J. Math. Soc. Japan
Vol. 28, No. 3, 1976

On the values of ray-class L-functions for real quadratic fields

To the memory of Taira Honda

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(Received June 20, 1974)
(Revised Feb. 14, 1975)

Let K be a real quadratic field and $\mathfrak{f}$ an integral ideal in K. Let χ be a character of the ray-class group mod $\mathfrak{f}$. We consider only the cases

(i) $$\chi((\alpha))=\chi(\alpha)\,, \qquad \alpha\in K$$

and

(ii) $$\chi((\alpha))=(N(\alpha)/|N(\alpha)|)\chi(\alpha)\,, \qquad \alpha\in K\,,$$

where χ in the right means, respectively, the character of the residue class group mod $\mathfrak{f}$, attached to the χ in the left.

In the present paper, we shall give the explicit formulas for $L(2k, \chi)$ in the case (i) and $L(2k+1, \chi)$ in the case (ii), k being a positive integer. The formulas for them are already given by Siegel [7] in other shapes (and by the different method from us). Our formulas, different from Siegel's, express explicitly the role of the totally positive units congruent to 1 mod $\mathfrak{f}$.

In his paper [2], Barner gave the explicit formulas for values of the ring-class L-functions of certain types at integral arguments. His tools are the representation of L-functions by the integrals of Eisenstein series, given by Siegel [6], and the transformation formulas of certain Lambert series under modular substitutions given in Apostol [1], S. Iseki [3]. The main point of his computation is the use of certain differential operators which connect Eisenstein series with Lambert series. Here we follow, with some necessary changes and supplies, the method of Barner. In the special case $\mathfrak{f}=(1)$, our formula coincides with Barner's.

In the course of the computation, fundamental is the representation of L-functions by the integral of Eisenstein series. This is clarified by Siegel [6] and will be formulated in somewhat general point of view in our Appendix.

NOTATION. As usual, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ are fields of rational, real and complex numbers, respectively. $\boldsymbol{Z}$ is the ring of rational integers.

We denote by $\boldsymbol{N}$ the set of natural numbers. $M_n(R)$ is the total matrix ring of order n with coefficients in a ring R. $\boldsymbol{H}$ means the upper-half plane. For a matrix $Y=(y_{ij})$ in $M_n(\boldsymbol{R})$, with $y_{ij}>0$, we put

$$\log Y = (\log y_{ij})$$

and for a column vector $\xi = {}^t(\xi_1, \xi_2, \cdots, \xi_n)$, we define the vector power of a matrix Y by

$$Y^{\xi} = \left(\prod_{j=1}^{n} y_{1j}^{\xi_j}, \cdots, \prod_{j=1}^{n} y_{nj}^{\xi_j}\right),$$

which is again a column vector.

$\sum_m{}'$ and $\sum_{m,n}{}'$ mean the sums over all non-zero m and all $(m, n) \neq (0, 0)$, respectively.

We denote by $\{x\}$ the fractional part of a real number x.

§ 1. Transformation formula.

Let u, v be real numbers such that $0 \leqq u, v < 1$. For a positive integer ν and for $z \in \boldsymbol{H}$, we put

$$F_{\nu}(u, v\,;\, \omega_1, \omega_2) = \omega_2^{-1} F(\nu\,;\, u, v\,;\, z), \tag{1.1}$$

$$F(\nu\,;\, u, v\,;\, z) = \sum_{n=-\infty}^{\infty}{}' \frac{e^{-2\pi i n(uz-v)}}{n^{\nu}(e^{-2\pi i n z}-1)}$$

with $z = \omega_1/\omega_2$.

For a real u, we define

$$P_s(u) = -s\,! \sum_{m=-\infty}^{\infty}{}' \frac{e^{2\pi i m u}}{(2\pi i m)^s}, \qquad s = 1, 2, 3, \cdots, (u, s) \neq (0, 1), \tag{1.2}$$

$$P_0(u) = 1.$$

Here for $s=1$, we understand that the sum means

$$\sum_{m=1}^{\infty} \frac{1}{2\pi i m}(e^{2\pi i m u} - e^{-2\pi i m u}).$$

For $0 < u < 1$, $P_s(u)$ gives the Fourier expansion of Bernoulli polynomial $B_s(u)$ of degree s. Thus $P_s(u)$ becomes a polynomial for $0 < u < 1$.

We define

$$S_{\nu+1}^{(r)}(\sigma, u, v) = \sum_{h=0}^{|c|-1} P_r(du/c + v + h/c) P_{\nu+1+r}(u/c + ah/c) \tag{1.3}$$

for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})$, $c \neq 0$. This is to be called a generalized higher Dedekind sum.

For rational numbers u, v, we define

$$\Gamma(u, v) = \left\{ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})\,;\, du + cv \equiv u \text{ and } bu + av \equiv v \pmod 1 \right\}. \tag{1.4}$$

Further we put

$$\delta(u)\begin{cases} =0 & u\notin \boldsymbol{Z} \\ =-\dfrac{1}{2} & u\in \boldsymbol{Z}. \end{cases}$$

Then we have

LEMMA 1. *Let u, v be rational numbers with $0\leqq u,\ v<1$. For $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ $\in\Gamma(u, v)$, put $\hat{\omega}_1=a\omega_1+b\omega_2$, $\hat{\omega}_2=c\omega_1+d\omega_2$. Assume that $(\nu, u, v)\neq(1, 0, 0)$. Then for $c\neq 0$,*

$$F_\nu(u, v\,;\hat{\omega}_1, \hat{\omega}_2)=(\operatorname{sgn} c)^{\nu-1}F_\nu(u, v\,;\omega_1, \omega_2)$$
$$-(2\pi i)^\nu \omega_2^{\nu-1}(\operatorname{sgn} c)\sum_{r=0}^{\nu+1}\frac{(-1)^{\nu+1-r}(cz+d)^{\nu-r}}{r\,!(\nu+1-r)\,!}S_{\nu+1}^{(r)}(\sigma, u, v)$$
$$-(2\pi i)^\nu \omega_2^{\nu-1}\frac{P_\nu(v)}{\nu\,!}\delta(u)((cz+d)^{\nu-1}-(\operatorname{sgn} c)^{\nu-1}).$$

To obtain the lemma, first we prove the inversion formula for $F(\nu\,; u, v)$:

$$\text{(1.5)}\qquad F(\nu\,; u, v, z)+(-1)^\nu z^{\nu-1}F(\nu\,; 1-v, u\,; -z^{-1})$$
$$=(-1)^{\nu+1}(2\pi i)^\nu z^{\nu-1}\frac{P_\nu(u)}{\nu\,!}\delta(v)-(2\pi i)^\nu\frac{P_\nu(v)}{\nu\,!}\delta(u)$$
$$-(2\pi i)^\nu\sum_{r=0}^{\nu+1}\frac{(-1)^{r-1}z^{r-1}}{r\,!(\nu+1-r)\,!}P_r(u)P_{\nu+1-r}(v),$$

where ν is a positive integer, u, v are real numbers such that $0\leqq u,\ v<1$ and we assume $(\nu, u, v)\neq(1, 0, 0)$.

We need the formula

$$\text{(1.6)}\qquad \frac{n^\nu-(-mxi)^\nu}{n-(-mxi)}=n^{\nu-1}+n^{\nu-2}(-mxi)+n^{\nu-3}(-mxi)^2+\cdots+(-mxi)^{\nu-1}.$$

The use of this formula in proving (1.5) was suggested by Siegel, to whom the author expresses his hearty thanks. Multiplying the both sides of (1.6) by

$$(mn)^{-\nu}e^{2\pi i(mv+nu)},$$

taking the sum over all non-zero integers m, n and changing the order of the sum with respect to m, n (this is permitted by the same reason as in [5]), we get

$$\text{(1.7)}\qquad \sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imv}}{m^\nu}\sum_{n=1}^{\infty}\Big(\frac{e^{2\pi inu}}{n+mxi}+\frac{e^{-2\pi inu}}{-n+mxi}\Big)$$
$$-(-xi)^\nu\sum_{n=-\infty}^{\infty}{}'\frac{e^{2\pi inu}}{n^\nu}\sum_{m=1}^{\infty}\Big(\frac{e^{2\pi imv}}{n+mxi}+\frac{e^{-2\pi imv}}{n-mxi}\Big)$$
$$=\sum_{r=1}^{\nu}\sum_{m,n=-\infty}^{\infty}{}'\frac{e^{2\pi inu}e^{2\pi imv}}{n^r m^{\nu+1-r}}(-xi)^{r-1}.$$

The right hand side of this is equal to

$$\sum_{r=1}^{\nu}(-xi)^{r-1}(2\pi i)^{\nu+1}\frac{P_r(u)}{r!}\frac{P_{\nu+1-r}(v)}{(\nu+1-r)!}$$

by (1.2). Using Kronecker's formula

$$\frac{e^{2\pi ux}}{e^{2\pi x}-1}=\delta(u)+\frac{1}{2\pi x}-\frac{1}{2\pi i}\sum_{n=1}^{\infty}\left\{\frac{e^{2\pi inu}}{xi+n}+\frac{e^{-2\pi inu}}{xi-n}\right\}$$

valid for $0\leqq u<1$, we have

$$\sum_{n=1}^{\infty}\left(\frac{e^{2\pi inu}}{n+mxi}+\frac{e^{-2\pi inu}}{-n+mxi}\right)=2\pi i\delta(u)+\frac{i}{mx}-\frac{2\pi i\, e^{2\pi mux}}{e^{2\pi mx}-1} \tag{1.8}$$

and

$$\sum_{m=1}^{\infty}\left(\frac{e^{2\pi imv}}{n+mxi}+\frac{e^{-2\pi imv}}{n-mxi}\right)=\frac{-2\pi\delta(v)}{x}-\frac{1}{n}+\frac{2\pi}{x}\frac{e^{2\pi n(1-v)/x}}{e^{2\pi n/x}-1}. \tag{1.9}$$

These formulas are valid for $z\in \boldsymbol{H}$ instead of xi, $x>0$, by the principle of analytic continuation. Then by (1.7, 8, 9), we get (1.5).

Now let u, v be rational numbers, $0\leqq u$, $v<1$ and $(\nu, u, v)\neq(1, 0, 0)$. For $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma(u, v)$, put $u^*=du+cv$, $v^*=bu+av$. Then by definition, $u^*\equiv u$, $v^*\equiv v \pmod 1$. For $c=0$, we easily have

$$F(\nu\,;u^*, v^*\,;\sigma(z))=(\operatorname{sgn} d)^{\nu+1}F(\nu\,;u, v\,;z)\,.$$

Assume $c>0$. Then decomposing the transformation $z\to\sigma(z)$ into

$$\sigma(z)=\frac{az+b}{cz+d}=\frac{a}{c}+\frac{z_1}{c},\quad -z_1^{-1}=z_2,\quad z_2=cz+d\,,$$

and following the same procedure as in [4], pp. 265-268, on the basis of the inversion formula (1.5), we have

$$\begin{aligned} &F(\nu\,;u^*, v^*\,;\sigma(z))\\ &=(-1)^{\nu-1}z_1^{\nu-1}\sum_{h=0}^{c-1}F\left(\nu\,;1-\left\{v^*-\frac{a}{c}(h+u^*)\right\}, \left\{\frac{h+u^*}{c}\right\};z_2\right)\\ &\quad-(2\pi i)^{\nu}\sum_{r=0}^{\nu+1}\frac{(-1)^{r+1}z_1^{r-1}}{r!\,(\nu+1-r)!}\sum_{h=0}^{c-1}P_r\left(\frac{u^*+h}{c}\right)P_{\nu+1-r}\left(v^*-\frac{a}{c}(h+u^*)\right), \end{aligned}$$

and then the transformation formula for $F(\nu\,;u, v\,;z)$ when $c>0$. When $c<0$, we put

$$a'=-a,\quad b'=-b,\quad c'=-c,\quad d'=-d$$

and follow the above computation with a', b', c', d'. Then we get the formula for $c<0$. Finally, going to $F_\nu(u, v\,;\omega_1, \omega_2)$, we get our Lemma 1.

§ 2. Characters.

Let K be an algebraic number field of degree n, $\mathfrak{o}$ the ring of integers in K and $\mathfrak{f}$ a fixed integral ideal in K.

Let $\mathfrak{G}_{\mathfrak{f}}$ be the group composed of fractional ideals whose denominator and numerator are both coprime to $\mathfrak{f}$. Define

$$\mathfrak{E}_{\mathfrak{f}}=\{(\alpha)\in\mathfrak{G}_{\mathfrak{f}}\mid \alpha\gg 0,\ \alpha\equiv 1\ (\mathrm{mod}^*\,\mathfrak{f})\}\,,$$

where $\alpha\gg 0$ means that all the real conjugates of α are positive and mod* means the usual multiplicative congruence. Then $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$ is called the ray-class group modulo $\mathfrak{f}$. Especially, if K is real quadratic and $\mathfrak{f}=(1)$, ray-class group modulo $\mathfrak{f}$ is nothing but the class in the narrow sense.

Let χ be a ray-class character modulo $\mathfrak{f}$, i.e., a character of the group $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$. A ray-class L-function with χ is defined by

$$L(s,\chi)=\sum_{(\mathfrak{a},\mathfrak{f})=1}\chi(\mathfrak{a})N(\mathfrak{a})^{-s}$$

where the sum is extended over all non-zero integral ideals $\mathfrak{a}$ coprime to $\mathfrak{f}$. The series is convergent absolutely for $\mathrm{Re}(s)>1$ and uniformly for $\mathrm{Re}(s)\geqq 1+\varepsilon$ with any $\varepsilon>0$. For $\chi=1$, $\mathfrak{f}=(1)$, $L(s,\chi)$ is nothing but the Dedekind zeta-function $\zeta_K(s)$ of K.

We denote by $G(\mathfrak{f})$ the group of prime residue classes modulo $\mathfrak{f}$. Let $w(\alpha)$ be a character of signature of $\alpha\in K$. Now let K be a real quadratic field. Then $w(\alpha)$ takes one of the following forms:

(i) $w(\alpha)=1$,

(ii) $w(\alpha)=\mathrm{sgn}(N(\alpha))$,

(iii) $w(\alpha)=\alpha/|\alpha|$,

(iv) $w(\alpha)=\alpha'/|\alpha'|$,

where α' is the conjugate of α over $\boldsymbol{Q}$.

Let χ be a ray-class character modulo $\mathfrak{f}$. Then there exists a character w such that the value $\chi((\alpha))/w(\alpha)$ depends only on the residue class of α. Denote it also by $\chi(\alpha)$, which is a character of $G(\mathfrak{f})$. We extend the definition of the new χ by putting $\chi(\alpha)=0$ for $(\alpha,\mathfrak{f})\neq 1$.

§ 3. Parameter λ.

In what follows, K is a real quadratic field $\boldsymbol{Q}(\sqrt{D})$. Take a $B\in\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$ and a $\mathfrak{b}_B\in B$. Let $[\alpha_1,\alpha_2]$ be an integral basis of $\mathfrak{b}_B$.

Γ being the group of all units in K, we put

$$\Gamma_{\mathfrak{f}}=\{\varepsilon\in\Gamma \mid \varepsilon\equiv 1 \pmod{\mathfrak{f}}\}$$

and

$$\Gamma_{\mathfrak{f}}^{*}=\{\varepsilon\in\Gamma_{\mathfrak{f}} \mid \varepsilon\gg 0\}\ . \tag{3.1}$$

$\Gamma_{\mathfrak{f}}^{*}$ is infinite cyclic. Let $\varepsilon_{\mathfrak{f}}$ be a generator of $\Gamma_{\mathfrak{f}}^{*}$. Then we can take as $\varepsilon_{\mathfrak{f}}>1$. Thus

$$\varepsilon_{\mathfrak{f}}\equiv 1 \pmod{\mathfrak{f}}\ ,\qquad N(\varepsilon_{\mathfrak{f}})=1\quad\text{and}\quad \varepsilon_{\mathfrak{f}}>\varepsilon_{\mathfrak{f}}'>0\ . \tag{3.2}$$

We can write

$$\varepsilon_{\mathfrak{f}}\alpha_2=d\alpha_2+c\alpha_1\ ,\qquad \varepsilon_{\mathfrak{f}}\alpha_1=b\alpha_2+a\alpha_1\ , \tag{3.3}$$

with $a, b, c, d\in\boldsymbol{Z}$, $c>0$ and $ad-bc=N(\varepsilon_{\mathfrak{f}})=1$. Put $\rho=\alpha_1/\alpha_2$. Then

$$\rho=\frac{a\rho+b}{c\rho+d}\ ,\qquad \rho'=\frac{a\rho'+b}{c\rho'+d}\ .$$

The transformation

$$z\longrightarrow \hat{z}=\frac{az+b}{cz+d}$$

is hyperbolic and has ρ, ρ' as its fixed points. We may take $\rho>\rho'$. Denote by C the semi-circle in $\boldsymbol{H}$ with ρ', ρ as its end points of diameter. The transformation $z\rightarrow\hat{z}$ fixes C. For $z\in C$, we put

$$z=\frac{\rho pi+\rho'}{pi+1}\quad\text{or}\quad pi=\frac{z-\rho'}{\rho-z}\ .$$

Then p is real positive and runs from 0 to ∞ as z does from ρ' to ρ on C. After K. Barner [2], we adopt

$$\lambda=\log p \tag{3.4}$$

as a parameter. λ runs from $-\infty$ to ∞. The transformation $z\rightarrow\hat{z}$ corresponds to

$$\lambda\longrightarrow\hat{\lambda}=\lambda+2\log\varepsilon_{\mathfrak{f}}\ . \tag{3.5}$$

We put

$$\omega_1=\omega_1(\lambda)=\rho e^{(1/2)\lambda}-i\rho' e^{-(1/2)\lambda},\quad \omega_2=\omega_2(\lambda)=e^{(1/2)\lambda}-ie^{-(1/2)\lambda} \tag{3.6}$$

and $z=\omega_1/\omega_2$. Then z belongs to C. Conversely any $z\in C$ can be written in the form $z=\omega_1/\omega_2$ with ω_1, ω_2 in (3.6).

§ 4. Siegel's results.

We shall quote some of Siegel's results from [6]. For $0 \leqq u,\ v<1$, we consider the following Eisenstein series:

$$g(z, s, u, v) = y^s \sum_{m,n\in \boldsymbol{Z}}{}' \frac{e^{2\pi i(mu+nv)}}{|m+nz|^{2s}} \tag{4.1}$$

where $y=\mathrm{Im}(z)$. The right hand converges absolutely for $\mathrm{Re}\, s=\sigma>1$ and uniformly for $\sigma\geqq 1+\varepsilon$, $\varepsilon>0$. g is extended analytically over the whole s-plane, with an exception $s=1$ for $u=v=0$, through the functional equation satisfied by g.

Let $K=\boldsymbol{Q}(\sqrt{D})$, $D>0$. Take and fix once for all a number γ of K such that the denominator of $(\gamma\sqrt{D})$ is exactly $\mathfrak{f}$. We denote by Tr the trace from K to $\boldsymbol{Q}$. Then $\mathrm{Tr}(\alpha\gamma)\in \boldsymbol{Z}$ for $\alpha\in\mathfrak{f}$.

Define for a character χ of $G(\mathfrak{f})$,

$$T_\chi = \sum_{\alpha \bmod \mathfrak{f}} \bar{\chi}(\alpha) e^{2\pi i\, \mathrm{Tr}(\alpha\gamma)},$$

where α runs over a complete set of representatives of residue classes modulo $\mathfrak{f}$. For $\mathfrak{f}=(1)$, we have $T_\chi=1$. T_χ does not depend on the choice of representatives α. Further, by the meaning of the character χ, we may consider α runs over only on $G(\mathfrak{f})$.

If χ is *primitive*, then

$$T_\chi \neq 0,$$

$$T_\chi \chi(\beta) = \sum_{\alpha \bmod \mathfrak{f}} \bar{\chi}(\alpha) e^{2\pi i \mathrm{Tr}(\beta\alpha\gamma)}$$

for $\beta\in\mathfrak{o}$, and

$$|T_\chi|^2 = N(\mathfrak{f}).$$

In what follows, we assume always χ is *primitive*. Let γ, $\mathfrak{b}_B$, $[\alpha_1, \alpha_2]$, ρ, ρ', $z=\omega_1(\lambda)/\omega_2(\lambda)$ be as above. We take rational numbers u, v as

$$u=u_B=\{\mathrm{Tr}(\alpha_1\gamma)\}, \qquad v=v_B=\{\mathrm{Tr}(\alpha_2\gamma)\}.$$

Let $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be the element of $\mathrm{SL}(2, \boldsymbol{Z})$ defined in (3.3). Then we have

$$du+cv = \mathrm{Tr}(d\alpha_1\gamma + c\alpha_2\gamma) = \mathrm{Tr}(\varepsilon_{\mathfrak{f}}\alpha_1\gamma) \equiv \mathrm{Tr}(\alpha_1\gamma) \equiv u \pmod 1$$

since $\varepsilon_{\mathfrak{f}}\equiv 1 \pmod{\mathfrak{f}}$. In the same way, we have

$$bu+av\equiv v \pmod 1.$$

Thus σ belongs to $\Gamma(u, v)$.

We consider two cases of $w(\alpha)$.

(i) The case $w(\alpha)=1$. Variable z being in C, we can view

$$g(z, s, u_B, v_B) \tag{4.2}$$

as a function of λ. This depends not only on B but also on the choice of $\mathfrak{b}_B$ and its integral basis, but for simplicity, (4.2) will be denoted by $g_B(z, s)$. Then Siegel showed

LEMMA 2 (Siegel [6]. Also see Appendix).

$$L(s, \chi)=\frac{D^{-(1/2)s}\Gamma(s)}{2T_\chi\Gamma^2((1/2)s)}\sum_B \bar{\chi}(\mathfrak{b}_B)\int_\lambda^{\hat{\lambda}} g_B(z, s)d\lambda . \tag{4.3}$$

Here B runs over $\mathfrak{G}_\mathfrak{f}/\mathfrak{E}_\mathfrak{f}$ and the integral on the right does not depend on the choice of $\mathfrak{b}_B$ and its integral basis.

(ii) The case $w(\alpha)=N(\alpha)/|N(\alpha)|$. In this case, $N(\varepsilon)$ must be 1 for a unit $\varepsilon\equiv 1 \pmod{\mathfrak{f}}$. We put

$$f_B(z, s)=-\frac{(z-\rho)(z-\rho')}{\rho-\rho'}\frac{i}{s}\frac{\partial g_B(z, s)}{\partial z} \tag{4.4}$$

(c. f., Barner [2]), which is considered as a function of λ for $z\in C$. Then Siegel showed

LEMMA 3.

$$L(s, \chi)=\frac{D^{-(1/2)s}\Gamma(s)}{2T_\chi\Gamma^2((1/2)(s+1))}\sum_B \bar{\chi}(\mathfrak{b}_B)w(\alpha_1)\int_\lambda^{\hat{\lambda}} f_B(z, s)d\lambda . \tag{4.5}$$

Here B runs over $\mathfrak{G}_\mathfrak{f}/\mathfrak{E}_\mathfrak{f}$ and the integral does not depend on the choice of $\mathfrak{b}_B$ and its integral basis.

§ 5. Barner's operators $\Lambda_k^{(e)}$, $\Lambda_k^{(o)}$.

We keep the notation D, $\mathfrak{b}_B$, $[\alpha_1, \alpha_2]$, λ, $\omega_1(\lambda)$, $\omega_2(\lambda)$ and put

$$J=J_B=N(\mathfrak{b}_B)\sqrt{D}=\text{abs.}\begin{vmatrix}\alpha_1 & \alpha_1' \\ \alpha_2 & \alpha_2'\end{vmatrix}.$$

Let $\kappa>0$, ν, $\mu>0$ be integers. In [2], K. Barner defined

$$a_{\mu,\nu}^{(\kappa)}\begin{cases} =\dfrac{\mu!(\kappa-\nu)!}{(\kappa-\mu)!(\mu-\nu)!\nu!}\dfrac{1}{2^{\mu-\nu}} & \text{for } 0\leqq\nu\leqq\mu , \\ =0 & \nu<0 \text{ or } \mu<\nu \end{cases} \tag{5.1}$$

and proved the following identities:

$$a_{\mu+1,\nu}^{(\kappa)}=\frac{\kappa-\mu-\nu}{2}a_{\mu,\nu}^{(\kappa)}+a_{\mu,\nu-1}^{(\kappa)} , \tag{5.2}$$

(5.3)
$$(\mu-\nu)a_{\mu,\nu}^{(\kappa)}=\frac{\mu(\kappa-\mu+1)}{2}\, a_{\mu-1,\nu}^{(\kappa)}\,.$$

Further, he proved, on the basis of (5.2), (5.3), the existence of rational numbers

$$b_{\mu,\nu}^{(\kappa)} \qquad \nu=0,1,\cdots,\mu$$

such that

(5.4)
$$\sum_{\nu=0}^{\mu} a_{\mu,\nu}^{(\kappa)}(CJ)^{\nu}\omega_2^{\kappa-\mu-\nu}\bar{\omega}_2^{\mu-\nu}e^{iCz}=\sum_{\nu=0}^{\mu} b_{\mu,\nu}^{(\kappa)}\frac{d^{\nu}}{d\lambda^{\nu}}\{\omega_2^{\kappa}e^{iCz}\}$$

holds for any real constant C and

(5.5)
$$b_{\mu,\nu}^{(\kappa)}=0 \qquad \text{for} \quad \nu\not\equiv\mu \pmod 2\,.$$

After Barner, we define differential operators

$$\Lambda_k^{(e)}=\prod_{l=1}^{k}\Big(\frac{d^2}{d\lambda^2}-(2l)^2\Big)$$

and

$$\Lambda_k^{(o)}=\prod_{l=1}^{k}\Big(\frac{d^2}{d\lambda^2}-(2l-1)^2\Big)\,.$$

Further put

(5.6)
$$\begin{cases}(-1)^{k-1-l}S_{k-1-l}^{(k)}=b_{2k-1,2l+1}^{(4k-2)} & l=0,1,\cdots,k-1\\ S_0^{(k)}=1\end{cases}$$

(5.7)
$$\begin{cases}(-1)^{k-l}T_{k-l}^{(k)}=b_{2k+1,2l+1}^{(4k)}-\dfrac{2k(2k+1)}{4}b_{2k-1,2l+1}^{(4k)} & l=0,1,\cdots,k-1\,,\\ T_0^{(k)}=b_{2k+1,2k+1}^{(4k)}=1\end{cases}$$

and

(5.8)
$$\begin{cases}(-1)^{k-l}U_{k-l}^{(k)}=b_{2k+1,2l+1}^{(4k)}+\dfrac{2k(2k+1)}{4}b_{2k-1,2l+1}^{(4k)} & l=0,1,\cdots,k-1\\ U_0^{(k)}=b_{2k+1,2k+1}^{(4k)}=1\,.\end{cases}$$

Then

(5.9)
$$z^{k-1}-S_1^{(k)}z^{k-2}+S_2^{(k)}z^{k-3}-\cdots+(-1)^{k-1}S_{k-1}^{(k)}=0$$

holds for $z=(2l)^2$, $l=1,\cdots,k-1$ and

(5.10)
$$z^{k}-T_1^{(k)}z^{k-1}+T_2^{(k)}z^{k-2}-\cdots+(-1)^{k}T_k^{(k)}=0$$

holds for $z=(2l)^2$, $l=1,\cdots,k$. Using (5.4)~(5.10), Barner proved

LEMMA 4.

$$\sum_{l=0}^{k-1}(-1)^{k-1-l}S_{k-1-l}^{(k)}\frac{d^{2l+1}}{d\lambda^{2l+1}}=\frac{d}{d\lambda}\Lambda_{k-1}^{(e)}\,.$$

LEMMA 5.

$$\sum_{l=0}^{k}(-1)^{k-l}T_{k-l}^{(k)}\frac{d^{2l+1}}{d\lambda^{2l+1}}=\frac{d}{d\lambda}\Lambda_k^{(e)}.$$

For simplicity, we put

(5.11) $$\Omega_r^{(4k)}=(c\omega_1(\lambda)+d\omega_2(\lambda))^{4k-1-r}\omega_2(\lambda)^{r-1}, \qquad r=0, 1, \cdots, 4k.$$

Then by (3.3) and (3.6), we have

(5.12) $$\Omega_r^{(4k)}=\sum_{j=0}^{2k-1}(-1)^j\Big\{\sum_{\substack{\mu+\nu=2j\\ \mu,\nu\geqq 0}}\binom{r-1}{\mu}\binom{4k-1-r}{\nu}\varepsilon_{\mathfrak{f}}^{4k-1-r-2\nu}\Big\}e^{(2k-1-2j)\lambda}$$
$$+i\sum_{j=1}^{2k-1}(-1)^j\Big\{\sum_{\substack{\mu+\nu=2j-1\\ \mu,\nu\geqq 0}}\binom{r-1}{\mu}\binom{4k-1-r}{\nu}\varepsilon_{\mathfrak{f}}^{4k-1-r-2\nu}\Big\}e^{2(k-j)\lambda}$$

for $1\leqq r\leqq 4k-1$ (see [2], p. 50). From this and (5.6), Barner obtained

(5.13) $$\Lambda_{k-1}^{(e)}\,\mathrm{Im}\,\Omega_r^{(4k)}=-2^{2k-3}\Gamma^2(k)\sum_{\mu=0}^{2k-1}\binom{r-1}{\mu}\binom{4k-1-r}{2k-1-\mu}\,\mathrm{Tr}\,(\varepsilon_{\mathfrak{f}}^{2\mu+1-r})$$

for $r=1, \cdots, 4k-1$. Note that this is rational and independent of λ. For $r=0$, Barner used the expression

(5.14) $$\mathrm{Im}\,\Omega_0^{(4k)}$$
$$=\frac{1}{1+e^{-2\lambda}}\sum_{\mu=0}^{2k-1}(-1)^\mu\Big\{\binom{4k-1}{2\mu}\varepsilon_{\mathfrak{f}}^{4k-1-4\mu}-\binom{4k-1}{2\mu+1}\varepsilon_{\mathfrak{f}}^{4k-3-4\mu}\Big\}e^{2(k-1-\mu)\lambda}$$

with $\dfrac{1}{1+e^{-2\lambda}}=\sum_{\nu=0}^{\infty}(-1)^\nu e^{-2\nu\lambda}$ and showed

(5.15) $$\lim_{\lambda\to\infty}\Lambda_{k-1}^{(e)}\,\mathrm{Im}\,\Omega_0^{(4k)}=2^{2(k-1)}\Gamma^2(k)\sum_{\mu=0}^{2k-1}(-1)^\mu\binom{4k-1}{\mu}\varepsilon_{\mathfrak{f}}^{4k-1-2\mu}.$$

For $r=4k$, he used the expression

(5.16) $$\mathrm{Im}\,\Omega_{4k}^{(4k)}=\frac{1}{1+\varepsilon_{\mathfrak{f}}^{-4}e^{-2\lambda}}\sum_{\mu=0}^{2k-1}(-1)^\mu\Big\{\binom{4k-1}{2\mu}\varepsilon_{\mathfrak{f}}^{-3}-\binom{4k-1}{2\mu+1}\varepsilon_{\mathfrak{f}}^{-1}\Big\}e^{2(k-1-\mu)\lambda}$$

with

$$\frac{1}{1+\varepsilon_{\mathfrak{f}}^{-4}e^{-2\lambda}}=\sum_{\nu=0}^{\infty}(-1)^\nu\varepsilon_{\mathfrak{f}}^{-4\nu}e^{-2\nu\lambda}$$

and proved

(5.17) $$\lim_{\lambda\to\infty}\Lambda_{k-1}^{(e)}\,\mathrm{Im}\,\Omega_{4k}^{(4k)}=2^{2(k-1)}\Gamma^2(k)\sum_{\mu=0}^{2k-1}(-1)^\mu\binom{4k-1}{\mu}\varepsilon_{\mathfrak{f}}^{2\mu+1-4k}.$$

Further consider

$$\Omega_r^{(4k+2)}=(c\omega_1(\lambda)+d\omega_2(\lambda))^{4k+1-r}\omega_2(\lambda)^{r-1}$$

for $r=0, 1, \cdots, 4k+2$. (This is formally obtained from (5.11) by replacing k by $k+1/2$.) By (3.3) and (3.6), we have, for $r=1, \cdots, 4k+1$,

$$\Omega_r^{(4k+2)} = \sum_{j=0}^{2k} (-1)^j \Big\{ \sum_{\substack{\mu+\nu=2j \\ \mu,\nu\geq 0}} \binom{r-1}{\mu}\binom{4k+1-r}{\nu} \varepsilon_{\mathfrak{f}}^{4k+1-r-2\nu} \Big\} e^{2(k-j)\lambda} \tag{5.18}$$

$$+ i \sum_{j=1}^{2k} (-1)^j \Big\{ \sum_{\substack{\mu+\nu=2j-1 \\ \mu,\nu\geq 0}} \binom{r-1}{\mu}\binom{4k+1-r}{\nu} \varepsilon_{\mathfrak{f}}^{4k+1-r-2\nu} \Big\} e^{(2k+1-2j)\lambda} .$$

Under (5.7), Barner showed

$$\Lambda_k^{(e)} \operatorname{Re} \Omega_r^{(4k+2)} = 2^{2k-1}\Gamma^2(k+1) \sum_{\mu=0}^{2k} \binom{r-1}{\mu}\binom{4k+1-r}{2\kappa-\mu} \operatorname{Tr}(\varepsilon_{\mathfrak{f}}^{2\mu+1-r}) \tag{5.19}$$

for $r=1, \cdots, 4k+1$.

For $r=0$, he used the expression

$$\operatorname{Re} \Omega_0^{(4k+2)} = \frac{1}{1+e^{-2\lambda}} \Big\{ \sum_{\mu=0}^{2k} (-1)^\mu \binom{4k+1}{2\mu} \varepsilon_{\mathfrak{f}}^{4k+1-4\mu} e^{2(k-\mu)\lambda}$$

$$- \sum_{\mu=1}^{2k+1} (-1)^\mu \binom{4k+1}{2\mu-1} \varepsilon_{\mathfrak{f}}^{4k+3-4\mu} e^{2(k-\mu)\lambda} \Big\}$$

with $1/(1+e^{-2\lambda}) = \sum_{\nu=0}^{\infty} (-1)^\nu e^{-2\nu\lambda}$, to get

$$\lim_{\lambda\to\infty} \Lambda_k^{(e)} \operatorname{Re} \Omega_0^{(4k+2)} = 2^{2k}\Gamma^2(k+1) \sum_{\mu=0}^{2k} (-1)^\mu \binom{4k+1}{\mu} \varepsilon_{\mathfrak{f}}^{4k+1-2\mu} . \tag{5.20}$$

Also for $r=4k+2$, he used

$$\operatorname{Re} \Omega_{4k+2}^{(4k+2)} = \frac{1}{1+\varepsilon_{\mathfrak{f}}^{-4} e^{-2\lambda}} \Big\{ \sum_{\mu=0}^{2k} (-1)^\mu \binom{4k+1}{2\mu} \varepsilon_{\mathfrak{f}}^{-1} e^{2(k-\mu)\lambda}$$

$$- \sum_{\mu=1}^{2k+1} (-1)^\mu \binom{4k+1}{2\mu-1} \varepsilon_{\mathfrak{f}}^{-3} e^{2(k-\mu)\lambda} \Big\}$$

to get

$$\lim_{\lambda\to\infty} \Lambda_k^{(e)} \operatorname{Re} \Omega_{4k+2}^{(4k+2)} = 2^{2k}\Gamma^2(k+1) \sum_{\mu=0}^{2k} (-1)^\mu \binom{4k+1}{\mu} \varepsilon_{\mathfrak{f}}^{2\mu-1-4k} . \tag{5.21}$$

§ 6. Further on $\Lambda_k^{(o)}$.

Besides Barner's results quoted in § 5, we need the analogous results for $U_{k-l}^{(k)}$ defined in (5.8). Namely

LEMMA 6.

$$\sum_{l=0}^{k} (-1)^{k-l} U_{k-l}^{(k)} \frac{d^{2l+1}}{d\lambda^{2l+1}} = \frac{d}{d\lambda} \Lambda_k^{(o)} .$$

Proof goes in the same way as in Barner [2]. We operate the left hand side of Lemma 6 to $\omega_2^{4k} = \omega_2(\lambda)^{4k}$. Then by (5.8), (5.4) and (5.5), we have

$$\sum_{l=0}^{k}(-1)^{k-l}U_{k-l}^{(k)}\frac{d^{2l+1}}{d\lambda^{2l+1}}\omega_2^{4k} \tag{6.1}$$

$$=\frac{k(4k)!}{(2k)!\,2^{2k}}(\omega_2^{2k-1}\bar{\omega}_2^{2k+1}+\omega_2^{2k+1}\bar{\omega}_2^{2k-1}).$$

Observing

$$\omega_2^{4k}=(e^{(1/2)\lambda}-ie^{-(1/2)\lambda})^{4k}$$

$$=(-1)^k\Big\{\sum_{j=1}^{k}(-1)^j\binom{4k}{2k-2j}2\cosh(2j\lambda)+\binom{4k}{2k}$$

$$-i\sum_{j=1}^{k}(-1)^j\binom{4k}{2k+1-2j}2\sinh((2j-1)\lambda)\Big\},$$

we see the left hand side of (6.1) is equal to

$$(-1)^k\Big\{\sum_{j=1}^{k}(-1)^j\binom{4k}{2k-2j}\sum_{l=0}^{k}(-1)^{k-l}U_{k-l}^{(k)}(2j)^{2l+1}2\sinh(2j\lambda)$$

$$-i\sum_{j=1}^{k}(-1)^j\binom{4k}{2k+1-2j}\sum_{l=0}^{k}(-1)^{k-l}U_{k-l}^{(k)}(2j-1)^{2l+1}2\cosh(2j-1)\lambda\Big\}.$$

The right, hence the left of (6.1) is real, and so the imaginary part of the above is identically zero: namely we get

$$\sum_{l=0}^{k}(-1)^{k-l}U_{k-l}^{(k)}(2j-1)^{2l+1}=0,\qquad \text{for}\quad j=1,\cdots,k,$$

which means

$$z^k-U_1^{(k)}z^{k-1}+U_2^{(k)}z^{k-2}-\cdots+(-1)^kU_k^{(k)}=0 \tag{6.2}$$

has k roots $z=(2l-1)^2$, $l=1,\cdots,k$. Hence the left of (6.2) equals

$$\prod_{l=1}^{k}(z-(2l-1)^2).$$

This completes our proof.

Further we need values

$$\lim \Lambda_k^{(o)}\operatorname{Im}\Omega_r^{(4k+2)}.$$

By (5.18), we have, for $r=1,\cdots,4k+1$,

$$\operatorname{Im}\Omega_r^{(4k+2)}=\sum_{j=1}^{2k}(-1)^j\Big\{\sum_{\substack{\mu+\nu=2j-1\\ \mu,\nu\geqq 0}}\binom{r-1}{\mu}\binom{4k+1-r}{\nu}\varepsilon_\dagger^{4k+1-r-2\nu}\Big\}e^{(2k+1-2j)\lambda}. \tag{6.3}$$

Replacing j by $k-j$, we have

$$\operatorname{Im}\Omega_r^{(4k+2)}=\sum_{j=-k}^{k-1}(-1)^{k-j}\Big\{\sum_{\substack{\mu+\nu=2k-2j-1\\ \mu,\nu\geqq 0}}\binom{r-1}{\mu}\binom{4k+1-r}{\nu}\varepsilon_\dagger^{4k+1-r-2\nu}\Big\}e^{(2j+1)\lambda}.$$

Observing that the exponent $(2j+1)\lambda$ of e is never zero ($\lambda\neq 0$) and using the formula just above and (6.2), we see that

$$\Lambda_k^{(o)} \operatorname{Im} \Omega_r^{(4k+2)}=0 \tag{6.4}$$

holds for $r=1, 2, \cdots, 4k+1$.

For $r=0$, we have

$$\operatorname{Im} \Omega_0^{(4k+2)}=\frac{1}{1+e^{-2\lambda}}\sum_{\mu=0}^{2k}(-1)^{\mu}\left\{\binom{4k+1}{2\mu}\varepsilon_{\mathfrak{f}}^{4k+1-4\mu}-\binom{4k+1}{2\mu+1}\varepsilon_{\mathfrak{f}}^{4k-1-4\mu}\right\}e^{(2k-1-2\mu)\lambda}.$$

Using $1/(1+e^{-2\lambda})=\sum_{\nu=0}^{\infty}(-1)^{\nu}e^{-2\nu\lambda}$, we can write the above in the following form:

$$\operatorname{Im} \Omega_0^{(4k+2)}=\sum_{j=-k}^{\infty}c_j e^{-(2j+1)\lambda}$$

with

$$c_j=\sum_{\mu+\nu-k=j}(-1)^{\nu+\mu}\{\cdots\cdots\}.$$

Now the exponent $-(2j+1)\lambda$, $\lambda\neq 0$, is never zero and

$$\Lambda_k^{(o)}\sum_{j=-k}^{-1}c_j e^{-(2j+1)\lambda}$$

vanishes by (6.2). Hence

$$\lim_{\lambda\to\infty}\Lambda_k^{(o)} \operatorname{Im} \Omega_0^{(4k+2)}=0. \tag{6.5}$$

For $r=4k+2$, we have

$$\operatorname{Im} \Omega_{4k+2}^{(4k+2)}=\frac{1}{1+\varepsilon_{\mathfrak{f}}^{-4}e^{-2\lambda}}\sum_{\mu=0}^{2k}(-1)^{\mu}\left\{\binom{4k+1}{2\mu}\varepsilon_{\mathfrak{f}}^{-3}-\binom{4k+1}{2\mu+1}\varepsilon_{\mathfrak{f}}^{-1}\right\}e^{(2k-1-2\mu)\lambda}. \tag{6.6}$$

In the same way as above, we see that there does not appear zero in the exponent of the infinite series expression of $\operatorname{Im} \Omega_{4k+2}^{(4k+2)}$ with respect to $e^{-\lambda}$. Hence we have

$$\lim_{\lambda\to\infty}\Lambda_k^{(o)} \operatorname{Im} \Omega_{4k+2}^{(4k+2)}=0. \tag{6.7}$$

§ 7. Main Theorems.

On the basis of the above results, we get the explicit formulas for ray-class L-functions for real quadratic fields. Here we recall the definition of S:

$$S_{\nu+1}^{(r)}(\sigma, u, v)=\sum_{h=0}^{|c|-1} P_r\left(\frac{du+h}{c}+v\right)P_{\nu+1-r}\left(\frac{ah+u}{c}\right)$$

for $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

First consider the case (i).

THEOREM 1. *Let* $\mathfrak{f}$ *be an integral ideal of the real quadratic field* $K=\boldsymbol{Q}(\sqrt{D})$, $D>0$. *Let* γ *be an element of* K *such that* $(\gamma\sqrt{D})$ *has exactly the denominator* $\mathfrak{f}$. *Let* χ *be a primitive character of the ray-class group modulo* $\mathfrak{f}$ *with* $w(\alpha)=1$, w *being the character of signature. Let* $\varepsilon_{\mathfrak{f}}$ (>1) *be a generator of the group of totally positive units congruent to* 1 mod $\mathfrak{f}$. *Take and fix an ideal* $\mathfrak{b}_B$ *from each ray-class* B. *Denote by* σ_B *the element of* SL$(2, \boldsymbol{Z})$ *corresponding to* $\varepsilon_{\mathfrak{f}}$ *with respect to the fixed integral basis* $[\alpha_1, \alpha_2]$ *of* $\mathfrak{b}_B$. *Put* $u_B=\{\mathrm{Tr}(\alpha_1\gamma)\}$, $v_B=\{\mathrm{Tr}(\alpha_2\gamma)\}$. *Then for positive integer* k,

$$L(2k, \chi)=\frac{2^{4k-3}\pi^{4k}}{T_\chi\sqrt{D}^{4k-1}}\sum_B\frac{\bar{\chi}(\mathfrak{b}_B)}{N(\mathfrak{b}_B)^{2k-1}}$$

$$\cdot\left\{\sum_{r=1}^{4k-1}\frac{(-1)^{r+1}S_{4k}^{(r)}(\sigma_B, u_B, v_B)}{r!\,(4k-r)!}\sum_{\mu=0}^{2k-1}\binom{r-1}{\mu}\binom{4k-1-r}{2k-1-\mu}\mathrm{Tr}\,(\varepsilon_{\mathfrak{f}}^{2\mu+1-r})\right.$$

$$+\sum_{\mu=0}^{2k-1}\frac{(-1)^\mu\binom{4k-1}{\mu}}{(4k)!}(\varepsilon_{\mathfrak{f}}^{4k-1-2\mu}S_{4k}^{(0)}(\sigma_B, u_B, v_B)$$

$$\left.+\varepsilon_{\mathfrak{f}}^{2\mu+1-4k}S_{4k}^{(4k)}(\sigma_B, u_B, v_B))\right\}$$

where in the sum $\sum_B$, B *runs over all ray-classes* mod $\mathfrak{f}$.

REMARK. When $\mathfrak{f}=(1)$, $\chi=1$, $u_B=v_B=0$, we have $T_\chi=1$ and $S_{4k}^{(0)}(\sigma_B, 0, 0)=S_{4k}^{(4k)}(\sigma_B, 0, 0)$ and the formula coincides with Barner's for $N(\varepsilon_{\mathfrak{f}})=1$.

Next consider the case (ii).

THEOREM 2. *All the notations being the same as in Theorem* 1, *we assume that the character* $w(\alpha)$ *of signature attached to* χ *is given by* $w(\alpha)=N(\alpha)/|N(\alpha)|$. *Then*

$$L(2k+1, \chi)=\frac{2^{4k}\pi^{4k+2}}{T_\chi\sqrt{D}^{4k+1}}\sum_B\frac{\bar{\chi}(\mathfrak{b}_B)w(\alpha_1)}{N(\mathfrak{b}_B)^{2k}}$$

$$\cdot\left\{\sum_{r=0}^{4k+2}\frac{(-1)^r S_{4k+2}^{(r)}(\sigma_B, u_B, v_B)}{r!\,(4k+2-r)!}\sum_{\mu=0}^{2k}\binom{r-1}{\mu}\binom{4k+1-r}{2k-\mu}\mathrm{Tr}\,(\varepsilon_{\mathfrak{f}}^{2\mu+1-r})\right.$$

$$+\sum_{\mu=0}^{2k}\frac{(-1)^\mu\binom{4k+1}{\mu}}{(4k)!}(\varepsilon_{\mathfrak{f}}^{4k+1-2}S_{4k+2}^{(0)}(\sigma_B, u_B, v_B)$$

$$\left.+\varepsilon_{\mathfrak{f}}^{2\mu-4k-1}S_{4k+2}^{(4k)}(\sigma_B, u_B, v_B))\right\}.$$

§ 8. Proof of Theorem 1.

We go on the same way as Barner's. First we rewrite $g(z, s, u, v)$. The necessary formulas are:

(8.1) Theta inversion formula.

$$\sum_{m=-\infty}^{\infty} e^{-\pi t(m+v)^2+2\pi imu}=t^{-1/2}\sum_{m=-\infty}^{\infty} e^{-\pi t^{-1}(m-u)^2+2\pi i(m-u)v}.$$

(8.2) Γ-integral

$$\int_0^\infty t^{s-1}e^{-t\pi n}dt=\frac{\Gamma(s)}{\pi^s n^s}$$

(8.3) Modified Bessel function

$$\int_0^\infty t^s e^{-\pi(a^2t+b^2t^{-1})}\frac{dt}{t}=2\Big(\frac{b}{a}\Big)^s K_s(2\pi ab), \qquad (a, b>0)$$

(8.4) $$K_{n+(1/2)}(z)=\Big(\frac{\pi}{2z}\Big)^{1/2}e^{-z}\sum_{\nu=0}^{n}\frac{(n+\nu)!}{\nu!(n-\nu)!(2z)^\nu}, \qquad n\in\boldsymbol{N}.$$

Put $s=2k$ with positive integer k. We have

(8.5) $$g(z, 2k, u, v)=y^{2k}\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{m^{4k}}+y^{2k}\sum_{n=-\infty}^{\infty}{}'\sum_{m=-\infty}^{\infty}\frac{e^{2\pi i(mu+nv)}}{|m+nz|^{4k}}.$$

The first sum is equal to

(8.6) $$-\frac{(2\pi)^{4k}}{(4k)!}P_{4k}(u)\Big(\frac{J}{\omega_2\bar{\omega}_2}\Big)^{4k}$$

for $z=\omega_1(\lambda)/\omega_2(\lambda)\in\mathfrak{C}$. This follows easily from the definition of $P_s(u)$ and the parameter λ. For the second sum of (8.5), we first consider the inner sum $\sum_{m=-\infty}^{\infty}$. Applying the Γ-integral (8.2) to the summand, we have

$$\frac{1}{|m+nz|^{4k}}=\frac{\pi^{2k}}{\Gamma(2k)}\int_0^\infty t^{2k-1}e^{-t\pi|m+nz|^2}dt.$$

Put this into the inner sum and change the order of the summation $\sum_{m=-\infty}^{\infty}$ and the integral $\int_0^\infty$ to get

$$\sum_{m=-\infty}^{\infty}\frac{e^{2\pi i(mu+nv)}}{|m+nz|^{4k}}=\frac{\pi^{2k}}{\Gamma(2k)}\int_0^\infty e^{-\pi ty^2n^2+2\pi inv}\sum_{m=-\infty}^{\infty}e^{-\pi t(m+nx)^2+2\pi imu}t^{2k-1}dt.$$

Further, apply the theta-inversion formula (8.1) to the infinite sum in the above and change the order of $\int_0^\infty$ and $\sum_{m=-\infty}^{\infty}$ again. Then we have

$$y^{2k}\sum_{n=-\infty}^{\infty}{}' \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i(mu+nv)}}{|m+nz|^{4k}}$$

$$=\frac{y^{2k}\pi^{2k}}{\Gamma(2k)}\sum_{n=-\infty}^{\infty}{}' \sum_{m=-\infty}^{\infty} e^{-\pi tn^2y^2-\pi t^{-1}(m-u)^2+2\pi i(m-u)nx+2\pi inv}t^{2k-(3/2)}dt.$$

We divide this double sum into five parts:

$$(8.7)\qquad \begin{aligned}&\sum_{n=-\infty}^{\infty}{}' \quad (m=0), \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}, \quad \sum_{n=1}^{\infty}\sum_{m=-\infty}^{-1},\\ &\sum_{n=-\infty}^{-1}\sum_{m=1}^{\infty}, \quad \sum_{n=-\infty}^{-1}\sum_{m=-\infty}^{-1}.\end{aligned}$$

(The last three can be transformed to $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}$ by changing $-m$, $-n$ to m, n.) Every summand can be expressed in terms of the modified Bessel function, by (8.3). Then we apply (8.4). For example,

(8.8) the part $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}$

$$=\frac{2\pi^{2k}y^{1/2}}{\Gamma(2k)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{2\pi inv+2\pi i(m-u)nx}\left(\frac{m-u}{n}\right)^{2k-(1/2)} K_{2k-(1/2)}(2\pi n(m-u)y)$$

$$=\frac{\pi}{2^{2k-1}\Gamma(2k)}\sum_{n=1}^{\infty}\frac{1}{n^{4k-1}}\sum_{m=1}^{\infty} e^{2\pi inv}$$

$$\cdot\sum_{\nu=0}^{2k-1}\frac{(2\pi n(m-u))^{2k-1-\nu}(2k-1+\nu)\,!}{\nu\,!\,(2k-1-\nu)\,!\,2^\nu y^\nu} e^{2\pi in(m-u)z}$$

$$=\frac{\pi}{(2J)^{2k-1}\Gamma(2k)}\sum_{n=1}^{\infty}\frac{1}{n^{4k-1}} e^{2\pi inv}$$

$$\sum_{m=1}^{\infty}\sum_{\nu=0}^{2k-1} a^{(4k-2)}_{2k-1,\nu}(2\pi n(m-u)J)^\nu(\omega_2\bar{\omega}_2)^{2k-1-\nu}e^{2\pi in(m-u)z}$$

By (5.2), this is equal to

$$\frac{\pi}{(2J)^{2k-1}\Gamma(2k)}\sum_{n=1}\frac{e^{2\pi inv}}{n^{4k-1}}\sum_{m=1}^{\infty}\sum_{\nu=0}^{2k-1} b^{(4k-2)}_{2k-1,\nu}\frac{d^\nu}{d\lambda^\nu}(\omega_2^{4k-2}e^{2\pi in(m-u)z}).$$

For the other double sums in (8.7), we can compute in the same way. Then by (8.5), (8.6) and by Lemma 4, we have

$$(8.9)\qquad g(z, 2k, u, v)=-\frac{J^{2k}(2\pi)^{4k}}{(4k)\,!}P_{4k}(u)(\omega_2\bar{\omega}_2)^{-2k}$$

$$+\frac{2^{2-2k}\pi}{\Gamma(2k)J^{2k-1}}\operatorname{Re}\left[\frac{d}{d\lambda}\Lambda^{(e)}_{k-1}(F_{4k-1}(u, v\,;\,\omega_1, \omega_2))\right].$$

The last formula corresponds to Barner's Hilfssatz 5. We combine (8.9) with

Lemma 2:

$$L(2k, \chi)=\frac{D^{-k}\Gamma(2k)}{2T_\chi\Gamma^2(k)} \sum_B \bar{\chi}(\mathfrak{b}_B)\int_\lambda^{\hat{\lambda}} g_B(z, 2k)d\lambda\,.$$

Then computation of the integral $\int_\lambda^{\hat{\lambda}} g_B(z, 2k)d\lambda$ is quite the same as Barner's and so we only sketch the outline: Put

$$\int_\lambda^{\hat{\lambda}} g_B(z, 2k)d\lambda=I_1(\lambda)+I_2(\lambda)$$

with

$$I_1(\lambda)=-\frac{J^{2k}(2\pi)^{4k}}{(4k)!}P_{4k}(u)\int_\lambda^{\hat{\lambda}}(\omega_2\bar{\omega}_2)^{-2k}d\lambda$$

and

$$I_2(\lambda)=\frac{2^{2-2k}\pi}{\Gamma(2k)J^{2k-1}}\int_\lambda^{\hat{\lambda}}\mathrm{Re}\Big[\frac{d}{d\lambda}\Lambda_{k-1}^{(e)}(F_{4k-1}(u, v, \omega_1, \omega_2))\Big]d\lambda\,.$$

First consicer $I_2(\lambda)$. We note that Re and $\int_\lambda^{\hat{\lambda}}$ are commutative, $d/d\lambda$ and $\Big]_\lambda^{\hat{\lambda}}$ are commutative (since $d/d\hat{\lambda}=d/d\lambda$) and further Re and $d/d\lambda$ are so. Then we have

$$I_2(\lambda)=\frac{2^{2-2k}\pi}{\Gamma(2k)J^{2k-1}}\ \mathrm{Re}\ \Lambda_{k-1}^{(e)}\Big[F_{4k-1}(u_B,\ v_B\ ;\ \omega_1,\ \omega_2)\Big]_\lambda^{\hat{\lambda}}.$$

Now by Lemma 1 for $u=u_B$, $v=v_B$, $\nu=4k-1$ and the definition of $\Omega_r^{(4k)}$, we have

$$\Big[F_{4k-1}(u_B, v_B\ ;\ \omega_1, \omega_2)\Big]_\lambda^{\hat{\lambda}}=-(2\pi i)^{4k-1}\sum_{r=0}^{4k}\frac{(-1)^r\Omega_r^{(4k)}}{r\,!\,(4k-r)\,!}S_{4k}^{(r)}(\sigma_B, u_B, v_B)$$

and

$$\mathrm{Re}\ \Lambda_{k-1}^{(e)}\Big[F_{4k-1}(u_B, v_B\ ;\ \omega_1, \omega_2)\Big]_\lambda^{\hat{\lambda}}$$
$$=\sum_{r=0}^{4k}\frac{(-1)^{r+1}}{r\,!\,(4k-r)\,!}\ S_{4k}^{(r)}(\sigma_B, u_B, v_B)\Lambda_{k-1}^{(e)}\,\mathrm{Im}\ \Omega_r^{(4k)}\,.$$

In (5.13), we already get the values

$$\Lambda_{k-1}^{(e)}\,\mathrm{Im}\ \Omega_r^{(4k)}\,,\qquad r=1, 2, \cdots, 4k-1\,,$$

which are independent of λ.

There remains the evaluation of $I_1(\lambda)$ and of the sum of terms for $r=0$, $4k$:

$$\text{(8.10)}\qquad \frac{-1}{(4k)\,!}(S_{4k}^{(0)}(\sigma_B, u_B, v_B)\Lambda_{k-1}^{(e)}\,\mathrm{Im}\ \Omega_0^{(4k)}$$
$$+S_{4k}^{(4k)}(\sigma_B, u_B, v_B)\Lambda_{k-1}^{(e)}\,\mathrm{Im}\ \Omega_{4k}^{(4k)})\,.$$

We evaluate them by tending λ to ∞. By the integral by parts, we have

$$I_1(\lambda)=\text{const.}\sum_{l=0}^{k-1}\Big\{\prod_{r=0}^{l}\frac{2k-2r}{2k-1-2r}\Big\}\Big[\frac{\tanh\lambda}{(\cosh\lambda)^{2(k-1-l)}}\Big]_{\lambda}^{\hat{\lambda}}$$

and hence

$$\lim_{\lambda\to\infty} I_1(\lambda)=0\,.$$

Further by (5.15), (5.17), we get easily the value of $\lim_{\lambda\to\infty}$ of (8.10).

Altogether, we finish the proof of Theorem 1.

§ 9. Proof of Theorem 2.

We start with the function

$$f(z, 2k+1\,;\,u, v)=\frac{\rho-\rho'}{\omega_2^2(\lambda)}\frac{1}{2k+1}\frac{\partial}{\partial z}g(z, 2k+1\,;\,u, v)$$

and use the series expression of $g(z, 2k+1\,;\,u, v)=g(z, 2k+1)$ as given in the proof of Theorem 1. We divide $g(z, 2k+1)$ into two parts as in (8.5):

$$g(z, 2k+1)=y^{2k+1}\sum_{m=-\infty}^{\infty}{}'\frac{e^{2\pi imu}}{m^{4k+2}}+y^{2k+1}\sum_{n=-\infty}^{\infty}{}'\sum_{m=-\infty}^{\infty}\frac{e^{2\pi i(mu+nv)}}{|m+nz|^{4k+2}}\,.$$

Then as in the case of Theorem 1, apply the Γ-integral (8.2) to the summands of $\sum_{m=-\infty}^{\infty}$ and change the order of $\sum_{m=-\infty}^{\infty}$ and $\int_0^{\infty}$. Further apply the theta-inversion formula (8.1) and again change the order of $\int_0^{\infty}$ and $\sum_{m=-\infty}^{\infty}$. We divide the double sum so obtained into five parts as in (8.7). Then as in (8.8), we have, by (8.3), (8.4),

the part $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}$

$$=\frac{\pi}{2^{2k}\Gamma(2k+1)}\sum_{n=1}^{\infty}\frac{e^{2\pi inv}}{n^{4k}}\sum_{m=1}^{\infty}\sum_{\nu=0}^{2k}\frac{(2\pi n(m-u))^{\nu}(4k-\nu)\,!}{(2k-\nu)\,!\,\nu\,!\,2^{2k-\nu}y^{2k-\nu}}e^{2\pi i\nu(m-u)z}$$

and get the series expression of $g(z, 2k+1)$. Then apply the operator

$$\frac{\rho-\rho'}{\omega_2^2(\lambda)}\frac{1}{2k+1}\frac{\partial}{\partial z}$$

to the series expression of $g(z, 2k+1)$. This time, we have to use (5.2), (5.3) in getting the formula for $f(z, 2k+1\,;\,u, v)$ analogous to (8.9). Then by (5.2) and the definition of $F_{4k+1}(u, v\,;\,\omega_1, \omega_2)$, we have

$$f(z,2k+1\,;u,v)=\frac{e^{\lambda}+2i-e^{-\lambda}}{2i}\frac{J^{2k+1}}{(\omega_2\bar{\omega}_2)^{2k+2}}\frac{(2\pi)^{4k+2}}{(4k+2)\,!}P_{4k+2}(u)$$

$$+\frac{\pi}{2^{2k}\Gamma(2k+2)J^{2k}}\Big\{\sum_{\nu=0}^{2k+1}b_{2k+1,\nu}^{(4k)}\frac{d^{\nu}}{d\lambda^{\nu}}(iF_{4k+1}(u,v\,;\omega_1,\omega_2))$$

$$-\frac{2k(2k+1)}{4}\sum_{\nu=0}^{2k+1}b_{2k+1,\nu}^{(4k)}\frac{d^{\nu}}{d\lambda^{\nu}}(\overline{iF_{4k+1}(u,v\,;\omega_1,\omega_2)})\Big\}.$$

For simplicity, in what follows, we denote $F_{4k+1}(u,v\,;\omega_1,\omega_2)$ only by F_{4k+1}. By (5.5), (5.7), (5.8), Lemma 5 and Lemma 6, it follows that

$$\operatorname{Re}f(z,2k+1\,;u,v)=\frac{J^{2k+1}}{(\omega_2\bar{\omega}_2)^{2k+2}}\frac{(2\pi)^{4k+2}}{(4k+2)\,!}P_{4k+2}(u)$$

$$+\frac{\pi}{2^{2k}\Gamma(2k+2)J^{2k}}\sum_{l=0}^{k}(-1)^{k-l}T_{k-l}^{(k)}\frac{d^{2l+1}}{d\lambda^{2l+1}}\operatorname{Re}(iF_{4k+1})$$

$$=\frac{J^{2k+1}}{(\omega_2\bar{\omega}_2)^{2k+2}}\frac{(2\pi)^{4k+2}}{(4k+2)\,!}P_{4k+2}(u)$$

$$+\frac{\pi}{2^{2k}\Gamma(2k+2)J^{2k}}\frac{d}{d\lambda}\Lambda_k^{(e)}\operatorname{Re}(iF_{4k+1})$$

and

$$\operatorname{Im}f(z,2k+1\,;u,v)=\frac{e^{-\lambda}-e^{\lambda}}{2}\frac{J^{2k+1}}{(\omega_2\bar{\omega}_2)^{2k+2}}\frac{(2\pi)^{4k+2}}{(4k+2)\,!}P_{4k+2}(u)$$

$$+\frac{\pi}{2^{2k}\Gamma(2k+2)J^{2k}}\sum_{l=0}^{k}(-1)^{k-l}U_{k-l}^{(k)}\frac{d^{2l+1}}{d\lambda^{2l+1}}\operatorname{Im}(iF_{4k+1})$$

$$=\frac{e^{-\lambda}-e^{\lambda}}{2}\frac{J^{2k+1}}{(\omega_2\bar{\omega}_2)^{2k+2}}\frac{(2\pi)^{4k+2}}{(4k+2)\,!}P_{4k+2}(u)$$

$$+\frac{\pi}{2^{2k}\Gamma(2k+2)J^{2k}}\frac{d}{d\lambda}\Lambda_k^{(o)}\operatorname{Im}(iF_{4k+1})\,.$$

Combining these formulas with Lemma 3 and taking $u=u_B$, $v=v_B$, we have

$$L(2k+1,\chi)=\frac{D^{-(1/2)(2k+1)}\Gamma(2k+1)}{2T_{\chi}\Gamma^2(k+1)}\sum_{B}\bar{\chi}(\mathfrak{b}_B)w(\alpha_1)\{J_1+J_2+J_3\}\,,$$

where

$$J_1=J_1(\lambda)=\frac{J^{2k+1}(2\pi)^{4k+2}}{(4k+2)\,!}P_{4k+2}(u)\int_{\lambda}^{\hat{\lambda}}\Big(1+\frac{e^{-\lambda}-e^{\lambda}}{2}i\Big)\frac{d\lambda}{(\omega_2\bar{\omega}_2)^{2k+2}},$$

$$J_2=J_2(\lambda)=\frac{\pi}{2^{2k}\Gamma(2k+2)J^{2k}}\int_{\lambda}^{\hat{\lambda}}\frac{d}{d\lambda}\Lambda_k^{(e)}\operatorname{Re}(iF_{4k+1})d\lambda$$

and

$$J_3=J_3(\lambda)=\frac{\pi}{2^{2k}\Gamma(2k+2)J^{2k}}\int_{\lambda}^{\hat{\lambda}}\frac{d}{d\lambda}\Lambda_k^{(o)}\operatorname{Im}(iF_{4k+1})d\lambda\,.$$

To compute $J_1(\lambda)$, we divide it into two parts:

$$J_1'(\lambda)=\int_{\lambda}^{\hat{\lambda}}\frac{d\lambda}{(\omega_2\bar{\omega}_2)^{2k+2}} \quad \text{and} \quad J_1''(\lambda)=\int_{\lambda}^{\hat{\lambda}}\frac{e^{\lambda}-e^{-\lambda}}{(\omega_2\bar{\omega}_2)^{2k+2}}d\lambda .$$

The first is computed in the same way as $I_1(\lambda)$ and we get

$$\lim_{\lambda\to\infty} J_1'(\lambda)=0 .$$

The evaluation of the second is easy and we also get

$$\lim_{\lambda\to\infty} J_1''(\lambda)=0 .$$

Now by the same reason as in the case of Theorem 1, we have, putting $c=\dfrac{2^{2k}\Gamma(2k+2)J^{2k}}{\pi}$,

$$cJ_2=\Lambda_k^{(e)}\operatorname{Re}(iF_{4k+1})\Big]_{\lambda}^{\hat{\lambda}}$$

and

$$cJ_3=i\Lambda_k^{(o)}\operatorname{Im}(iF_{4k+1})\Big]_{\lambda}^{\hat{\lambda}} .$$

Hence by Lemma 1, computation of J_2 and J_3 is reduced to that of

$$\Lambda_k^{(e)}\operatorname{Re}\Omega_r^{(4k+2)}, \qquad r=0, 1, \cdots, 4k+2 ,$$

and

$$\Lambda_k^{(o)}\operatorname{Im}\Omega_r^{(4k+2)}, \qquad r=0, 1, \cdots, 4k+2$$

whose values are already given in (5.19), (5.20), (5.21), (6.4), (6.5) and (6.7). Summing up, we get Theorem 2.

Appendix

In his lecture at Tata Institute [**6**], Siegel showed that zeta-functions with Grössen-characters of a real quadratic field can be obtained as Fourier coefficients (with respect to the parameter λ) of non-holomorphic Eisenstein series. The special cases of this result are quoted as Lemma 2, 3 in § 4. Here we shall give a general formulation of Siegel's result for a finitely generated abelian group with some condition on its representation and apply our results to an algebraic number field K in order to get zeta-functions of K with Grössen-characters as Fourier coefficients of "Eisenstein-Epatein" series.

The author thanks Professors T. Ono and M. Kuga very much for their kind advices on this subject during his stay in Philadelphia, U. S. A.

1. Consider the algebra $\boldsymbol{R}^r$ (multiplication is defined element-wise) and its (r-dimensional) first octant $\boldsymbol{H}^r = y = {}^t(y_1, \cdots, y_r)$, $y_j > 0$. Let Λ be the mapping from $\boldsymbol{H}^r$ to $\boldsymbol{R}^r$ defined by

$$\Lambda(y) = L \log y \tag{1}$$

with an L in GL $(r, \boldsymbol{R})$. Λ is a diffeomorphism of $\boldsymbol{H}^r$ onto $\boldsymbol{R}^r$. We define the operation $\circ$ of $\xi \in \boldsymbol{R}^r$ on $\boldsymbol{H}^r$ by putting

$$\xi \circ y = Y^{\xi} \cdot y \tag{2}$$

with a matrix $Y \in M_r(\boldsymbol{R})$ whose columns belong to $\boldsymbol{H}^r$. Here $\cdot$ on the right means the multiplication in the algebra $\boldsymbol{R}^r$. It is easily see that for any $\xi \in \boldsymbol{R}^r$ and for any $y \in \boldsymbol{H}^r$, we have

$$\Lambda(\xi \circ y) = \Lambda(Y^{\xi}) + \Lambda(y) . \tag{3}$$

If Y satisfies the condition

$$\det \log Y \neq 0 , \tag{4}$$

then we have

$$\Lambda(Y^{\boldsymbol{R}^r}) = \boldsymbol{R}^r , \tag{5}$$

since the jacobian determinant of $\xi \to \Lambda(Y^{\xi})$ is essentially equal to det (log Y). Hereafter we always assume (4). From (5) it follows that $\Lambda(Y^{\boldsymbol{Z}^r})$ is a lattice in $\boldsymbol{R}^r$.

Let $\mathfrak{L}$ be a lattice in $\boldsymbol{R}^n$. Let G be a finitely generated abelian group and ρ a representation of G into GL $(\mathfrak{L})$. The group G is identified with $C \times I_1 \times I_2 \times \cdots \times I_r$, where C is a direct product of a finite number of finite cyclic groups and I_j is an infinite cyclic group. Let ι be an isomorphism of $I_1 \times \cdots \times I_r$ onto $\boldsymbol{Z}^r$. We write every element a of G as $a = a_0 a_\infty$ with $a_0 \in C$ and $a_\infty \in I_1 \times \cdots \times I_r$.

Let $P(x, y)$ be a map of $\boldsymbol{R}^r \times \boldsymbol{H}^r$ into $\boldsymbol{C}$ such that

(P.1) $\qquad x \neq 0 \;\Rightarrow\; P(x, y) \neq 0 \quad$ for any $\; y \in \boldsymbol{H}^r$,

(P.2) $\qquad$ for $\; a \in G, \;\; \gamma \in \mathfrak{L} \;$ and $\; y \in \boldsymbol{H}^r$,

$$P(\rho(a)\gamma, y) = P(\rho(a_0)\gamma, \; \iota(a_\infty) \circ y) .$$

Put

$$P'(x, v) = P(x, \Lambda^{-1}(v)) . \tag{6}$$

Then we have

$$P'(\rho(a)\gamma, v) = P'(\rho(a_0)\gamma, \; v + \varepsilon(a_\infty)) , \tag{7}$$

where $v=\Lambda(y)$ and $\varepsilon(a_\infty)=\Lambda(Y^{\iota(a_\infty)})$. Because, we have

$$P'(\rho(a_0)\gamma, v+\varepsilon(a_\infty))=P'(\rho(a_0)\gamma, \Lambda(Y^{\iota(a_\infty)}\circ y))$$
$$=P(\rho(a_0)\gamma, \iota(a_\infty)\circ y)=P(\rho(a)\gamma, y)=P'(\rho(a)\gamma, v).$$

For $y\in \boldsymbol{H}^r$, we put

$$\Phi(y, P, s)=\sum_{\gamma\in\mathfrak{L}}{}' P(\gamma, y)^{-s} \tag{8}$$

and assume that

(K) this converges absolutely and uniformly for $\operatorname{Re} s>\sigma_0$ with some $\sigma_0>0$.
By (6) we can view Φ as a function Φ' of v: namely

$$\Phi'(v, P', s)=\sum_{\gamma\in\mathfrak{L}}{}'P'(\gamma, v)^{-s}=\Phi(y, P, s)$$

with $\Lambda(y)=v$. Then

$$\Phi' \text{ is periodic with respect to } v. \tag{9}$$

Because, we have

$$\Phi'(v+\varepsilon, P', s)=\sum_{\gamma\in\mathfrak{L}}{}'P'(\gamma, v+\varepsilon)^{-s}$$

and since we can write $\varepsilon=\varepsilon(a_\infty)=\Lambda(Y^{\iota(a_\infty)})$ with $a_\infty\in I_1\times\cdots\times I_r$, it is equal to, with $v=\Lambda(y)$,

$$=\sum_{\gamma\in\mathfrak{L}}{}'P'(\gamma, \Lambda(Y^{\iota(a_\infty)}\circ y))^{-s}$$
$$=\sum_{\gamma\in\mathfrak{L}}{}'P(\rho(a_\infty)\gamma, y)^{-s} \qquad \text{(by (7))}$$
$$=\sum_{\gamma\in\mathfrak{L}}{}'P(\gamma, y)^{-s} \qquad (\rho(a_\infty)\in \mathrm{GL}(\mathfrak{L}))$$
$$=\Phi'(v, P', s).$$

Thus we can ask what the Fourier coefficients of Φ' are.

For simplicity, assume that

$$\Lambda(Y^{\boldsymbol{Z}^r})=\boldsymbol{Z}^r. \tag{10}$$

Then for a vector $\kappa\in\boldsymbol{Z}^r$, the '$\kappa$-th' Fourier coefficient of Φ' is given by

$$f_\kappa(s, P)=\int_0^1\cdots\int \Phi'(v, P', s)e^{-2\pi i t\kappa\cdot v}dv. \tag{11}$$

We define the equivalence relation in $\mathfrak{L}$ with respect to ρ as follows:

$$\gamma\sim\gamma_1 \Longleftrightarrow \text{there exists } a\in G \text{ such that } \rho(a)\gamma=\gamma_1.$$

Denote by $\{\gamma\}$ the class of γ with respect to this $\sim$.

We define subgroups of G as follows:

$$K_P(\gamma)=\{a\in G \mid P(\rho(a)\gamma, y)=P(\gamma, y)\}\,, \tag{12}$$

$$I(\gamma)=\{a\in G \mid \rho(a)\gamma=\gamma\}\,,$$

and denote by K_ρ the kernel of ρ. Then we have

$$K_P(\gamma)\supset I(\gamma)\supset K_\rho \qquad \text{for every } \gamma \tag{13}$$

and

$$K_P(\gamma_1)=K_P(\gamma),\quad I(\gamma_1)=I(\gamma) \qquad \text{if} \quad \gamma\sim\gamma_1\,. \tag{14}$$

From now on, we assume that

$$K_P(\gamma) \text{ is finite for every } \gamma\,. \tag{F}$$

Then by (13), $I(\gamma)$ and K_P are both finite. We can view $K_P(\gamma)$, $I(\gamma)$ and K_ρ are contained in C (finite part of G). By (14), the order of $K_P(\gamma)$ is defined for the class of γ. Also the order of $I(\gamma)$ is defined for the class. We denote by $n_{\{\gamma\}}$ and $g_{\{\gamma\}}$ the order of $K_P(\gamma)$ and $I(\gamma)$, respectively.

Now we compute $f_\kappa(s, P)$. By the assumption (K), the order of the integral $\int\cdots\int$ and the summation $\sum'$ can be changed. Then

$$\begin{aligned}
f_\kappa(s, P) &= \sum_{\{\gamma\}\neq 0}\frac{1}{g_{\{\gamma\}}}\sum_{a\in G}\int\cdots\int\frac{e^{-2\pi i t\kappa\cdot v}}{P'(\rho(a)\gamma, v)^s}\,dv \\
&= \sum_{\{\gamma\}\neq 0}\frac{1}{g_{\{\gamma\}}}\sum_{a_\infty\in I_1\times\cdots\times I_r}\ \sum_{a_0\in C}\int\cdots\int\frac{e^{-2\pi i t\kappa\cdot v}dv}{P'(\rho(a_\infty)\rho(a_0)\gamma, v)^s} \\
&= \sum_{\{\gamma\}\neq 0}\frac{n_{\{\gamma\}}}{g_{\{\gamma\}}}\sum_{a_\infty\in I_1\times\cdots\times I_r}\ \sum_{a_0\in C/K_P(\gamma)}\int\cdots\int\cdots \\
&= \sum_{\{\gamma\}\neq 0}\frac{n_{\{\gamma\}}}{g_{\{\gamma\}}}\sum_{a_0\in C/K_P(\gamma)}\ \sum_{\xi\in Z^r}\int\cdots\int\frac{e^{-2\pi i t\kappa\cdot v}dv}{P'(\rho(a_0), v+\varepsilon(\xi))^s} \\
&= \sum_{\{\gamma\}\neq 0}\frac{n_{\{\gamma\}}}{g_{\{\gamma\}}}\sum_{a_0\in C/K_P(\gamma)}\ \sum_{{}^t(l_1,\cdots,l_r)\in Z^r}\int\cdots\int\frac{e^{-2\pi i t\kappa\cdot v}dv}{P'(\rho(a_0)\gamma, v+\sum_{j=1}^r l_j\varepsilon_j)^s}
\end{aligned}$$

where $\xi=\iota(a_\infty)$, $\varepsilon_j={}^t(0,\cdots,\overset{j}{1},\cdots,0)$ and $\varepsilon(\xi)=\sum_{j=1}^r l_j\varepsilon_j$. Then we change variable v back to y. If v varies in the unit cube U, $v+\sum_{j=1}^r l_j\varepsilon_j$ varies in the cube which is the translation of U by $\sum_{j=1}^r l_j\varepsilon_j$. y varies in the domain $D(l_1,\cdots,l_r)$ $=\Lambda^{-1}(U+\sum_{j=1}^r l_j\varepsilon_j)$ in $\boldsymbol{H}^r$. If ${}^t(l_1,\cdots,l_r)$ runs over all $\boldsymbol{Z}^r$, $v+\sum_{j=1}^r l_j\varepsilon_j$ runs over all $\boldsymbol{R}^n$ and $D(l_1, \cdots, l_r)$ covers the whole $\boldsymbol{H}^r$ without gaps and overlaps. Thus y runs over all $\boldsymbol{H}^r$. Hence we have

$$(15)\qquad f_\kappa(s, P)=\sum_{\{\gamma\}\neq 0}\frac{n_{\{\gamma\}}}{g_{\{\gamma\}}}\sum_{a_0\in C/K_P(\gamma)}\int\cdots\int_{\boldsymbol{H}^r}\frac{e^{-2\pi i t\kappa\cdot A(y)}J(v, y)dy}{P(\rho(a_0)\gamma, y)^s},$$

where $J(v, y)$ is the jacobian determinant.

We shall call $f_\kappa(s, P)$ the Siegel's zeta-function for the data $(G, \mathfrak{L}, P, Y, A, \rho)$.

2. Let K be an algebraic number field of finite degree n. As usual, we denote by $K^{(1)}=K, \cdots, K^{(r_1)}$ the real conjugates of K and $K^{(r_1+1)}, \cdots, K^{(r_1+r_2)}$, $K^{(r_1+r_2+1)}=\overline{K^{(r_1+1)}}, \cdots, K^{(r_1+2r_2)}=\overline{K^{(r_1+r_2)}}$ the complex conjugates of K. Thus $n=r_1+2r_2$.

Let $\mathfrak{a}$ be an ideal of K and $\omega_1, \cdots, \omega_n$ a basis of $\mathfrak{a}$. Then $K=\boldsymbol{Q}\omega_1+\cdots+\boldsymbol{Q}\omega_n$. We denote by M the regular representation of K with respect to $\omega_1, \cdots, \omega_n$: namely if we put

$$\boldsymbol{\Omega}=\begin{pmatrix}\omega_1^{(1)}, & \cdots\cdots\cdots, & \omega_n^{(1)}\\ \vdots & & \vdots\\ \omega_1^{(n)}, & \cdots\cdots\cdots, & \omega_n^{(n)}\end{pmatrix}$$

then for $\alpha\in K$, $M(\alpha)$ is defined by

$$(16)\qquad \begin{pmatrix}\alpha^{(1)} & & & \\ & \alpha^{(2)} & & \\ & & \ddots & \\ & & & \alpha^{(n)}\end{pmatrix}\Omega=\Omega M(\alpha).$$

If $\alpha=q_1\omega_1+\cdots+q_n\omega_n\in K$ with $q_i\in\boldsymbol{Q}$, $M(\alpha)=\sum_{i=1}^n q_iM(\omega_i)$.

Let $A_{K,\infty}$ be the infinite part of the adelization of K. We understand an element u of $A_{K,\infty}$ as a vector ${}^t(u_1, \cdots, u_n)$ with $\bar{u}_{r_1+j}=u_{r_1+r_2+j}$. Put $\Omega^{-1}\,{}^t(u_1, \cdots, u_n)={}^t(v_1, \cdots, v_n)$. Then we can extend M to $A_{K,\infty}$ by putting

$$M(u)=\sum_{i=1}^n v_iM(\omega_i).$$

Let Γ be the group of all units in K. Define the set

$$(17)\qquad T=\{u\in A_{K,\infty}\mid \prod_{i=1}^n u_i=1\}.$$

Then $\Gamma^2=\{\varepsilon^2\mid\varepsilon\in\Gamma\}$ is contained in T, where the imbedding of α of K is the one considered as $\alpha={}^t(\alpha^{(1)}, \cdots, \alpha^{(n)})\in A_{K,\infty}$.

Define $X=\{{}^t\bar{g}g\mid g\in\mathrm{GL}(n, \boldsymbol{C})\}$. The operation of $g\in\mathrm{SL}(n, \boldsymbol{C})$ on X is given by $X\ni S\to{}^t\bar{g}Sg=S[g]$. We take $S={}^t\bar{\Omega}\Omega$ and consider the $M(T)$-orbit of S on X. For $u\in T$, we have

$$ {}^{t}M(u)SM(u) = {}^{t}M(u){}^{t}\bar{\Omega}\Omega M(u) $$

$$ = {}^{t}\bar{\Omega}\begin{pmatrix} u_1^2 & & & & & \\ & \ddots & & & & \\ & & u_{r_1}^2 & & & \\ & & & |u_{r_1+1}|^2 & & \\ & & & & \ddots & \\ & & & & & |u_n|^2 \end{pmatrix}\Omega . $$

Put $y_1^*=u_1^2, \cdots, y_{r_1}^*=u_{r_1}^2$, $y_{r_1+1}^*=|u_{r_1+1}|^2, \cdots, y_n^*=|u_n|^2$. Hence y_j^* is real and positive for every $j=1, \cdots, n$ and $y_{r_1+j}^*=y_{r_1+r_2+j}^*$ for every $j=1, \cdots, r_2$. We define $e_j=1$ for $j\leqq r_1$ and $=2$ for $j>r_1$. For $m={}^t(m_1, \cdots, m_n)\in \boldsymbol{Z}^n$, we have

$$ {}^{t}M(u)SM(u)[m] = {}^{t}\bar{\Omega}\begin{pmatrix} y_1^* & & \\ & \ddots & \\ & & y_n^* \end{pmatrix}\Omega[m] = \sum_{j=1}^{r_1+r_2} e_j |\mu^{(j)}|^2 y_j^* , $$

where we write

$$ \mu = (\omega_1, \cdots, \omega_n)m , $$

which is an element of $\mathfrak{a}$.

Let $\eta_1, \cdots, \eta_r$ be a system of fundamental units in Γ, where as usual $r=r_1+r_2-1$. Let w be the number of roots of 1 in K. Every η of Γ can be written uniquely as

$$ \eta = \nu^c \eta_1^{c_1} \cdots \eta_r^{c_r} , $$

with $c_i\in \boldsymbol{Z}$, where ν is a root of 1 and $0\leqq c\leqq w-1$. We take Γ as G in **1** and so $C=\{\nu\}$, $I_j=\{\eta_j\}$. $\boldsymbol{Z}^n$ is taken as $\mathfrak{L}$. We define ρ by $\rho(\eta)=M(\eta)$. Define

$$ (18) \qquad Y = \begin{pmatrix} \dfrac{|\eta_1^{(1)}|^2}{|\eta_1^{(r+1)}|} , & \cdots\cdots , & \dfrac{|\eta_r^{(1)}|^2}{|\eta_r^{(r+1)}|} \\ \vdots & & \vdots \\ \dfrac{|\eta_1^{(r)}|^2}{|\eta_1^{(r+1)}|} , & \cdots\cdots , & \dfrac{|\eta_r^{(r)}|^2}{|\eta_r^{(r+1)}|} \end{pmatrix} . $$

By the straightforward calculation, we can show that $\det(\log Y)$ is equal to non-zero constant times the regulator. Hence $\log Y$ satisfies the condition (4) and we put $L=(\log Y)^{-1}$. By this L, we define the map Λ of $\boldsymbol{H}^r$ onto $\boldsymbol{R}^r$. Also we define the operation $\circ$ of $\xi\in \boldsymbol{Z}^r$ on $\boldsymbol{H}^r$ (see (2)) by

$$ \xi\circ y = Y^{\xi}\cdot y , \qquad y\in \boldsymbol{H}^r . $$

With these data, we have

$$ \Lambda(Y^{\boldsymbol{Z}^r}) = \boldsymbol{Z}^r . $$

We write

$$ Q(y^*) = {}^{t}\bar{\Omega}\begin{pmatrix} y_1^* & & \\ & \ddots & \\ & & y_n^* \end{pmatrix}\Omega $$

and consider the expression

(19) $$\frac{|\det Q(y^*)|^{(1/2)s}}{(Q(y^*)[m])^{(1/2)ns}},$$

which is equal to

$$\frac{d_a(s)\prod_{j=1}^{r_1+r_2} y_j^{*(1/2)e_js}}{(\sum_{j=1}^{r_1+r_2} e_j|\mu^{(j)}|^2y_j^*)^{(1/2)ns}},$$

where $d_a(s)=|\det {}^t\bar{\Omega}\Omega|^{(1/2)s}$. $y^*={}^t(y_1^*, \cdots, y_n^*)$ belongs to T and we introduce new variables $y_j=y_j^*/y_{r+1}^*$, $j=1, \cdots, r$. Then $y={}^t(y_1, \cdots, y_r)$ belongs to $\boldsymbol{H}^r$. With these y_j, we can write the above in the following form:

(20) $$\frac{d_a(s)\prod_{j=1}^{r} y_j^{(1/2)e_js}}{(\sum_{j=1}^{r} e_j|\mu^{(j)}|^2y_j+e_{r+1}|\mu^{(r+1)}|^2)^{(1/2)ns}}.$$

We define this as $P(m, y)^{-s}$ and put

(21) $$\Phi(P, y, s)=\sum_{m\in\boldsymbol{Z}^n}{}' P(m, y)^{-s}.$$

This is nothing but an Epstein zeta-function and it is well-known that (21) satisfies the condition (K) with $\sigma_0=1$.

The group $G=\Gamma$ is diagonalized by Ω. It is easily seen that for such G, we have

(22) $$K_\rho=I(m)=1 \qquad \text{for any } m\in\boldsymbol{Z}^n.$$

Moreover, by the choice of P, we can show that

(23) $$K_P(m)=\{\nu\}=C \qquad \text{for any } m\in\boldsymbol{Z}^n.$$

Therefore, $n_{\{m\}}=w$ and $g_{\{m\}}=1$ for any m.

We shall show that P of (20) satisfies the condition (P.2): namely

(24) $$P(\rho(\eta)m, y)=P(\rho(\eta_0)m, \iota(\eta_\infty)\circ y) \qquad \text{for } \eta\in\Gamma.$$

By (22), we only need to show this for $\eta\in\{\eta_1, \cdots, \eta_r\}$.

The transformation $m\to\rho(\eta)m$ corresponds to the transformation $\mu\to\eta\mu$, since $\mu=(\omega_1, \cdots, \omega_n)m$. Then observe that $Q(y^*)[m]$ is written in terms of μ. Then the above transformation gives rise to the transformation $y_j^*\to|\eta^{(j)}|^2y_j^*$, which gives

$$y_j \longrightarrow |\eta^{(j)}/\eta^{(r+1)}|^2y_j.$$

This means $y\to Y^{\iota(\eta)}\cdot y=\iota(\eta)\circ y$, which completes the proof of (24).

Now our P satisfies the condition (P.1). This is obvious from (20). Thus we can consider the Siegel's zeta-function $f_\kappa(s, P)$ for our data $\Xi_k=(G=\Gamma, \mathfrak{L}=\boldsymbol{Z}^n, P, Y, \Lambda, \rho=M)$. We shall show that our Siegel's zeta-function is essentially equal to the zeta-function $\zeta_K(s, \tilde{\chi})$ of K with some Grössen-character $\tilde{\chi}$.

First we recall the definition of $\zeta_K(s, \tilde{\chi})$. Let $\tilde{\chi}$ be a Grössen-character of K. We put

$$\zeta_K(s, \tilde{\chi})=\sum_{\mathfrak{b}:\ \text{integral}}\tilde{\chi}(\mathfrak{b})N(\mathfrak{b})^{-s}.$$

We have

$$\zeta_K(s, \tilde{\chi})=\sum_{A}\sum_{0\neq\mathfrak{b}\in A}\tilde{\chi}(\mathfrak{b})N(\mathfrak{b})^{-s},$$

where A runs over all classes in the wide sense. Taking $\mathfrak{a}\in A^{-1}$, we have

$$\begin{aligned}\zeta_K(s, \tilde{\chi}, A)&=\sum_{0\neq\mathfrak{b}\in A}\tilde{\chi}(\mathfrak{b})N(\mathfrak{b})^{-s} \\ &=\tilde{\chi}(\mathfrak{a})^{-1}N(\mathfrak{a})^{-s}\sum_{0\neq(\mu)\subset\mathfrak{a}}\tilde{\chi}((\mu))\,|N(\mu)|^{-s}.\end{aligned} \tag{25}$$

Now we have

$$e^{-2\pi i t\kappa\cdot\Lambda(y)}=\prod_{j=1}^{r}y_j^{-2\pi i h_j}$$

with

$$(h_1, \cdots, h_r)={}^t\kappa\cdot L,$$

$$J(v, y)=|\det L|\prod_{j=1}^{r}y_j^{-1},$$

$$\prod_{j=1}^{r}e_j^{2\pi i h_j}/e_{r+1}^{2\pi i\sum_{j=1}^{r}h_j}=1$$

and

$$\prod_{j=1}^{r}e_j^{(1/2)e_j s}=2^{r_2 s}.$$

Put

$$c=w\,|\det L|\,d_{\mathfrak{a}}(s)$$

and define a Grössen-character $\tilde{\chi}_\kappa$ by

$$\tilde{\chi}_\kappa((\mu))=\prod_{j=1}^{r}|\mu^{(j)}|^{4\pi i h_j}/|\mu^{(r+1)}|^{4\pi i\sum_{j=1}^{r}h_j}.$$

Let A be the inverse class of $\mathfrak{a}$. Then by (15), (20) and (24), we have

$$\begin{aligned}f_\kappa(s, P)&=c\sum_{0\neq(\mu)\in\mathfrak{a}}\int_0^\infty\!\!\cdots\!\int\frac{\prod_{j=1}^{r}y_j^{(1/2)e_j s-2\pi i h_j}y_j^{-1}dy_j}{\left(\sum_{j=1}^{r}e_j|\mu^{(j)}|^2y_j+e_{r+1}|\mu^{(r+1)}|^2\right)^{(1/2)ns}} \\ &=c\sum_{0\neq(\mu)\subset\mathfrak{a}}\frac{1}{e_{r+1}^{(1/2)ns}|\mu^{(r+1)}|^{ns}}\int_0^\infty\!\!\cdots\!\int\frac{\prod_{j=1}^{r}y_j^{(1/2)e_j s-2\pi i h_j}y_j^{-1}dy_j}{\left(\sum_{j=1}^{r}\frac{e_j}{e_{r+1}}\left|\frac{\mu^{(j)}}{\mu^{(r+1)}}\right|^2y_j+1\right)^{(1/2)ns}}\end{aligned}$$

$$=c2^{r_2 s}\sum_{0\neq(\mu)\subset\mathfrak{a}}\tilde{\chi}_\kappa((\mu))\,|N(\mu)|^{-s}\Gamma(\kappa, s)\,,$$

with

$$\Gamma(\kappa, s)=\int_0^\infty\cdots\int\frac{\prod_{j=1}^{r}t_j^{(1/2)e_js-2\pi ih_j}t_j^{-1}dt_j}{(\sum_{j=1}^{r}t_j+1)^{(1/2)ns}}$$

$$=\frac{\prod_{j=1}^{r}\Gamma((1/2)e_js-2\pi ih_j)\Gamma((1/2)e_{r+1}s+2\pi i\sum_{j=1}^{r}h_j)}{\Gamma((1/2)ns)}\,.$$

In the above computation, we used that $n_{(m)}=w$, $g_{(m)}=1$ and $C=K_P(m)$ for every $m\in\boldsymbol{Z}^n$.

Let d be the discriminant of K. Then $d^{(1/2)s}=d_\mathfrak{a}(s)N(\mathfrak{a})^s$. We sum up the above result in

THEOREM. *Let K be an algebraic number field of degree n and d the discriminant of K. Let $\mathfrak{a}$ be an ideal of K. Then the Siegel's zeta function for the data $\Xi_K=(\Gamma, \boldsymbol{Z}^n, P, Y, \Lambda, M)$ is equal to*

$$w\,|\det L\,|\,d^{(1/2)s}\tilde{\chi}_\kappa(\mathfrak{a})\Gamma(\kappa, s)\zeta_K(s, \tilde{\chi}_\kappa, A)\,,$$

where w is the number of roots of 1 in K, L is $(\log Y)^{-1}$, A is the inverse class of $\mathfrak{a}$.

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