frac{v}{w}\right|^{1/2}e^{-\frac{\pi}{2}i\,\mathrm{Ind}\,\gamma}\,. \tag{7}$$

Here $\mathrm{Ind}\,\gamma$ is the Maslov index of the curve γ whose definition will be given in the appendix. Explicitly $\mathrm{Ind}\,\gamma=(n-1)\cdot\left[\left[\dfrac{\text{length of }\gamma+\frac{\pi}{2}}{\pi}\right]\right]$, where $[[x]]$ is the greatest integer smaller than x.

REMARK 1. If $\chi_t(y, \eta)=(x(t), \xi(t))$, we have

$$\chi_s(x(t), \xi(t))=(x(t+s), \xi(t+s)) \tag{8}$$

and

$$e_0(t+s, x(t+s), \eta)=e_0(t, x(t), \eta)e_0(s, x(t+s), \xi(t))\,. \tag{9}$$

REMARK 2. $S(t, x, \eta)$ is a generating function of $G(\chi)$ at (t, x, η).

We shall introduce a local coordinates system around a focal point in $G(\chi)$.

Let $0=\zeta_1{}^j, \zeta_2{}^j, \zeta_3{}^j, \cdots$ be the lattice points of $T^*_{g_j}S^n$ whose momentum coordinates are multiples of, say, $\frac{\delta}{4\sqrt{n}}$. $T^*_{g_j}S^n$ is covered by open cubes C_{jk} of side length $\frac{\delta}{2\sqrt{n}}$ centered at one of the ζ's. We set $\omega_{jk}=U_\delta(g_j)\times C_{jk}$. Since the portion $\rho^{-1}U_\pi(g_j)\subset T^*S^n$ is identified with $U_\pi(g_j)\times T^*_{g_j}S^n$, ω_{jk} is identified with an open set in the initial space. We set $\Omega_{jk}(t)=G(\chi_t)\cap(\omega_{jk}\times\chi_t(\omega_{jk}))$ and $\Omega_{jk}=\bigcup_t\{(E, t, \Omega_{jk}(t))\}$. A point $(y, \eta, x, \xi)\in\Omega_{jk}(t)$ is focal if and only if x lies on the geodesic emanating from y with the direction of $\pm\eta$ and dist $(x, y)=\frac{\pi}{2}$. Let $g_{l(jk)}$ be one of the g's that are closest to x. Then the following lemma is obvious.

LEMMA 2.1. *If δ is sufficiently small and if $\chi_t(\omega_{jk})$ contains a focal point, then we can find an $l(jk)$ such that*

$$\text{dist}\,(\rho\chi_t(\omega_{jk}),\ \partial U_{\pi/2}(g_{l(jk)}))>5\delta\,. \tag{10}$$

This lemma implies that the trivialization of $\rho^{-1}U_{\pi/2}(g_{l(jk)})$ of the image space is valid in the 5δ-neighbourhood of $\rho\chi_t(\omega_{jk})$ and that the momentum coordinates ξ in the image space and the momentum coordinates η of the initial space can be used as local coordinates in this open subset of $G(\chi_t)$. Let p be the momentum coordinates of $\chi_t(g_j, \zeta_k{}^j)$. Then we shall define

$$\tilde{S}(t, \xi, \eta)=S(t, x, \eta)-\sum_{j=1}^{n}x^j(\xi_j-p_j)\,, \tag{11}$$

where the independent variables are ξ and η. We have a momentum expression of the oscillatory function

$$\tilde{e}_0(t, \xi, \eta)e^{i\nu\tilde{S}(t,\xi,\eta)}\,, \tag{12}$$

where

$$\tilde{e}_0(t, \xi, \eta)=(2\pi)^{\frac{n}{2}}\nu^{\frac{n}{2}}\left|\frac{g(y(\xi, \eta))}{g(x(\xi, \eta))}\right|^{\frac{1}{2}}\left|\frac{dy^1\wedge\cdots\wedge dy^{n-1}}{d\xi_1\wedge\cdots\wedge d\xi_{n-1}}\right|^{\frac{1}{2}}e^{\frac{\pi}{4}i(n-2\,\text{Ind}\,\gamma)}. \tag{13}$$

Here γ is the curve joining the initial point $(0, y, \eta)$ to the focal image point (t, x, ξ). We can express this oscillatory function by position coordinates, at least formally, as

$$(2\pi)^{-n}\nu^{\frac{n}{2}}\int_{R^n}\tilde{e}_0(t, \xi, \eta)e^{i\nu(\tilde{S}(t,\xi,\eta)+x_0(\xi-p))}d\xi\,. \tag{14}$$

(See Maslov [**14**] for the details.)

We shall construct our global parametrix from this oscillatory function. Let $\{\varphi_j(y)\}$ be a partition of unity subordinate to the open covering $\{U_\delta(g_j)\}$ and $\phi_{jk}(\eta)$ be a partition of unity subordinate to the open covering C_{jk} of

$T^*_{g_j}S^n$. We set $e_{jk}=\varphi_j(y)\phi_{jk}(\eta)$. This is a C^∞ function in Ω_{jk} if Ω_{jk} is parametrized by t and the coordinates (y, η) of the initial space. $\{e_{jk}\}_{jk}$ is a partition of unity subordinate to the open covering $\{\Omega_{jk}\}_{j,k}$ of $G(\chi)$. We set

$$a_{jk}(t, x, \eta)=e_{jk}(t, x, \eta)e_0(t, x, \eta) \tag{15}$$

and

$$E_{jk}(t, x, \eta)=a_{jk}(t, x, \eta)e^{i\nu S(t,x,\eta)}, \tag{16}$$

if $\Omega_{jk}(t)$ does not contain focal points. If $\Omega_{jk}(t)$ does contain focal points, e_{jk} is a function of (ξ, η) and we set

$$\tilde{a}_{jk}(t, \xi, \eta)=e_{jk}(t, \xi, \eta)\tilde{e}_0(t, \xi, \eta) \tag{17}$$

and

$$\tilde{E}_{jk}(t, x, \eta)=\tilde{a}_{jk}(t, \xi, \eta)e^{i\nu\tilde{S}(t,\xi,\eta)}. \tag{18}$$

The expression by position coordinates x of this function is

$$E_{jk}(t, x, \eta)=(2\pi)^{-n}\nu^{\frac{n}{2}}\int \tilde{a}_{jk}(t, \xi, \eta)e^{i\nu(\tilde{S}(t,\xi,\eta)+x\cdot(\xi-p))}d\xi. \tag{19}$$

Note that we have an asymptotic expansion

$$E_{jk}(t, x, \eta)\sim\left|\frac{v}{w}\right|^{\frac{1}{2}}e^{-\frac{\pi}{2}i\,\mathrm{Ind}\,\gamma}e^{i\nu S(t,x,\eta)}+O\left(\frac{t}{\nu}\right), \tag{20}$$

as $\frac{t}{\nu}\to 0$, if (y, η, x, ξ) is not a focal point in $\Omega_{jk}(t)$. Here, γ is a curve joining y to x along the geodesic.

We define a function on $\Omega_{jk}\times U_\delta(g_l)$ by

$$\begin{aligned} E_{jkl}(t, x, \eta, z)&=(2\pi)^{-n}E_{jk}(t, x, \eta)\varphi_l(z)e^{-i\nu z\cdot\eta} \\ &=(2\pi)^{-n}a_{jk}(t, x, \eta)\varphi_l(z)e^{i\nu(S(t,x,\eta)-z\cdot\eta)} \end{aligned} \tag{21}$$

where $z\cdot\eta=\sum_{j=1}^n z^j\eta_j$. We define this only if $\Omega_{jk}(t)$ does not contain focal points and if $U_\delta(g_l)\cap U_\delta(g_j)\neq\emptyset$. If $U_\delta(g_l)\cap U_\delta(g_j)\neq\emptyset$ but $\Omega_{jk}(t)$ contains focal points, we define

$$\begin{aligned} \tilde{E}_{jkl}(t, \xi, \eta, z)&=(2\pi)^{-n}\tilde{E}_{jk}(t, \xi, \eta)\varphi_l(z)e^{-i\nu z\cdot\eta} \\ &=(2\pi)^{-n}\tilde{a}_{jk}(t, \xi, \eta)\varphi_l(z)e^{i\nu(\tilde{S}(t,\xi,\eta)-z\cdot\eta)}. \end{aligned} \tag{22}$$

We set

$$E_{jkl}(t, x, \eta, z)=(2\pi)^{-n}\psi_{jk}(t, x)\int_{R^n}\tilde{E}_{jkl}(t, \xi, \eta, z)e^{i\nu x\cdot(\xi-p)}d\xi, \tag{23}$$

where $\psi_{jk}(t, x)$ is a C^∞ function on $R\times S^n$ and $\psi_{jk}\equiv 1$ on $\rho\chi_t(\omega_{jk})$. Summing these functions, we define our parametrix

$$E(t, x, \eta, z)=\sum_{j,k}\sum_{l\in J(j)}E_{jkl}(t, x, \eta, z) \tag{24}$$

where $J(j)=\{l \mid U_\delta(g_l)\cap U_\delta(g_j)\neq\emptyset\}$.

PROPOSITION 2.2. 1) *If $\Omega_{jk}(t)$ does not contain focal points, we have*

$$\Big(i\nu\partial_t+\frac{1}{2}\Delta\Big)E_{jkl}(t, x, \eta, z)=b_{jk}e^{i\nu(S(t,x,\eta)-\eta\cdot z)}$$

where $b_{jk}(t, x, \eta, z)=(2\pi)^{-n}2^{-1}\Delta_x(e_0(t, x, \eta)e_{jk}(t, x, \eta))$.

2) *If $\Omega_{jk}(t)$ contains focal points, then*

$$\begin{aligned}
&\Big(i\nu\partial_t+\frac{1}{2}\Delta\Big)E_{jkl}(t, x, \eta, z)\\
&=\Big(i\nu\partial_t+\frac{1}{2}\Delta\Big)\psi_{jk}(t, x)\int\tilde{E}_{jkl}(t, \xi, \eta, z)e^{i\nu x\cdot(\xi-p)}d\xi\\
&\quad+\sum_{r,s}g^{r,s}(x)\partial_{x_r}\psi_{jk}(t, x)\int(\xi_s-p_s)\tilde{E}_{jkl}(t, \xi, \eta, z)e^{i\nu x\cdot(\xi-p)}d\xi\\
&\quad+\psi_{jk}(t, x)\int\Big(i\nu\partial_t+\frac{1}{2}\tilde{\Delta}(\xi)\Big)\tilde{E}_{jkl}(t, \xi, \eta, z)e^{i\nu x\cdot(\xi-p)}d\xi\,,
\end{aligned}$$

where the operator $\tilde{\Delta}(\xi)$ is a pseudo-differential operator acting on functions of ξ:

$$\begin{aligned}
\tilde{\Delta}(\xi)=&\sum_{r,s}(i\nu(\xi-p)_r)(i\nu(\xi-p)_s)\tilde{g}^{r,s}((i\nu)^{-1}\partial_\xi)\\
&-\sum_{r,s}(i\nu(\xi-p)_r)\partial_{x_s}\tilde{g}^{r,s}\Big(\frac{1}{i\nu}\partial_\xi\Big)\\
&-\sum_{r,s}(i\nu(\xi-p)_r)\Big(\tilde{g}^{r,s}\sqrt{\tilde{g}}\partial_s\frac{1}{\sqrt{\tilde{g}}}\Big)\Big(\frac{1}{i\nu}\partial_\xi\Big)\\
&+\sum_{r,s}\partial_{x_r}\Big(\partial_{x_s}\Big(\frac{1}{\sqrt{g}}\Big)\cdot\tilde{g}^{jk}\sqrt{\tilde{g}}\Big)\Big(\frac{1}{i\nu}\partial_\xi\Big)\,,
\end{aligned}$$

and

$$\tilde{g}^{r,s}(x)=\tau(x)g^{r,s}(x)$$

with $\tau\in C_0^\infty(S^n)$ and $\tau\equiv 1$ in some neighbourhood of $\operatorname{supp}\psi_{jk}(t, x)$.

PROOF OF PROPOSITION. We have only to prove 1). Let $a(x)$ be a C^∞ function of x. Then

$$\begin{aligned}
&\Big(i\nu\partial_t+\frac{1}{2}\Delta\Big)(a(x)e^{i\nu S(t,x,\eta)})\\
&=e^{i\nu S(t,x,\eta)}\Big\{(i\nu)^2\Big(\partial_t S+\frac{1}{2}(\operatorname{grad}S)^2\Big)\\
&\quad+(i\nu)\Big(\partial_t a+\operatorname{grad}S\cdot\operatorname{grad}a+\frac{1}{2}\Delta S\cdot a\Big)+\frac{1}{2}\Delta a\Big\}
\end{aligned}$$

$$= e^{i\nu S(t,x,\eta)}\Big\{(i\nu)\Big(L_{\mathfrak{X}}+\frac{1}{2}\Delta S\Big)a+\frac{1}{2}\Delta a\Big\} ,$$

where $L_{\mathfrak{X}}=\partial_t+\operatorname{grad} S\cdot\operatorname{grad}$. It is well known that

$$L_{\mathfrak{X}}\Big(\frac{1}{\sqrt{g(x)}}\Big|\det\Big(\frac{\partial y(x, t)}{\partial x}\Big)\Big|^{\frac{1}{2}}\Big)=-\frac{1}{2}\Delta S\Big(\frac{1}{\sqrt{g(x)}}\Big|\det\Big(\frac{\partial y(t, x)}{\partial x}\Big)\Big|^{\frac{1}{2}}\Big).$$

Consequently we have $L_{\mathfrak{X}}e_{jk}(t, x, \eta)=0$. This proves the proposition.

Since the functions $\tilde{g}^{r,s}(x_{(j)})$ are bounded in $C_0^\infty(R^n)$, they are bounded in the class $S^0_{1,0}(R^n)$ in the terminology of Hörmander [9]. If $\Omega_{jk}(t)$ contains focal points, we can choose a point $\bar{x}_j=(\bar{x}_j^1, \cdots, \bar{x}_j^n)$ with $|\bar{x}_j|=(\sum_j(\bar{x}_j^k)^2)^{1/2}<10\pi$ such that $\sum_{j=1}^{n-1}(\partial_{\xi_k}\tilde{S}(t, \xi, \eta)-\bar{x}_j^k)^2$ has a positive lower bound, because $\sum_j|\partial_{\xi_j}\tilde{S}(t, \xi, \eta)|^2\leqq(2\pi)^2$ at any point of $\Omega_{jk}(t)$. The functions $\tilde{g}^{r,s}(x_{(j)}+\bar{x}_j)$ are bounded in $S^0_{1,0}(R^n)$. Therefore, we can use Hörmander's Theorem 3.3 in [8] and obtain an asymptotic expansion in the space $\mathcal{E}(R^n)$ when $\nu\to\infty$,

$$\begin{aligned}
&e^{-i\nu(\tilde{S}(t,\xi,\eta)-z\cdot\eta)}\Big(\tilde{g}^{r,s}\Big(\frac{1}{i\nu}\partial_\xi\Big)\tilde{E}_{jkl}(t, \xi, \eta, z)\Big)\\
&\quad=e^{-i\nu(\tilde{S}-z\cdot\eta-\bar{x}(j)\xi)}(2\pi)^{-n}\varphi_l(z)\tilde{g}^{r,s}\Big(\frac{1}{i\nu}(\partial_\xi+\bar{x}_{(j)})\Big)(e^{i\nu(\tilde{S}-\bar{x}(j)\xi)}\tilde{a}_{jk}(t, \xi, \eta))\\
&\quad\sim(2\pi)^{-n}\varphi_l(z)(\tilde{g}^{r,s}(\partial_\xi\tilde{S})\tilde{a}_{jk}(t, \xi, \eta)\\
&\qquad+\frac{1}{i\nu}\sum_r\partial_{x_r}\tilde{g}^{r,s}(\partial_\xi\tilde{S})\cdot\partial_{\xi_r}\tilde{a}_{jk}(t, \xi, \eta)+O(\nu^{-2}) .
\end{aligned}$$

If one carefully follows Hörmander's proof, one can easily see that the asymptotic expansion is valid also in the topology of $\mathcal{D}_{L^\infty}(R^n)$ in our situation. Therefore, we have an asymptotic expansion in $\mathcal{D}_{L^\infty}(R^n)$:

$$\begin{aligned}
&e^{-i\nu(S-z\cdot\eta)}\tilde{\Delta}(\xi)\tilde{E}_{jkl}(t, \xi, \eta, z)\\
&\quad=(2\pi)^{-n}\varphi_l(z)\Big[(i\nu)^2\sum_{r,s}(\xi-p)_r(\xi-p)_s\tilde{g}^{r,s}(\partial_\xi\tilde{S})\tilde{a}_{jk}(t, \xi, \eta)\\
&\qquad+i\nu\Big\{\sum_{r,s}(\xi-p)_r\partial_{x_s}\tilde{g}^{r,s}(\partial_\xi\tilde{S})+\sum_{r,s}(\xi-p)_r\Big(\tilde{g}^{r,s}\sqrt{\tilde{g}}\,\partial_{x_s}\frac{1}{\sqrt{\tilde{g}}}\Big)(\partial_\xi\tilde{S})\Big\}\tilde{a}_{jk}(t, \xi, \eta)\\
&\qquad+(i\nu)\sum_r\partial_{x_r}\tilde{g}^{r,s}(\partial_\xi\tilde{S})\partial_{\xi_s}\tilde{a}_{jk}(t, \xi, \eta)+O(1)\Big].
\end{aligned}$$

Since $\tau\equiv1$ on the support of $\tilde{a}_{jk}$, the definitions of $\tilde{a}_{jk}$ and $\tilde{S}$ imply that

$$\Big(i\nu\partial_t+\frac{1}{2}\tilde{\Delta}(\xi)\Big)\tilde{E}_{jkl}(t, \xi, \eta, z)=b(t, \xi, \eta, z)e^{i\nu(\tilde{S}-z\cdot\eta)} ,$$

where $b(t, \xi, \eta, z)$ is bounded in $\mathcal{D}_{L^\infty}(R^n)$ as $\nu\to\infty$. This is uniformly bounded when ξ, η and z vary. Taking derivatives with respect to t, η and z, we can easily see that all derivatives of $b(t, \xi, \eta, z)$ with respect to (t, ξ, η, z) are all uniformly bounded as $\nu\to\infty$. The techniques used here we have adopted from Hörmander [8]. Thus we have proved the following proposition.

PROPOSITION 2.3. $\left(i\nu\partial_t+\frac{1}{2}\tilde{\Delta}(\xi)\right)\tilde{E}_{jkl}(t, \xi, \eta, z)=b_{jkl}(t, \xi, \eta, z)e^{i\nu(\tilde{S}(t,\xi,\eta)-z\cdot\eta)}$, *where* $b_{jkl}(t, \xi, \eta, z)$ *is bounded in* $\mathcal{D}_{L^\infty}(R\times R^n\times R^n\times R^n)$ *as* $\nu\to\infty$.

§3. Two methods to construct the fundamental solutions.

We shall start with some technical lemmata.

LEMMA 3.1. *Given a time interval* $[-T, T]$, $T>0$, *we have the estimates*

$$|\mathrm{grad}_x(S(t, x, \eta)-S(t, x, \eta'))|\geqq C\,\mathrm{dist}\,(\eta, \eta') \tag{1}$$

$$|\mathrm{grad}_\eta(S(t, x, \eta)-S(t, x', \eta))|\geqq C\,\mathrm{dist}\,(x, x') \tag{2}$$

$$|\partial_x^\alpha\partial_\eta^\beta S(t, x, \eta)|\leqq C_{\alpha\beta}\,, \qquad |\alpha|\geqq 1,\ and\ |\beta|\geqq 1\,, \tag{3}$$

for points in Ω_{jk} *if* $\chi_t(\omega_{jk})$ *does not contain focal points.*

PROOF. We shall make use of the notation $(x, \xi)=\chi_t(y, \eta)$. We can easily see that

$$\left|\det\left(\frac{\partial^2}{\partial x^j\partial\eta_k}S(t, x, \eta)\right)\right|=\left|\det\left(\frac{\partial\xi_j}{\partial\eta_k}\right)\right|=\left|\det\left(\frac{\partial y_k}{\partial x^j}\right)\right|>C>0\,. \tag{4}$$

Inequalities (1) and (2) then follow from this. Proof of (3) is omitted.

Similarly, we can prove

LEMMA 3.2. *If* $\Omega_{jk}(t)$ *contains focal points, we have*

$$|\mathrm{grad}_\xi(\tilde{S}(t, \xi, \eta)-\tilde{S}(t, \xi, \eta'))|\geqq C\,\mathrm{dist}\,(\eta, \eta')\,, \tag{5}$$

$$|\mathrm{grad}_\eta(\tilde{S}(t, \xi, \eta)-\tilde{S}(t, \xi, \eta))|\geqq C\,\mathrm{dist}\,(\xi, \xi')\,, \tag{6}$$

$$|\partial_\xi^\alpha\partial_\eta^\beta\tilde{S}(t, \xi, \eta)|\leqq C_{\alpha\beta}\,, \qquad |\alpha|\geqq 1,\ and\ |\beta|\geqq 1\,. \tag{7}$$

The proof of the following two lemmata is also similar to that of Lemma 3.1.

LEMMA 3.3. *For any multi-indices* α *and* β, *there exists a constant* $C_{\alpha\beta}$ *such that*

$$|\partial_x^\alpha\partial_\eta^\beta a_{jk}(t, x, \eta)|\leqq C_{\alpha\beta}\,, \tag{8}$$

for any point in $\Omega_{jk}(t)$ *if* $\Omega_{jk}(t)$ *contains no focal points.*

LEMMA 3.4. *For any multi-indices* α *and* β, *there exists a constant* $C_{\alpha\beta}>0$

such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \tilde{a}_{jk}(t, \xi, \eta)| \leqq C_{\alpha\beta}, \tag{9}$$

for any point in Ω_{jk} if $\Omega_{jk}(t)$ contains a focal point.

Let us define the linear mappings

$$E_{jkl}f(x) = \iint E_{jkl}(t, x, \eta, z) f(z) dz d\eta \tag{10}$$

$$j, l = 1, 2, \cdots, m, \qquad k = 1, 2, 3, \cdots.$$

Our first fundamental lemma is

LEMMA 3.5. *There exists a positive constant C such that*

$$\|E_{jkl}(t)\| \leqq C\nu^{-n/2} \|\phi_{jk} \hat{f}_l\| \tag{11}$$

where

$$\hat{f}_l(\eta) = \nu^{-n/2} \int_{R^n} e^{-i\nu\eta\cdot y} \varphi_l(y) f(y) dy, \tag{12}$$

for $f \in C^\infty(S^n)$.

PROOF. This is an immediate consequence of our previous work [7].

Next we have

LEMMA 3.6. 1° *Let f be a function in $C^\infty(S^n)$. Then*

$$\operatorname{supp} E_{jkl}(t) f \subseteqq \rho \chi_t(\omega_{jk}), \qquad \omega_{jk} = U_\delta(g_j) \times C_{jk}. \tag{13}$$

2° *There exists a positive constant C such that*

$$|(E_{jkl}(t)f, E_{jk'l}(t)g)| \leqq C(1 + \nu|\zeta_k^j - \zeta_{k'}^j|)^{-2(n+1)} \|\phi_{jk}\hat{f}_l\| \|\phi_{jk'}\hat{g}_l\|. \tag{14}$$

PROOF. 1° is obvious.

2° If $\operatorname{supp} E_{jkl}(t)f \cap \operatorname{supp} E_{jk'l}(t)g \neq \emptyset$, then $\rho(\chi_t(\omega_{jk})) \cap \rho(\chi_t(\omega_{jk'})) \neq \emptyset$. We shall first treat the case where both $\chi_t(\omega_{jk})$ and $\chi_t(\omega_{jk'})$ do not contain focal points. We can write

$$(E_{jkl}(t)f, E_{jk'l}(t)g) = (2\pi)^{-2n} \iiint a_{jk}(t, x, \eta) \overline{a_{jk'}(t, x, \xi)} \hat{f}_l(\eta) \overline{\hat{g}_l(\xi)} \times e^{i\nu(S(t,x,\eta) - S(t,x,\xi))} d\xi d\eta dx. \tag{15}$$

The equality $\operatorname{grad}_x(S(t, x, \eta) - S(t, x, \xi)) = 0$ implies that $\xi = \eta$. This does not occur if $|\zeta_k^j - \zeta_{k'}^j| \geqq 10\delta$. In this case, we can define a linear partial differential operator

$$L = \sum_j \frac{\theta_j(t, x, \xi, \eta)}{\Theta^2} \partial_{x_j},$$

where $\theta_j(t, \xi, x, \eta) = \partial_{x_j}(S(t, x, \eta) - S(t, x, \xi))$ and $\Theta = [\sum_{j=1}^n \theta_j(t, \xi, x, \eta)^2]^{1/2}$. The adjoint L^* is $L^*=-L-\sum_{j=1}^{n}\partial_{x_j}\left(\frac{\theta_j}{\Theta^2}\right)$. Since $(L-i\nu)e^{i\nu(S(t,x,\eta)-S(t,x,\xi))}=0$, (15) turns out to be

(16) $$(E_{jkl}(t)f, E_{jk'l}(t)g)$$
$$=(2\pi)^{-2n}(i\nu)^{-l}\iint b_l(t, \xi, x, \eta)e^{i\nu(S(t,x,\eta)-S(t,x,\xi))}\hat{f}_l(\eta)\overline{\hat{g}_l(\xi)}d\eta dx d\xi$$

for $l=0, 1, 2, \cdots$. Here we write

(17) $$b_l(t, \xi, x, \eta)=L^{*l}(a_{jk}(t, x, \eta)\overline{a_{jk'}(t, x, \xi)}).$$

We claim that for any multi-indices α, β, γ there exists a positive constant C such that

(18) $$\sup_{x,\eta}\int_{R^n}|\partial_\xi^\alpha\partial_x^\beta\partial_\eta^\gamma b_l(t, \xi, x, \eta)|d\xi\leqq C|\zeta_k^j-\zeta_{k'}^j|^{-l}.$$

In fact, this is true for $l=0$. Assume that (18) is true for $l=k$, then

$$b_{k+1}(t, \xi, x, \eta)=L^*b_k(t, \xi, x, \eta).$$

Since $\Theta\geqq C|\xi-\eta|$ (Lemma 3.1), we obtain

$$\sup_{x,\eta}\int_{R^n}|b_{k+1}(t, \xi, x, \eta)|d\xi\leqq C|\zeta_k^j-\zeta_{k'}^j|^{-k-1}.$$

Simple calculation explicitly gives the commutators

$$[\partial_{x_j}, L]=\sum_k\partial_{x_j}\left(\frac{\theta_k}{\Theta^2}\right)\partial_{x_k}$$
$$=\sum_k(\Theta^{-2}\partial_{x_j}\theta_k-\sum_l 4\Theta^{-4}\theta_k\theta_l\partial_{x_j}\theta_l)\partial_{x_k}.$$
$$[\partial_{\xi_j}, L]=\sum_k\partial_{\xi_j}(\Theta^{-2}\theta_k)\partial_{x_k}.$$
$$[\partial_{\eta_j}, L]=\sum_k\partial_{\eta_j}(\Theta^{-2}\theta_k)\partial_{x_k}.$$
$$[\partial_{x_k}, \partial_{x_j}(\Theta^{-2}\theta_j)]=\partial_{x_k}\partial_{x_j}(\Theta^{-2}\theta_j).$$
$$[\partial_{\xi_k}, \partial_{x_j}(\Theta^{-2}\theta_j)]=\partial_{\xi_k}\partial_{x_j}(\Theta^{-2}\theta_j).$$
$$[\partial_{\eta_k}, \partial_{x_j}(\Theta^{-2}\theta_j)]=\partial_{\eta_k}\partial_{x_j}(\Theta^{-2}\theta_j).$$

By induction we can prove that for any multi-indices α, β, γ there exists a positive constant C such that

$$|\partial_\xi^\alpha\partial_x^\beta\partial_\eta^\gamma\theta_j|\leqq C\Theta,$$

and that

$$|\partial_\xi^\alpha \partial_x^\beta \partial_\eta^\gamma(\Theta^{-2}\theta_j)| \leqq C\Theta^{-1}.$$

Since

$$\partial_{x_j} b_{l+1}(t, \xi, x, \eta) = L^* \partial_{x_j} b_l(t, \xi, x, \eta) + [\partial_{x_j}, L^*] b_l(t, \xi, x, \eta),$$

we have

$$|\partial_{x_j} b_{k+1}(t, \xi, x, \eta)| \leqq \Theta^{-1}\{[\sum_k |\partial_{x_k}\partial_{x_j} b_l(t, \xi, x, \eta)|^2]^{1/2}$$
$$+[\sum_k |\partial_{x_k} b_l(t, \xi, x, \eta)|^2]^{1/2}\}.$$

Consequently we obtain

$$\sup_{x,\eta}\int |\partial_{x_j} b_{k+1}(t, \xi, x, \eta)| d\xi \leqq C(1+\nu|\zeta_k^j-\zeta_{k'}^j|)^{-k-1}. \tag{19}$$

Similar discussions prove (18) for $l=k+1$. Therefore (18) holds for any l.

We apply Theorem (A-II) of Appendix II to (16). Then the estimate (18) implies that

$$|(E_{jkl}(t)f, E_{jk'l}(t)g)| \leqq C|\zeta_k^j-\zeta_{k'}^j|^{-l}$$

for $l=0, 1, 2, \cdots$.

Next we shall treat the case where one of $\chi_t(\omega_{jk})$ and $\chi_t(\omega_{jk'})$ is focal. We may assume that $\rho\chi_t(\omega_{jk}) \cap \rho\chi_t(\omega_{jk'}) \neq \emptyset$. This implies that the focal coordinates are valid in some neighbourhood of $\overline{\Omega_{jk}(t)} \cup \overline{\Omega_{jk'}(t)}$ if δ is sufficiently small. Therefore we have

$$(E_{jkl}(t)f, E_{jk'l}(t)g) = \int \phi_{jk}(t, x)\phi_{jk'}(t, x)\sqrt{g(x)}\, dx\, e^{i\nu x\cdot(\xi-\xi')} \tag{20}$$
$$\times \int \tilde a_{jk}(t, \xi, \eta)\tilde a_{jk'}(t, \xi', \eta')\tilde f_l(\eta)\overline{\tilde g_l(\eta')}$$
$$\times e^{i\nu(\tilde S_1(t,\xi,\eta)-\tilde S(t,\xi',\eta'))} d\xi' d\eta' d\xi d\eta$$
$$= \int \tilde\phi_{jkk'}(t, \xi-\xi')\tilde a_{jk}(t, \xi, \eta)\tilde a_{jk'}(t, \xi', \eta')\hat f_l(\eta)\overline{\hat g_l(\eta')}$$
$$\times e^{i\nu(\tilde S_1(t,\xi,\eta)-\tilde S_1(t,\xi,\eta'))} d\xi' d\eta' d\xi d\eta$$

where

$$\tilde S_1(t, \xi, \eta) = \tilde S(t, \xi, \eta) - \sum_j x^j p_j = S(t, x, \eta) - \sum_j x^j \xi_j$$

and

$$\tilde\phi_{jkk'}(t, \zeta) = \int \phi_{jk}(t, x)\phi_{jk'}(t, x)\sqrt{g(x)}\, e^{i\nu x\cdot\zeta} dx.$$

Since ϕ_{jk} and $\phi_{jk'}$ belong to a bounded set in $C_0^\infty(\rho\chi_t(\omega_{jk}) \cup \rho\chi_t(\omega_{jk'}))$, there exists a positive constant C independent of k, k' such that

$$|\tilde\phi_{jkk'}(t, \zeta)| \leqq C(1+\nu|\zeta|)^{-2(n+1)}. \tag{21}$$

Let

$$\tilde{E}_{jkl}(t)f(\xi)=\int \tilde{a}_{jk}(t,\xi,\eta)\hat{f}_l(\eta)e^{i\nu\tilde{S}(t,\xi,\eta)}d\eta\,. \tag{22}$$

Then the support of $\tilde{E}_{jkl}(t)f$ is compact and we obtain the estimate (see [7])

$$\|\tilde{E}_{jkl}(t)f\|\leqq C\|\phi_{jk}\hat{f}_l\|\,. \tag{23}$$

If $\xi\in\operatorname{supp}\tilde{E}_{jkl}(t)f$ and $\xi'\in\operatorname{supp}\tilde{E}_{jk'l}(t)g$, then

$$|\tilde{\varphi}_{jkk'}(t,\xi-\xi')|\leqq C(1+\nu|\xi-\xi'|)^{-2(n+1)}\,. \tag{24}$$

This and the expression

$$(E_{jkl}(t)f,E_{jk'l}(t)g)=\int\tilde{\varphi}_{jkk'}(t,\xi-\xi')\tilde{E}_{jkl}(t)f(\xi)\overline{\tilde{E}_{jk'l}(t)g(\xi')}d\xi'd\xi$$

imply that

$$\begin{aligned}|(E_{jkl}(t)f,E_{jk'l}(t)g)|&\leqq C(1+\nu|\zeta_k^j-\zeta_{k'}^j|)^{-2(n+1)}\\&\qquad\times\|E_{jkl}(t)f\|\|E_{jk'l}(t)g\|\\&\leqq C(1+\nu|\zeta_k^j-\zeta_{k'}^j|)^{-2(n+1)}\|\phi_{jk}\hat{f}_l\|\|\phi_{jk'}\hat{g}_l\|\,.\end{aligned}$$

Therefore, Lemma 3.6 has been proved.

We now define

$$E(t)f(x)=\iint E(t,x,\eta,z)f(z)dzd\eta\,. \tag{25}$$

The precise meaning of the right hand side of (25) is

$$E(t)f(x)=\sum_{j,k}\sum_{l\in J(j)}E_{jkl}f(x)\,. \tag{26}$$

As a consequence of Lemma 3.5 and Lemma 3.6 we can prove

THEOREM 1. *For any $T>0$, there exists a positive constant C such that we have the estimate*

$$\|E(s)f\|\leqq C\nu^{-n/2}\|f\| \tag{27}$$

for any $s\in[-T,T]$ and $f\in C^\infty(S^{n-1})$.

PROOF.

$$\begin{aligned}\|E(s)f\|^2&=\|\sum_{j,k,l}E_{jkl}(s)f\|^2\\&=\sum_{j,k}\sum_{l\in J(j)}\|E_{jkl}(s)f\|^2\\&\quad+\sum_{jkl}\sum_{(j'k'l')\neq(jkl)}(E_{jkl}(s)f,E_{j'k'l'}(s)f)\,.\end{aligned}$$

The first term is estimated in the following manner:

$$\sum_{jkl}\|E_{jkl}(s)f\|^2 \leqq C\nu^{-n}\sum_{jkl}\|\phi_{jk}\hat{f}_l\|^2 \leqq C\nu^{-n}\sum_{jl}\|\hat{f}_l\|^2 \leqq C\nu^{-n}\|f\|^2, \tag{28}$$

because $\{\phi_{jk}(\eta)\}$ is a partition of unity subordinate to the open covering C_k^j of $T_{g_j}S^n$.

The second term is

$$\sum_{jkl}\sum_{(j'k'l')\neq(jkl)}|(E_{jkl}(s)f, E_{j'k'l'}(s)f)| \leqq C\sum_{jkl}\sum_{j'k'l'}(1+\nu|\zeta_k^j-\zeta_{k'}^{j'}|)^{-N}\|\phi_{jk}\hat{f}_l\|\|\phi_{j'k'}\hat{f}_{l'}\| \tag{29}$$

because of Lemma 3.6. Making use of Schur's inequality, we can prove that this does not exceed

$$C\sum_{jl}\sum_{j'l'}\nu^{-n}(\sum_k\|\phi_{jk}\hat{f}_l\|^2)^{1/2}(\sum_{k'}\|\phi_{jk'}\hat{f}_{l'}\|^2)^{1/2} \leqq C\nu^{-n}(\sum_{jkl}\|\phi_{jk}\hat{f}_l\|^2)^{1/2}(\sum_{j'k'l'}\|\phi_{j'k'}\hat{f}_{l'}\|^2)^{1/2} \leqq C\nu^{-n}\|f\|^2. \tag{30}$$

Theorem 1 has been proved.

Our next result is

THEOREM 2. *For any* $f\in L^2(S^n)$,

$$s\text{-}\lim_{s\to 0} E(s)f = f. \tag{31}$$

PROOF. By virtue of the Banach-Steinhaus theorem we have only to prove (31) for $f\in C^\infty(S^n)$. It is obvious that

$$s\text{-}\lim_{s\to 0} E_{jkl}(s)f = E_{jkl}(0)f. \tag{32}$$

Using Lebesgue's dominated convergence theorem, we can prove (31) from (32).

Let us denote

$$F(t) = \left(i\nu\partial_t + \frac{1}{2}\Delta\right)E(t). \tag{33}$$

This is the sum

$$F(t) = \sum_{jkl} F_{jkl}(t) \tag{34}$$

where

$$F_{jkl}(t)=\Big(i\nu\partial_t+\frac{1}{2}\Delta\Big)E_{jkl}(t)\,. \tag{35}$$

LEMMA 3.7. *For any $T>0$, there exists a positive constant C such that*

$$\|F(s)f\|\leqq C\nu^{-n/2}\|f\| \tag{36}$$

for $s\in[-T, T]$.

PROOF. If $\Omega_{jk}(t)$ does not contain focal points, then

$$F_{jkl}(t)f(x)=\iint b_{jk}(t, x, \eta, z)e^{i\nu(S(t,x,\eta)-\eta\cdot z)}f(z)dzd\eta\,,$$

where

$$b_{jk}(t, x, \eta, z)=(2\pi)^{-n}2^{-1}\Delta_x(e_0(t, x, \eta)e_{jk}(t, x, \eta))\,.$$

We can replace $E_{jkl}(t)$ by $F_{jkl}(t)$ in Lemma 3.5 and Lemma 3.1. Thus there only remains the case where $\Omega_{jk}(t)$ contains focal points. In this case, $F_{jkl}(t)$ is divided into three parts,

$$F_{jkl}(t)=F_{jkl}^{(1)}(t)+F_{jkl}^{(2)}(t)+F_{jkl}^{(3)}(t)\,,$$

where

$$F_{jkl}^{(1)}(t)f(x)=\Big(\Big(i\nu\partial_t+\frac{1}{2}\Delta\Big)\phi_{jk}(t, x)\Big)E_{jkl}(t)f(x)\,, \tag{37}$$

$$F_{jkl}^{(2)}(f)f(x)=\sum_{r,s}g^{r,s}(x)\partial_{x_r}\phi_{jk}(t, x)\int(i\nu(\xi-p))_s\tilde{E}_{jkl}(t, \xi, \eta, z)\cdot e^{i\nu x\cdot(\xi-p)}f(z)dzd\xi d\eta\,, \tag{38}$$

$$F_{jkl}^{(3)}(t)f(x)=\phi_{jk}(t, x)\int\Big(i\nu\partial_t+\frac{1}{2}\tilde{\Delta}(\xi)\Big)\tilde{E}_{jkl}(t, \xi, \eta, z)e^{i\nu x\cdot(\xi-p)}f(z)dzd\eta d\xi\,. \tag{39}$$

The discussions for $F_{jkl}^{(1)}(t)$ and $F_{jkl}^{(3)}(t)$ are similar to those for $E_{jkl}(t)$ because of Proposition 2.2. At the stationary point of the phase function of the integral (38),

$$z=\partial_\eta\tilde{S}\,,\qquad x+\partial_\xi\tilde{S}=0$$

must hold. This means that $\chi_t(z, \eta)=(x, \xi)$. The amplitude function vanishes at this point. Just as we did in the proof of Lemma 3.6, we define

$$M=\frac{\sum_j^n(\partial_{t_j}\tilde{S}-z^j)\partial_{\eta_j}+\sum_{j=1}^n(\partial_{\xi_j}\tilde{S}+x_j)\partial_{\xi_j}}{(|\partial_\eta\tilde{S}-z|^2+|\partial_\xi\tilde{S}+x|^2)}\,.$$

And we obtain

$$F_{jkl}^{(2)}(t)f(x)=(iv)^{-N}\sum_{r,s}g^{r,s}(x)\partial_{x_r}\phi(t, x)\iint M^{*N}(i\nu(\xi-p)_s\tilde{a}_{jk}(t, \xi, \eta, z))\cdot e^{i\nu x\cdot(\xi-p)}f(z)dzd\eta d\xi\,. \tag{40}$$

Consequently, we can manage with $F^{(2)}_{jkl}(t)$ just as $F^{(1)}_{jkl}(t)$ and $F^{(3)}_{jkl}(t)$.

In the following we shall construct an iterated parametrices. Let $A(s)$ and $B(s)$ be continuous functions with values in the space of bounded linear mappings in $L^2(S^n)$. Then we shall denote

(41) $$A \# B(t)=\int_0^t A(t-s)B(s)ds\,.$$

We set

(42) $$F_1(t)=F(t)$$

and

(43) $$F_l(t)=F \# F_{l-1}(t)\,, \qquad l \geqq 2\,.$$

The iterated parametrices are defined by

(44) $$E_N(t)=E(t)+E \# G_N(t)\,, \qquad {}^{\forall}N \geqq 1\,,$$

where $G_N(t)=\sum_{j=1}^{N}(-i\nu)^{-j}F_j(t)$.

LEMMA 3.8. *We have*

(45) $$\left(i\nu\partial_t+\frac{1}{2}\Delta\right)E_N(t)=(-i\nu)^{-N}F_{N+1}(t)\,.$$

The proof is omitted here.

LEMMA 3.9. *For any $T>0$, there exists a positive constant C, such that*

(47) $$\|F_j(t)\| \leqq C\frac{\nu^{-n/2}}{\Gamma(j)}(C\nu^{-n/2}|t|)^{j-1}$$

for $t\in[-T, T]$ and $j=1, 2, 3, \cdots$.

PROOF. The estimate for $j=1$ is obvious. Assume that (47) holds for $j \leqq r$. Then

$$\begin{aligned}\|F_{r+1}(t)\| &\leqq \int_0^{|t|}\|F(t-s)\|\,\|F_r(s)\|d|s| \\ &\leqq C\frac{\nu^{-n/2}}{\Gamma(r)}(C\nu^{-n/2})^r\int_0^{|t|}s^{r-1}ds \\ &\leqq C\frac{\nu^{-n/2}}{\Gamma(r+1)}(C\nu^{-n/2}|t|)^r\,.\end{aligned}$$

The Lemma is thus proved by induction.

We shall denote by $U(t)$ the fundamental solution of (1) in § 2, this is, for any $f\in C^\infty(S^n)$, $u(t, x)=(U(t)f)(x)$ satisfies

(48) $$\begin{cases}\left(i\nu\partial_t+\frac{1}{2}\Delta\right)u(t, x)=0 \\ u(0, x)=f(x)\,.\end{cases}$$

Now we shall majorize the difference

$$R_N(t)=U(t)-E_N(t)\,. \tag{49}$$

LEMMA 3.9. *For any $T>0$, there exists a positive constant C such that*

$$\|R_N(t)\| \leqq 2C|\nu|^{-N}\frac{(\nu^{-n/2}|t|)^{N+1}}{(N+1)!}$$

for any $t\in[-T, T]$.

PROOF. For any $f\in C^\infty(S^n)$, $R_N(t)$ satisfies

$$\begin{cases} \left(i\nu\partial_t+\frac{1}{2}\Delta\right)R_N(t)f=-(-i\nu)^{-N}F_{N+1}(t)f \\ R_N(0)f=0\,. \end{cases} \tag{50}$$

This yields

$$\begin{aligned}\frac{d}{dt}\|R_N(t)f\|^2 &= 2\,\mathrm{Re}\left(\frac{d}{dt}R_N(t)f(t),\ R_Nf\right)\\ &= -2\,\mathrm{Re}\,((i\nu)^{-1}2^{-1}\Delta R_N(t)f,\ R_N(t)f)\\ &\quad -2\,\mathrm{Re}\,(-i\nu)^{-N}(F_{N+1}(t)f,\ R_N(t)f)\,.\end{aligned}$$

We obtain from this

$$\frac{d}{dt}\|R_N(t)f\|^2 \leqq 2\nu^{-N}\|F_{N+1}(t)f\|\,\|R_N(t)f\|\,.$$

Therefore

$$\frac{d}{dt}\|R_N(t)f\| \leqq 2\nu^{-N}\frac{C\nu^{-n/2}}{N!}(C\nu^{-n/2}|t|)^N\,.$$

Consequently

$$\|R_N(t)f\| \leqq 2\nu^{-N}\frac{C}{(N+1)!}(\nu^{-n/2}|t|)^{N+1}\,.$$

We can now prove

THEOREM 3. *As N goes to ∞, the iterated parametrix $E_N(t)$ defined by (44) tends to the fundamental solution $U(t)$. More precisely,*

$$\lim_{N\to\infty}\|U(t)-E_N(t)\|=0 \tag{51}$$

uniformly for t in every compact set in R.

PROOF. Obvious from Lemma 3.9.

Now we shall discuss the second method of constructing the fundamental solution $U(t)$ of the Schrödinger's equation (1) of § 2. The idea for this is due to Feynman [5].

Let $\Delta: s=t_0<t_1<\cdots<t_k=s'$ be a division of the time interval $[s_1, s']$. We shall denote

$$\delta_j = t_j - t_{j-1}, \qquad j = 1, 2, \cdots, k,$$
$$\delta = \max_j \delta_j .$$

Note that

(52)
$$\sum_{j=1}^{k} \delta_j = s' - s .$$

Our final aim is to prove

THEOREM 4. *If* $N \geqq 1$, *then*

(53)
$$\lim_{\delta \to 0} \| E_N(s' - t_{k-1}) E_N(t_{k-1} - t_{k-2}) \cdots E(t_1 - s) - U(s' - s) \| = 0 .$$

Convergence is uniform for s *and* s' *in every compact set of* R^1 *and for* $\nu \geqq 1$.

PROOF. We may assume that $s=0$ and $s'>0$. We define

(54)
$$U_N^{\Delta}(t) = E_N(t - t_j) E_N(t_j - t_{j-1}) \cdots E_N(t_1)$$

for $t \in [t_j, t_{j+1}]$.

This is a strongly continuous function with values in the space of bounded linear operators in $L^2(S^n)$. This function satisfies

(55)
$$\Big(i\nu\partial_t + \frac{1}{2}\Delta\Big) U_N^{\Delta}(t) = (-i\nu)^{-N} F_{N+1}(t - t_j) U_N^{\Delta}(t_j), \qquad t \in [t_j, t_{j+1}]$$
$$U_N^{\Delta}(0) = \text{Identity} .$$

Note that

(56)
$$\sup_{0 \leqq t \leqq s'} \| U_N^{\Delta}(t) \| = M < \infty .$$

In fact, we have

$$\begin{aligned}
&\| E_N(t_j - t_{j-1}) \cdots E_N(t_1) \| \\
&\quad = \| (U(t_j - t_{j-1}) + R(t_j - t_{j-1})) \cdots (U(t_1) + R_N(t_1)) \| \\
&\quad \leqq \| U(t_j) \| + \sum_{\substack{1 \leqq p \leqq k \\ j'_l \leqq j_{l+1}}} \| R_N(t_j - t_{j'}) \| \cdots \| R(t_{j_p} - t_{j_{p'}}) \| \\
&\quad \leqq 1 + \sum (C | t_{j_1} - t_{j_{1'}} |) \cdots (C | t_{j_p} - t_{j_{p'}} |) \\
&\quad \leqq (1 + C | t_1 - t_2 |) \cdots (1 + C | t_{k-1} - t_k |) \\
&\quad \leqq e^{C \Sigma \delta j} \\
&\quad \leqq e^{C | s' - s |} .
\end{aligned}$$

Hence we have (56) with $M = e^{C|s'-s|}$. We set $V_N^{\Delta}(t) = U(t) - U_N^{\Delta}(t)$. Then

(57)
$$\begin{cases} V_N^{\Delta}(0) = 0 \\ \Big(i\nu\partial_t + \dfrac{1}{2}\Delta\Big) V_N^{\Delta}(t) = -(-i\nu)^{-N} F_{N+1}(t - t_j) U_N^{\Delta}(t_j) . \end{cases}$$

We can prove, just as in the proof of Lemma 3.9, that

$$\frac{d}{dt}\|V_N^A(t)f\| \leqq \nu^{-N-1}\|F_{N+1}(t-t_j)\|\|U_N^A(t_j)\|\|f\|.$$

Hence

$$\|V_N^A(t)f\| \leqq M\nu^{-N-1}\delta^N.$$

Consequently, $\lim_{\delta\to 0}\|V_N^A(t)\|=0$. This proves the Theorem.

Appendix I. A remark on the Maslov index.

Let M be a Riemannian manifold of dimension n. The cotangent bundle T^*M has a natural connection, i.e., at any point α of T^*M, the tangent space $T_\alpha T^*M$ splits into the sum of the horizontal space λ_h and the vertical space λ_v. (See [11].) On the other hand T^*M has the natural canonical structure σ.

LEMMA A. *Both λ_h and λ_v are Lagrangean planes in $T_\alpha T^*M$ for any $\alpha\in T^*M$.*

PROOF. What we have to prove is that σ vanishes on $\lambda_h\times\lambda_h$ and $\lambda_v\times\lambda_v$. Let $(x^1\cdots x^n)$ be local coordinates around $\pi(\alpha)$, where $\pi: T^*M\to M$ is the projection. Any local cotangent vector field can be written as

$$\xi=\sum_{j=1}^n \xi_j dx^j$$

around $\pi(\alpha)$. We adopt $(x^1\cdots x^n, \xi_1\cdots\xi_n)$ as local coordinates of T^*M near α. The coordinate expression of σ is

$$\sigma=\sum_{j=1}^n d\xi_j\wedge dx^j.$$

Since λ_v is the tangent space of the fibre, σ vanishes on $\lambda_v\times\lambda_v$, this is, λ_v is Lagrangean.

Note that we can form a local cross-section N of T^*M which consists of parallel displacements of $\alpha\in T^*M$ along the geodesics emanating from $\pi(\alpha)$. The horizontal space λ_h is the tangent space of N at α. We shall make use of the geodesic coordinates $(x^1\cdots x^n)$ of M centered at $\pi(\alpha)$. Assume that α is expressed as $(0,\cdots,0,\xi,\cdots,\xi)$. Then the parallel displacement $\beta=(x^1\cdots x^n, \eta_1,\cdots,\eta_n)$ is

$$x^j=sa^j+O(s^2)$$

$$\eta_j=O(s^2)$$

where $s=\text{dist}(\pi(\alpha), z)$. Hence $\lambda_h=\left\{\sum_{j=1}^n a^j\left(\frac{\partial}{\partial x^j}\right); a=(a^1\cdots a^n)\in R^n\right\}$. Therefore $\sigma=\sum d\eta_j\wedge dx^j$ vanishes on $\lambda_h\times\lambda_h$.

Since λ_h and λ_v are transversal, they are dual by the bilinear mapping σ. Hence we can define a linear isomorphism $J: \lambda_v \to \lambda_h$ such that

$$g(x, y) = \sigma(x, Jy) \qquad \forall x, y \in \lambda_v$$

where g is the metric on $T_\alpha T^*M$. We can extend J to the whole of $T_\alpha T^*M$ by the rule that $J^2 = -$identity. By this complex structure, $T_\alpha T^*M$ becomes a hermitian vector space with a hermitian inner product

$$h(x, y) = g(x, y) + i\sigma(x, y)$$

for $\forall x, y \in T_\alpha T^*M$.

LEMMA B. *There exists a symplectic mapping*

$$\Phi_\alpha : T_\alpha T^*M \longrightarrow \boldsymbol{C}^n$$

which maps λ_h and λ_v onto real and pure imaginary subspaces respectively and preserves the hermitian inner product.

PROOF. Obvious.

Mappings Φ_α are not defined globally when α runs the whole of T^*M.

LEMMA C. *Let Φ_α and Φ'_α be two mappings stated in Lemma B. Then $\Phi'_\alpha \Phi_\alpha^{-1} \in O(n, R)$.*

PROOF. Obvious.

We shall denote by $\Lambda(T_\alpha T^*M)$ the space of all Lagrangean subspaces of $T_\alpha T^*M$. Then we have

THEOREM A-I. *We can define a map*

$$\psi_\alpha : \Lambda(T_\alpha T^*M) \longrightarrow \Lambda(n),$$

where $\Lambda(n)$ is the space of all Lagrangean subspaces of $\boldsymbol{C}^n$. The mapping is uniquely defined for $\alpha \in T^*M$ and depends continuously on α and the Riemannian structure of M.

PROOF. This is an immediate consequence of Lemma C and Arnold [**2**].

Let X be a Lagrangean submanifold of T^*M. Then by the mapping above, we can define a mapping $\rho: X \ni x \to \rho(x) \in \Lambda(n)$.

DEFINITION D. Let X be a Lagrangean submanifold of T^*M and ρ be the above mapping. Let γ be a curve in X. Then $\rho(\gamma)$ is a curve in $\Lambda(n)$. We define as the Maslov index of γ=Maslov index of $\rho(\gamma)$.

If the curve γ joins two non-focal points $\gamma(0)$ and $\gamma(1)$, then, using the terminology of Leray [**14**],

$$\operatorname{Ind} \gamma = \operatorname{Inert}(\lambda_v{}^0, \lambda_h{}^0, \lambda_0) - \operatorname{Inert}(\lambda_v{}^1, \lambda_h{}^1, \lambda_1)$$

where $\lambda_j = T_{\gamma(j)}X$ and $\lambda_v{}^j$, $\lambda_h{}^j$ $(j=0, 1)$ are vertical and horizontal spaces at point $\gamma(j)$.

REMARK. Maslov's index of a curve does not depend on the particular metric that is used to define it as above because the totality of Riemannian metrics on M forms a convex set and Maslov's index is a homotopy invariant. See Leray [13] for the details in this context.

Appendix II.

Our assumptions are
(A-0) Let $S_j(x, \xi)$, $j=1$ and 2, are real C^∞ functions of $(x, \xi)\in R^n\times R^n$.
(A-I) There exist positive constants C and σ such that we have

$$\Phi\equiv|\mathrm{grad}_x(S_j(x, \xi)-S_j(x, \eta))|\geqq C\theta(|\xi-\eta|)\,. \tag{1}$$

$$\Psi\equiv|\mathrm{grad}_\xi(S_j(x, \xi)-S_j(y, \eta))|\geqq C\theta(|x-y|)\,, \tag{2}$$

$$\text{where}\quad \theta(t)=\frac{t}{(1+t)^{1+\sigma}}\qquad \text{for}\quad t\geqq 0\,.$$

(A-II) For any multi-index α with $|\alpha|\geqq 1$, there exists a constant $C>0$ such that

$$|\alpha_x{}^\alpha(S_j(x, \xi)-S_j(x, \eta))|\leqq C\Phi \tag{3}$$

and

$$|\partial_\xi{}^\alpha(S_j(x, \xi)-S_j(y, \eta))|\leqq C\Psi\,. \tag{4}$$

(A-III) $a(\xi, x, \eta)\in C^\infty(R^n\times R^n\times R^n)$ and for any multi-indices α, β, γ, there exists a positive constant C such that

$$\int_{R^n}|\partial_\xi{}^\alpha\partial_x{}^\beta\partial_\eta{}^\gamma a(\xi, x, \eta)|\,d\xi< C\,. \tag{5}$$

Define a linear operator A by

$$Af(\xi)=(2\pi)^{-n}\iint_{R^n\times R^n}a(\xi, x, \eta)e^{i\nu(S_1(x,\xi)-S_2(x,\eta))}f(\eta)\,d\eta\, dx \tag{6}$$

which is well defined at least for $f\in C_0^\infty(R^n)$.

THEOREM A-II. *Assume that* (A-0), (A-I), (A-II) *and* (A-III) *hold. Then there exists a positive constant C such that we obtain*

$$\|Af\|\leqq C\|f\| \tag{7}$$

for any $f\in C_0^\infty(R^n)$. Here

$$C\leqq \max_{|\alpha|+|\beta|\leqq 2n}\ \sup_{\xi,\eta}\int(1-\Delta_\xi)^{3n}\partial_x{}^\alpha\partial_\eta{}^\beta a(\xi, x, \eta)\,d\xi\,.$$

PROOF. By Fourier's integral formula

$$a(\xi, x, \eta)=(2\pi)^{-n}\int_{R^n}\hat{a}(y, x, \eta)e^{i\xi\cdot y}dy\,, \tag{8}$$

where $\hat{a}(y, x, \eta)=\int a(\xi, x, \eta)e^{-i\xi\cdot y}d\xi$ satisfies estimate

$$|y^\alpha\partial_x{}^\beta\partial_\eta{}^\gamma\hat{a}(y, x, \eta)|\leqq C\,. \tag{9}$$

by virtue of (A-III).

For any f, $g\in C_0^\infty(R^n)$, (Af, g) turns out to be

$$\begin{aligned}(Af, g)&=(2\pi)^{-n}\int_{R^n}\overline{g(\xi)}d\xi\iint_{R^n\times R^n}a(\xi, x, \eta)e^{i\nu(S_1(x,\xi)-S_2(x,\eta))}f(\eta)d\eta dx\\ &=(2\pi)^{-2n}\int dy\int\hat{a}(y, x, \eta)e^{i\xi\cdot y}e^{i\nu(S_1(x,\xi)-S_2(x,\eta))}\overline{g(\xi)}f(\eta)d\eta dxd\xi \qquad (10)\\ &=(2\pi)^{-2n}\int_{R^n}dy\int_{R^n}\overline{T_yg(x)}A_yf(x)dx\,,\end{aligned}$$

where

$$A_yf(x)=\int_{R^n}\hat{a}(y, x, \eta)e^{-i\nu S_2(x,\eta)}f(\eta)d\eta \tag{11}$$

and

$$T_yg(x)=\int_{R^n}e^{i\nu S_1(x,\xi)}e^{i\xi\cdot y}g(\xi)d\xi\,. \tag{12}$$

By virtue of our assumptions, we can apply our previous result [7] and obtain estimates

$$\|A_yf\|\leqq C_y\|f\| \tag{13}$$

and

$$\|T_yg\|\leqq C\|g\| \tag{14}$$

where C is a positive constant independent of y and

$$C_y=\operatorname*{Max}_{\substack{|\beta|\leqq 3n\\ |\gamma|\leqq 3n}}\ \sup_{x,\eta}|\partial_x{}^\beta\partial_\eta{}^\gamma\hat{a}(y, x, \eta)|\,.$$

(9) means that

$$C_y\leqq C(1+|y|)^{-2n}\,. \tag{15}$$

If we apply the Schwarz inequality to (10) we have

$$|(Af, g)|\leqq(2\pi)^{-2n}\int_{R^n}\|T_yg\|\|A_yf\|dy\,.$$

Making use of (13), (14) and (15), we obtain

$$
\begin{aligned}
|(Af, g)| &\leqq C\int_{R^n} C_y \|f\| \|g\| dy \\
&\leqq C\|f\|\|g\|\int_{R^n}(1+|y|)^{-2n}dy \\
&\leqq C\|f\|\|g\| .
\end{aligned}
$$

Thus Theorem A-II has been proved.

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Daisuke FUJIWARA
Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Bunkyo-ku
Tokyo, Japan