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Homeomorphism between the open unit disk and a Gleason part

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§ 1. Introduction.

Let $P(m)$ be the non-trivial Gleason part which contains a complex homomorphism m of a uniform algebra A on a compact space X, and suppose that m has a unique positive representing measure on X (for the definitions see § 2). Then, it is known that there is a one-one continuous map τ of the open unit disk D in the complex plane onto $P(m)$ (in the Gelfand topology) such that for every $f\in A$, $\tau(t)(f)$ is analytic in D (Wermer's embedding theorem). But τ is not necessarily a homeomorphism. Such examples are found in Wermer [**10**], p. 443, Hoffman [**6**], p. 109 and others. The purpose of this paper is to establish some conditions for τ to be a homeomorphism. In § 2 some preliminaries are given. In § 3 we state and prove our results, and an example is studied in relation to our main Theorem 3.2.

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§ 2. Preliminaries.

For a commutative Banach algebra B over the complex numbers, let $\mathcal{M}(B)$ be the maximal ideal space (or the space of complex homomorphisms) of B which has the Gelfand topology, and let $\hat{f}$ be the Gelfand transform of $f\in B$.

Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space X and let A be a *uniform algebra* on X, that is, a closed (by supremum norm $\|\ \|$) subalgebra in $C(X)$ containing constants and separating points of X. For φ in $\mathcal{M}(A)$, $M_{\varphi}=M_{\varphi}(A)$ denotes the set of representing measures on X for φ, i.e., the set of all probability measures μ on X such that $\varphi(f)=\int fd\mu$ for all $f\in A$.

Given φ and θ in $\mathcal{M}(A)$, we set

$$(2.1)\qquad \sigma(\varphi, \theta)=\sup\{|\varphi(f)| : f\in A,\ \|f\|\leqq 1,\ \theta(f)=0\},$$

and write $\varphi\sim\theta$ if and only if $\sigma(\varphi, \theta)<1$. Then $\sim$ is an equivalence relation in $\mathcal{M}(A)$, and an equivalence class $P(m)=\{\varphi\in\mathcal{M}(A):\varphi\sim m\}$ $(\supsetneqq\{m\})$ is called the *(non-trivial) Gleason part* of m in $\mathcal{M}(A)$ (cf. Gleason [**4**]). It is known that the function $\sigma(\varphi, \theta)$ is a (part) metric on $P(m)$ (cf. König [**8**]).

When φ $(\in\mathcal{M}(A))$ has a unique representing measure, we will use the same symbol φ to denote its representing measure. *Throughout this paper we suppose that m $(\in\mathcal{M}(A))$ has a unique representing measure m and that $P(m)$ is a non-trivial Gleason part.* It is known that if φ belongs to $P(m)$ then there is an invertible function h in $L^\infty(m)$ such that $M_\varphi=\{hm\}$ (cf. Gamelin [**3**], p. 143).

Denote by A_m the kernel of a complex homomorphism m. Let $H^\infty(m)$ and H^∞_m be the weak-star closures of A and A_m in $L^\infty(m)$ respectively, and for $1\leqq p<\infty$ let $H^p(m)$ and H^p_m be the closures of A and A_m in $L^p(m)$-norm respectively. Let $\hat{H}^\infty=\{\hat{f}:f\in H^\infty(m)\}$ be the Gelfand transform of the Banach algebra $H^\infty(m)$, and let $\tilde{H}^\infty$ be the restriction of $\hat{H}^\infty$ to Y $(=\mathcal{M}(L^\infty(m))$. Then it is known that $\tilde{H}^\infty$ is a logmodular algebra on Y (cf. Hoffman [**5**] and Browder [**1**], p. 212). Sometimes we shall identify $H^\infty(m)$ with $\tilde{H}^\infty$. Functions in $H^\infty(m)$ of unit modulus are called *inner functions.*

DEFINITION. A map $\rho(t)$ of the open unit disk D in the complex plane onto $P(m)$ (in the relative Gelfand topology) is called an *analytic map* if $\rho(t)$ is a one-one continuous map and the composition $\hat{f}(\rho(t))$ is analytic in D, for every f in A.

THEOREM 2.1 (Wermer's Embedding Theorem). *Let A be a uniform algebra on a compact space X. Suppose that $m\in\mathcal{M}(A)$ has a unique representing measure m on X, and that the Gleason part $P(m)$ of m is non-trivial. Then we have the following.*

(i) *There is an inner function Z such that $ZH^\infty(m)=H^\infty_m$.*

(ii) *For $\varphi\in P(m)$, set $\hat{Z}(\varphi)=\int Zd\varphi$. Then $\hat{Z}$ is a one-one map of the part $P(m)$ onto the open unit disk D in the plane, and the inverse map τ of $\hat{Z}$ is an analytic map.* (Cf. Gamelin [**3**], p. 158.)

If φ belongs to $P(m)$, then it is easily seen that $H^\infty(m)=H^\infty(\varphi)$, and thus that the functional $\tilde{\varphi}$ defined on $H^\infty(m)$ by

$$\tilde{\varphi}(f)=\int fd\varphi$$

is a well-defined element of $\mathcal{M}(H^\infty(m))$. We call $\tilde{\varphi}$ the *measure extension* of φ in $P(m)$.

PROPOSITION 2.2. *Let A, m, $P(m)$ and Z be as in Theorem 2.1. Let $\mathcal{P}=\mathcal{P}(\tilde{m})$ be the set of all measure extensions of elements of $P(m)$. Then we have the following.*

(i) *$\mathcal{P}$ is the non-trivial Gleason part of $\tilde{m}$ in $\mathcal{M}(H^\infty(m))$.*

(ii) *Let $\hat{Z}|\mathcal{P}$ be the restriction of $\hat{Z}$ to $\mathcal{P}$ and let $\tilde{\tau}$ be the inverse map of*

$\hat{Z}|\mathcal{P}$. *Then* $\tilde{\tau}$ *is an analytic map and a homeomorphism of the open unit disk* D *onto* $\mathcal{P}$. (Cf. Kishi [7], Proposition.)

Let Z, P, $\mathcal{P}$, τ and $\tilde{\tau}$ be as in Theorem 2.1 and Proposition 2.2. Then we have, for $\tilde{\tau}(t)=\tilde{\varphi}$,

$$\sigma(\tau(t),\ \tau(s))=\sigma(\tilde{\tau}(t),\ \tilde{\tau}(s))=\left|\tilde{\varphi}\left(\frac{Z-s}{1-\bar{s}Z}\right)\right|=\sigma(t,\ s)\,, \tag{2.2}$$

where $\sigma(t,\ s)=\left|\frac{t-s}{1-\bar{s}t}\right|$ is the pseudo-hyperbolic metric in D. Therefore τ is an isometry of the open unit disk D with the pseudo-hyperbolic metric $\sigma(t,\ s)$ onto the Gleason part $P(m)$ with the part metric $\sigma(\varphi,\ \theta)$. The similar result is true for $\tilde{\tau}$ (see Kishi [7], Theorem 3).

If $\tilde{\varphi}$ belongs to $\mathcal{P}$ (or φ belongs to $P(m)$) and a number ε $(0<\varepsilon<1)$ is given, then there is a constant c $(0<c<1)$ such that

$$\{\tilde{\theta}\in\mathcal{P}(\tilde{m}):\sigma(\tilde{\varphi},\ \tilde{\theta})<\varepsilon\}\subset\{\tilde{\theta}\in\mathcal{P}(\tilde{m}):\sigma(\tilde{m},\ \tilde{\theta})\leqq c<1\} \tag{2.3}$$

and

$$\{\theta\in P(m):\sigma(\varphi,\ \theta)<\varepsilon\}\subset\{\theta\in P(m):\sigma(m,\ \theta)\leqq c<1\}\,. \tag{2.4}$$

In fact, there is a point $t_0\in D$ such that $\tilde{\varphi}=\tilde{\tau}(t_0)$, and in D we have $\{t:\sigma(t_0,\ t)<\varepsilon\}\subset\{t:\sigma(0,\ t)\leqq c<1\}$ for some constant c, so that, by (2.2), we have $\{\tilde{\tau}(t):\sigma(\tilde{\tau}(t_0),\ \tilde{\tau}(t))<\varepsilon\}\subset\{\tilde{\tau}(t):\sigma(\tilde{\tau}(0),\ \tilde{\tau}(t))\leqq c<1\}$, i.e., we obtain (2.3). By (2.2) and (2.3) we have (2.4).

From (2.2) we have, for $\tilde{\tau}(t)=\tilde{\varphi}$,

$$\begin{aligned}\sigma(0,\ t)&=\sigma(m,\ \tau(t))=\sigma(\tilde{m},\ \tilde{\tau}(t))\\&=|\tilde{\varphi}(Z)|.\end{aligned} \tag{2.5}$$

Hence we have $\mathcal{P}\subset\{\Phi:\Phi\in\mathcal{M}(H^\infty(m)),\ |\Phi(Z)|<1\}$. On the other hand, from (i) of Theorem 2.1, if Φ belongs to $\mathcal{M}(H^\infty(m))-\mathcal{P}$, then we have

$$\begin{aligned}1&=\sup\{|\Phi(f)|:f\in H^\infty(m),\ \|f\|\leqq 1,\ \tilde{m}(f)=0\}\\&=\sup\{|\Phi(Z)\Phi(g)|:g\in H^\infty(m),\ \|g\|\leqq 1\}\\&=|\Phi(Z)|.\end{aligned}$$

Hence we have

$$\mathcal{P}=\{\Phi:\Phi\in\mathcal{M}(H^\infty(m)),\ |\Phi(Z)|<1\}\,, \tag{2.6}$$

and we see that $\mathcal{P}$ is an open set in the space $\mathcal{M}(H^\infty(m))$. By Proposition 2.2 and (2.2), we see that, for any $\tilde{\varphi}$ in $\mathcal{P}$, $V_\varepsilon(\tilde{\varphi})=\{\tilde{\theta}:\sigma(\tilde{\varphi},\ \tilde{\theta})<\varepsilon<1\}$ is an open set in the space $\mathcal{M}(H^\infty(m))$, and that $\{V_\varepsilon(\tilde{\varphi}):0<\varepsilon<1\}$ is a fundamental neighborhood system of $\tilde{\varphi}$ in the subspace $\mathcal{P}$ of $\mathcal{M}(H^\infty(m))$.

For every Φ in $\mathcal{M}(H^\infty(m))$, denote by $\pi(\Phi)$ the restriction of Φ to A, i.e., $\pi(\Phi)=\Phi|A$, and denote by π_1 the restriction of π to $\mathcal{P}$, i.e., $\pi_1=\pi|\mathcal{P}$. Then π is a continuous map of $\mathcal{M}(H^\infty(m))$ into $\mathcal{M}(A)$. It is easily seen that $\pi(\mathcal{P})=P$

and $\pi(\bar{\mathscr{P}})=\bar{P}$, where $\bar{\mathscr{P}}$ and $\bar{P}$ are the closures of $\mathscr{P}$ and P in $\mathscr{M}(H^\infty(m))$ and $\mathscr{M}(A)$ respectively.

§ 3. Results.

First we shall prove the following lemma.

LEMMA 3.1. *Let A, m, $P(m)$, Z and τ be as in Theorem* 2.1. *Then the following are equivalent.*

(i) *τ^{-1} is continuous at a point φ in the subspace $P(m)$ of the space $\mathscr{M}(A)$.*

(ii) *π_1^{-1} is continuous at a point φ in the subspace $P(m)$.*

(iii) *There are an open neighborhood $V(\varphi)$ of a point φ in the subspace $P(m)$ and a positive constant c such that*

$$V(\varphi)\subset\{\theta\in P(m):\sigma(m,\theta)\leqq c<1\}.$$

PROOF. Let $\mathscr{P}$ and $\tilde{\tau}$ be as in Proposition 2.2. By using $\tau=\pi_1\circ\tilde{\tau}$ the equivalence of (i) and (ii) is easily seen.

(ii) $\Rightarrow$ (iii). For each neighborhood $V_\varepsilon(\tilde{\varphi})=\{\tilde{\theta}:\sigma(\tilde{\varphi},\tilde{\theta})<\varepsilon<1\}$ of $\tilde{\varphi}$ in the subspace $\mathscr{P}$ of the space $\mathscr{M}(H^\infty(m))$ there is an open neighborhood $V(\varphi)$ of φ in the subspace $P(m)$ such that $\pi_1^{-1}(V(\varphi))\subset V_\varepsilon(\tilde{\varphi})$. On the other hand, by (2.3), there is a constant c such that $V_\varepsilon(\tilde{\varphi})\subset\{\tilde{\theta}\in\mathscr{P}:\sigma(\tilde{m},\tilde{\theta})\leqq c<1\}$. Therefore, by (2.2), we have $V(\varphi)\subset\{\theta\in P(m):\sigma(m,\theta)\leqq c<1\}$.

(iii) $\Rightarrow$ (ii). Let $\tilde{\tau}(t_0)=\tilde{\varphi}$ and let $V_\varepsilon(\tilde{\varphi})=\left\{\tilde{\theta}:\sigma(\tilde{\varphi},\tilde{\theta})=\left|\tilde{\theta}\left(\frac{Z-t_0}{1-\bar{t}_0Z}\right)\right|<\varepsilon<1\right\}$ be any neighborhood of $\tilde{\varphi}$ in the subspace $\mathscr{P}$ (see (2.2)). Moreover, put $V(\varphi)=\{\varphi_\lambda:\lambda\in\Lambda\}$ and $\varphi=\varphi_{\lambda_0}$. Now take $\varphi_\lambda\in V(\varphi)$, then we have $\tau(s)=\varphi_\lambda$ for some complex number $s\in D$ and $M_{\varphi_\lambda}=\left\{\frac{1-|s|^2}{|1-\bar{s}Z|^2}m\right\}=\{h_\lambda m\}$ (cf. Gamelin [3], p. 133). So, by (2.5), we have $\frac{1-c}{1+c}\leqq h_\lambda\leqq\frac{1+c}{1-c}$.

Put $F=\frac{Z-t_0}{1-\bar{t}_0Z}$. Then, by Hoffman-Wermer theorem (cf. Browder [1], Theorem 4.2.5), there is a sequence $\{f_n\}$ in A such that $\|f_n\|\leqq 1$ and $f_n\to F$ a.e. (dm) as $n\to\infty$. If n_0 is sufficiently large, then we have, for all $\lambda\in\Lambda$,

$$|(\tilde{\varphi}-\tilde{\varphi}_\lambda)(F-f_{n_0})|=\left|\int(F-f_{n_0})(h_{\lambda_0}-h_\lambda)dm\right|$$

$$\leqq\frac{2(1+c)}{1-c}\int|F-f_{n_0}|dm<\varepsilon/2.$$

Set $W(\varphi)=\{\varphi_\lambda:\varphi_\lambda\in V(\varphi),\ |\varphi_\lambda(f_{n_0})-\varphi(f_{n_0})|<\varepsilon/2\}$. Then $W(\varphi)$ is an open neighborhood of φ in the subspace $P(m)$, and we have, for all $\varphi_\lambda\in W(\varphi)$,

$$|(\tilde{\varphi}-\tilde{\varphi}_\lambda)(F)|\leqq|(\tilde{\varphi}-\tilde{\varphi}_\lambda)(F-f_{n_0})|+|(\tilde{\varphi}-\tilde{\varphi}_\lambda)(f_{n_0})|<\varepsilon.$$

That is, we obtain $\pi_1^{-1}(W(\varphi)) \subset V_\varepsilon(\tilde{\varphi})$. Thus we have completed the proof of Lemma 3.1.

We are now in a position to state and prove our main result in this paper.

THEOREM 3.2. *Let A be a uniform algebra on a compact Hausdorff space X. Suppose that m ($\in \mathcal{M}(A)$) has a unique representing measure m on X and that the part P of m is non-trivial. Let $\mathcal{P}$, Z, τ and $\tilde{\tau}$ be as in Theorem 2.1 and Proposition 2.2. Then the following are equivalent.*

(i) *An analytic map $\rho(t)$ is a homeomorphism of the open unit disk D onto the subspace P of the space $\mathcal{M}(A)$.*

(ii) *$\tau(t)$ is a homeomorphism of D onto the subspace P.*

(iii) *π_1 is a homeomorphism of the subspace $\mathcal{P}$ of the space $\mathcal{M}(H^\infty(m))$ onto the subspace P.*

(iv) *Every φ in the subspace P has a unique extension Φ in the closure $\bar{\mathcal{P}}$ of a set $\mathcal{P}$ in the space $\mathcal{M}(H^\infty(m))$, i.e., there exists a unique point Φ in $\bar{\mathcal{P}}$ such that $\pi(\Phi)=\varphi$.*

(v) (a) *There exist a point φ ($\in P$) and an open neighborhood $V(\varphi)$ of φ in the subspace P such that the closure $\overline{V(\varphi)}$ of $V(\varphi)$ in the subspace P is compact (i.e., P is locally compact at some point φ in P).*

(b) *If U_1 and U_2 are homeomorphic subsets of the subspace P, and U_1 is open in P, then U_2 is also open in P.*

PROOF. The equivalence of (i) and (ii) is proved in Kishi [7], Theorem 2, and the equivalence of (ii) and (iii) is obvious.

(iii) $\Rightarrow$ (iv). Suppose that there exist φ in P and Φ in $\partial\mathcal{P} = \bar{\mathcal{P}} - \mathcal{P}$ such that $\pi(\Phi)=\varphi$. For every open neighborhood $V(\varphi)$ of φ in the subspace P there exists an open neighborhood $W(\varphi)$ of φ in the subspace $\bar{P}$ such that $V(\varphi) = W(\varphi) \cap P$. Now put $\pi_2 = \pi | \bar{\mathcal{P}}$. Then π_2 is a continuous map of $\bar{\mathcal{P}}$ onto $\bar{P}$, so there is an open neighborhood $V(\Phi)$ of Φ in the subspace $\bar{\mathcal{P}}$ such that $\pi_2(V(\Phi)) \subset W(\varphi)$. On the other hand, since Φ belongs to $\bar{\mathcal{P}}$, we can find a net $\{\tilde{\varphi}_j\}$ ($\subset \mathcal{P} \cap V(\Phi)$) such that $\tilde{\varphi}_j(f) \to \Phi(f)$ for every $f \in H^\infty(m)$. Since, by (2.5) and (2.6), we have $\sigma(\tilde{m}, \tilde{\varphi}_j) = |\tilde{\varphi}_j(Z)|$ and $|\Phi(Z)|=1$, it follows that $\sigma(\tilde{m}, \tilde{\varphi}_j) \to 1$. Therefore, by $\pi(\tilde{\varphi}_j) = \varphi_j \in P \cap W(\varphi) = V(\varphi)$ and (2.2), we obtain $\sup\{\sigma(m, \theta) : \theta \in V(\varphi)\} = 1$. Thus, by Lemma (3.1), π_1^{-1} is not continuous at φ. This contradicts (iii).

(iv) $\Rightarrow$ (iii). Suppose that π_1^{-1} is not continuous at some point $\varphi \in P$. Then there is a net $\{\varphi_j\}$ ($\subset P$) such that $\varphi_j \to \varphi$ but $\tilde{\varphi}_j$ does not converge to $\tilde{\varphi}$. But, since $\bar{\mathcal{P}}$ is a compact subset of the space $\mathcal{M}(H^\infty(m))$, there is a subnet $\{\tilde{\varphi}_{j(k)}\}$ of $\{\tilde{\varphi}_j\}$ such that $\tilde{\varphi}_{j(k)} \to \Phi$ ($\in \bar{\mathcal{P}}$), $\Phi \neq \tilde{\varphi}$. Then we have $\varphi_{j(k)}(f) = \tilde{\varphi}_{j(k)}(f) \to \varphi(f) = \Phi(f)$ for every f in A, and hence we see that $\Phi \in \partial\mathcal{P}$ and $\pi(\Phi) = \varphi$. This contradicts (iv).

(ii) $\Rightarrow$ (v). If W_1 and W_2 are homeomorphic subsets of D, and W_1 is open in D, then W_2 is also open in D (Brouwer's theorem on the invariance of domain). (Cf. S. Eilenberg and N. Steenrod [**2**], p. 303.) Therefore (ii) implies (v).

(v) $\Rightarrow$ (ii). Set $S=\tau^{-1}(V(\varphi))(\subset D)$ and let $\{V_\varepsilon(t): t\in S\}$ be a covering of S, where $V_\varepsilon(t)=\{s\in D: \sigma(t, s)<\varepsilon<1\}$. Then, there is a countable set $\{t_1, t_2, \cdots, t_n, \cdots\}$ in S such that $S\subset \bigcup_{n=1}^{\infty} V_\varepsilon(t_n)$ (Lindelöf's covering theorem). Hence we have $V(\varphi)\subset \bigcup_{n=1}^{\infty} V_\varepsilon(\varphi_n)$, where $\varphi_n=\tau(t_n)$ and $V_\varepsilon(\varphi_n)=\{\theta\in \mathcal{M}(A): \sigma(\varphi_n, \theta)<\varepsilon\}$ (see (2.2)). Put $V(\varphi)=\bigcup_{n=1}^{\infty} V_n$, where $V_n=V(\varphi)\cap V_\varepsilon(\varphi_n)$. Then, since $V(\varphi)$ is a locally compact (sub-)space, there is a set $V_{n_0}\in\{V_n\}$ such that the interior W of the closure $\bar{V}_{n_0}$ of V_{n_0} in the space $V(\varphi)$ is not empty (Baire's category theorem). Then W is an open subset of a subspace $V(\varphi)$ and $V(\varphi)$ is an open subset of the subspace P, so W is an open subset of the subspace P. Since $\{\theta: \sigma(\varphi_{n_0}, \theta)\leqq\varepsilon\}$ is a compact set in the space $\mathcal{M}(A)$, we have $\bar{V}_{n_0}\subset V(\varphi)\cap \{\theta: \sigma(\varphi_{n_0}, \theta)\leqq\varepsilon\}$, and hence we obtain $W\subset V(\varphi)\cap\{\theta: \sigma(\varphi_n, \theta)\leqq\varepsilon\}$. By (2.4), there is a constant c such that $W\subset\{\theta: \sigma(\varphi_n, \theta)\leqq\varepsilon\}\subset\{\theta: \sigma(m, \theta)\leqq c<1\}$. Therefore, by Lemma 3.1, τ^{-1} is continuous in W, so the map τ is a homeomorphism of an open set $\tau^{-1}(W)$ in D onto an open set W in the subspace P.

For a (fixed) point t_0 in $\tau^{-1}(W)$ we can find a constant η $(0<\eta<1)$ such that $\{t: \sigma(t_0, t)<\eta\}$ is contained in $\tau^{-1}(W)$. Then the map τ is a homeomorphism of $\{t: \sigma(t_0, t)<\eta\}$ in D onto an open set $\{\tau(t): \sigma(\tau(t_0), \tau(t))<\eta\}$ in the subspace P (see (2.2)). And if s_0 is any point in D, then we can find a homeomorphism κ of $\{t: \sigma(t_0, t)<\eta\}$ onto $\{t: \sigma(s_0, t)<\eta\}$. On the other hand, since τ is a homeomorphism of a compact subspace $\{t: \sigma(s_0, t)\leqq\eta\}$ of D onto a compact subspace $\{\tau(t): \sigma(\tau(s_0), \tau(t))\leqq\eta\}$ of P, so τ is a homeomorphism of $\{t: \sigma(s_0, t)<\eta\}$ onto $\{\tau(t): \sigma(\tau(s_0), \tau(t))<\eta\}$. Put $\tau(t_0)=\varphi_0$ and $\tau(s_0)=\theta_0$.

Now set $T=\tau\circ\kappa\circ\tau^{-1}$. Then T is a homeomorphism of an open set $\{\theta: \sigma(\varphi_0, \theta)<\eta\}$ in the subspace P onto a set $\{\theta: \sigma(\theta_0, \theta)<\eta\}$ in P, so the set $\{\theta: \sigma(\theta_0, \theta)<\eta\}$ is an open set in the subspace P. (Here we use the hypothesis (b) of (v)). Then, since we have $\{\theta: \sigma(\theta_0, \theta)<\eta\}\subset\{\theta: \sigma(m, \theta)\leqq c'<1\}$ for some constant c' (see (2.4)), it follows from Lemma (3.1) that τ^{-1} is continuous at θ_0. Therefore Theorem 3.2 is proved.

COROLLARY 3.3. *Let P be as in Theorem* 3.2. *If the subspace P is locally euclidean of dimension* 2, *then P is homeomorphic to D.*

PROOF. If the subspace P is locally euclidean of dimension 2, then P satisfies the assertion (v) of Theorem 3.2 (cf. S. Eilenberg and N. Steenrod [**2**], p. 303). So we get the corollary.

EXAMPLE. Let X be the torus, represented as the space of pairs (θ, φ), $0\leqq\theta, \varphi\leqq 2\pi$, with the natural identifications. Fix a positive irrational number α. Let A be the algebra of all continuous functions on X which admit Fourier series of the form:

$$\sum_{n+m\alpha\geqq 0} c_{nm}e^{in\theta}e^{im\varphi}.$$

Then A is a dirichlet algebra (for details of this algebra see Wermer [**10**]). Let M' denote the set of points (z, w) in the space C^2 of two complex variables with $|w|=|z|^{\alpha}$ and $|z|$, $|w|\leqq 1$. Then we can identify $\mathcal{M}(A)$ and M' as topological spaces.

Fix a real number b. Let S_b be the analytic surface: $w=e^{ib}z^{\alpha}$, $0<|z|<1$. Then S_b is a non-trivial Gleason part, and S_b is dense in $\mathcal{M}(A)$ with the Gelfand topology.

Let m be a (fixed) point in S_b, and let $A=\{\varphi: \sigma(m, \varphi)\leqq r\}$ for a real number r $(0\leqq r<1)$, and let $B=\{\varphi: r<\sigma(m, \varphi)<1\}$. Then $S_b=A\cup B$ and $\mathcal{M}(A)=\bar{S}_b=\bar{A}\cup\bar{B}=A\cup\bar{B}$. (For $X\subset\mathcal{M}(A)$, denote by $\bar{X}$ the closure of X in the space $\mathcal{M}(A)$). So $\bar{B}$ contains $S_{b'}$, where $S_{b'}$ is a different analytic surface from S_b. Since $\bar{S}_{b'}=\mathcal{M}(A)$, we have $\bar{B}=\mathcal{M}(A)$. Hence, for every point φ in $S_b=P$ there exists a net $\{\varphi_j\}\subset P$ such that $\varphi_j\to\varphi$ and $\lim\sigma(m, \varphi_j)=1$. Therefore, from Lemma 3.1, π_1^{-1} is not continuous at φ. From this fact and the proof of (v)$\Rightarrow$(ii) in Theorem 3.2, we see that the subspace $P=S_b$ does not satisfy the condition (a) of the assertion (v) in Theorem 3.2.

I don't know whether the above example satisfies the condition (b) of the assertion (v) in Theorem 3.2.

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