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Lie algebra of vector fields and complex structure

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It was shown by [1] (also by [2] in compact case) that the structure of a smooth manifold M with countable basis is completely determined by the algebraic structure of the Lie algebra of smooth vector fields on M. In connection with this, K. Shiga posed the problem: whether or not the complex structure of a complex manifold is determined by the structure of the Lie algebra of vector fields of type (1, 0). The present paper is to give the affirmative answer to the problem together with some generalization. In this paper, all manifolds are assumed to have countable bases.

Let M be a complex manifold and $z_i=x_i+\sqrt{-1}\,y_i$ $(i=1, 2, \cdots, n)$ complex analytic coordinate in a neighbourhood of a point p of M. Complexified tangent vector at p is said to be of type (1, 0) if it is a complex linear combination of

$$\frac{\partial}{\partial z_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}-\sqrt{-1}\frac{\partial}{\partial y_i}\right) \qquad (i=1, 2, \cdots, n).$$

The set of all the tangent vectors of type (1, 0) costitutes a complex subbundle of the complexified tangent bundle of M. Smooth sections of this subbundle are called vector fields of type (1, 0), the totality of which forms a subalgebra $\mathfrak{A}_{\partial}(M)$ of the Lie algebra $\mathfrak{A}(M)$ of complex valued vector fields on M.

Now our main result can be formulated as follows:

THEOREM 1. *Let M and M' be complex manifolds and φ a Lie algebraic isomorphism of $\mathfrak{A}_{\partial}(M)$ to $\mathfrak{A}_{\partial}(M')$. Then there exists a biholomorphic map σ of M onto M' such that φ is induced by σ, that is,*

$$\varphi=\sigma_* .$$

Let us consider a more general situation. Let M be a smooth manifold. We denote by $C^\infty(M)$ the set of all real valued smooth functions on M. A real subalgebra A of $\mathfrak{A}(M)$ is said to be a quasi-foliation of M, if A satisfies the following conditions:

i) A is a module over $C^\infty(M)$, i.e., $X\in A$ implies $fX\in A$ for every $f\in C^\infty(M)$.

ii) For any point p of M, there exists $X\in A$ with $X_p\neq 0$.

iii) If $X_i\in A$ for $i=1, 2, \cdots$ and their supports forms a locally finite family, then the sum $X=\sum_i X_i$ belongs to A.

For a real or complex subbundle E of the complexified tangent bundle of M, $\Gamma(E)$, the space of smooth sections of E, is a quasi-foliation if and only if $\Gamma(E)$ forms a subalgebra of $\mathfrak{A}(M)$. Thus a usual foliation on M (considered as the set of smooth sections), the set of all smooth vector fields, and the set of all vector fields of type (1, 0) on a complex manifold are special cases of quasi-foliation.

Now Theorem 1 can be generalized into

THEOREM 2. *Let A and A' be quasi-foliations of smooth manifolds M and M' respectively and φ an isomorphism of A to A'. Then there exists a diffeomorphism σ of M onto M' such that $\sigma_*(A)=A'$ and φ coincides with σ_* on A.*

Theorem 1 is a consequence of Theorem 2. In fact, the latter guarantees the existence of a diffeomorphism σ of M to M' with $\sigma_*(\mathfrak{A}_\partial(M))=\mathfrak{A}_\partial(M')$. Therefore it is only necessary to show that σ is biholomorphic. This is an immediate consequence of the fact that a smooth function f is holomorphic in an open set U whenever $X(\bar{f})$ vanishes on U for every $X\in\mathfrak{A}_\partial(M)$.

The proof of Theorem 2 is based on the following fact.

THEOREM 3. *Let A be a quasi-foliation of a smooth manifold M. Then a subalgebra B of A is a maximal proper subalgebra of finite codimension if and only if B coincides with $N_p(A)=\{X\in A\,;\, X_p=0\}$ for some $p\in M$.*

PROOF. It is obvious that $N_p(A)$ is a proper subalgebra of A with finite codimension. Therefore it suffices to prove that every proper subalgebra B of A with finite codimension is contained in $N_p(A)$ for some point $p\in M$. Now we suppose $B\not\subset N_p(A)$ for every $p\in M$ and will show that it leads to a contradiction.

For an open set U of M, A_U denotes the set of all elements of A with the supports contained in U; U is said to be admissible if there exist $Y\in B$ and $f\in C^\infty(M)$ such that $Y(f)$ does not vanish in U. Then we will prove:

(i) If U is admissible, then $A_U\subset B$, and

(ii) M is covered by a finite number of admissible open sets.

If (i) and (ii) are true and $M=U_1\cup U_2\cup\cdots\cup U_k$, each U_i being admissible, then we have $A=A_{U_1}+A_{U_2}+\cdots+A_{U_k}\subset B$ contradicting the properness of B.

Proof of (i). Put $B'=\{X\in B\,;\,[X, Y]\in B$ for every $Y\in A\}$. Then B' is evidently an ideal of B. For $X\in B$, $adX: Y\to[X, Y]$ induces a linear transformation T_X of the finite-dimensional space A/B. Thus B', as the kernel of the map $X\to T_X$ of B into the space of endomorphisms of A/B, is of finite codimension in B and hence in A.

By assumption, there exist $Y\in B$ and $f\in C^\infty(M)$ such that $Y(f)$ does not vanish in U. The set of $g\in C^\infty(M)$ with $gY\in B'$ and $fgY\in B'$ is a subspace of $C^\infty(M)$ with finite codimension, since B' is of finite codimension in A as shown above. Put $E=\{g\in C^{\infty}(M); gA_U\in B\}$. First we will prove that the codimension of E in $C^{\infty}(M)$ is finite by showing that $gY\in B'$ and $fgY\in B'$ imply $g\in E$.

Now we have, for any $X\in A_U$,

$$B\ni[fgY, X]=f[gY, X]-X(f)gY$$

$$B\ni[gY, fX]=f[gY, X]+gY(f)X$$

and hence

$$B\ni X(f)gY+gY(f)X\,. \tag{*}$$

Substituting $(1/Y(f))X(f)Y$ for X in (*), we obtain

$$B\ni X(f)gY$$

which, combined with (*), gives

$$B\ni gY(f)X\,.$$

Again substituting $(1/Y(f))X$ for X, we have finally

$$gX\in B\,.$$

Thus we have proved that $gA_U\in B$, or $g\in E$.

If $g\in E$, then we have, for any $X\in A_U$,

$$B\ni[gX, Y]=g[X, Y]-Y(g)X$$

and, since $g[X, Y]\in gA_U\subset B$,

$$B\ni Y(g)X\,.$$

Thus $g\in E$ implies $Y(g)\in E$. Since E is of finite codimension in $C^{\infty}(M)$ as proved above, we can find a non-zero polynomial P such that $P(f)\in E$. Then, by the fact just proved, we have also $Y(P(f))=P'(f)Y(f)\in E$. Since $1/Y(f)$ exists in U, we have $(1/Y(f))X\in A_U$ for every $X\in A_U$ and hence $P'(f)Y(f)(1/Y(f))X\in B$, showing $P'(f)\in E$. Applying this argument successively, we have $P''(f)\in E$ etc. and finally we obtain

$$1\in E$$

which is equivalent to $A_U\subset B$. Thus (i) is proved.

Proof of (ii). For each $p\in M$, since $B\not\subset N_p(A)$, we can find $Y\in B$ with $Y_p\neq 0$. Consequently, there exists a neighbourhood U_p of p such that U_p is admissible. According to dimension theory, M admits finite open covering $\{U, \cdots\}$ such that each member U is the union of mutually disjoint open sets U_i, $i=1, 2, \cdots$, where $\{U_i\}$ constitutes a locally finite family of subsets in M

and each U_i is contained in some U_p and hence admissible. Moreover we can replace each U_i by an open set V_i with $\bar{V}_i \subset U_i$ so that we have a finite open covering $\{V, \cdots\}$ of M replacing each U by $V = \bigcup_i V_i$.

Since U_i is admissible, we can find $Z_i \in B$ and $g_i \in C^\infty(M)$ such that $Z_i(g_i)$ does not vanish in U_i. Then multiplying a suitable smooth function to both z_i and g_i, we obtain $Y_i \in B$ and $f_i \in C^\infty(M)$ such that the supports of Y_i and f_i are both contained in U_i and $Y_i(f_i)$ does not vanish in V_i. Now we can define a smooth function $f = \sum_i f_i$ and, for any sequence α_i of real numbers, $\sum_i \alpha_i Y_i$ which belongs to A by the condition (iii) of quasi-foliation.

Since B is of finite codimension in A, we can find a non-zero polynomial P such that $\sum_i P(i) Y_i \in B$. If m is so chosen that $m \leqq i$ implies $P(i) \neq 0$, then, since all Y_i and $P(i)Y_i$ belong to A_{U_i} and hence, by (i), to B, we have

$$Y = \sum_{i=1}^{m-1} Y_i + \sum_{i=m}^{\infty} P(i) Y_i \in B$$

and $Y(f)$ does not vanish in V, which shows that V is admissible.

Thus the proof is completed.

The rest of the paper is devoted to the

PROOF OF THEOREM 2. Let A, A', M, M', and φ be as stated in the theorem.

Let us call $N_p(A)$ a point-subalgebra of A. Since this notion is purely algebraic by Theorem 3, φ maps any point-subalgebra of A to a point-subalgebra of A'. Therefore there exists a map σ of M to M' such that

$$\varphi(N_p(A)) = N_{\sigma(p)}(A') \qquad \text{for every } p \in M.$$

It is obvious that σ is bijective.

For $f \in C^\infty(M)$, put $g = f \circ \sigma^{-1}$, then, for every $p \in M$ and every $X \in A$, we have

$$fX - f(p)X \in N_p(A)$$

and hence

$$\varphi(fX) - g(\sigma(p))\varphi(X) = \varphi\{fX - f(p)X\} \in N_{\sigma(p)}(A')$$

which shows

$$(*) \qquad \varphi(fX) = g\varphi(X).$$

Since $\varphi(X)$ can be any element of A', g is a smooth function on M'. This means that σ^{-1} is a smooth map and, similarly, σ is also smooth. Hence σ is a diffeomorphism.

σ induces the isomorphism σ_* of $\mathfrak{A}(M)$ to $\mathfrak{A}(M')$ so that its restriction to A is an isomorphism of A to $A'' = \sigma_*(A)$. Then the iteration $\sigma_* \circ \varphi^{-1}$ is an isomorphism of A' to A'' which maps $N_{p'}(A')$ to $N_{p'}(A'')$ for every $p' \in M'$. Therefore it is sufficient to consider the special case where $M = M'$ and σ is the identity; we have only to prove that $A = A'$ and that φ is the identity.

Now for $f \in C^{\infty}(M)$ and $X \in A$, we have by (*)

$$\varphi(fX) = f\varphi(X).$$

Therefore we have

$$\begin{aligned} X(f)\varphi(X) &= \varphi(X(f)X) \\ &= \varphi[X, fX] \\ &= [\varphi(X), \varphi(fX)] \\ &= [\varphi(X), f\varphi(X)] \\ &= \varphi(X)(f)\varphi(X) \end{aligned}$$

and, $\varphi(X) \in A'$ being arbitrary,

$$X(f) = \varphi(X)(f).$$

Here f is also arbitrary and hence we can conclude

$$X = \varphi(X).$$

Thus the proof is completed.

References

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