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# On the prolongation of local holomorphic solutions of nonlinear partial differential equations

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## § 1. Introduction.

One of important problems in the theory of partial differential equations in the complex domain is the following: When can the holomorphic solutions of a partial differential equation defined in some domain be continued to a larger domain? For linear partial differential equations this question has been answered by several authors. (See e.g. [3] and the references quoted therein.) One of the main results regarding this question is a theorem of M. Zerner [5] which states that the solutions of a linear partial differential equation can be continued holomorphically over any non-characteristic hypersurfaces.

In this paper we are concerned with holomorphic continuation of solutions of general nonlinear partial differential equations in the complex domain. Our purpose here is to present a continuation theorem which corresponds to that of Zerner [5] for linear equations. It can be stated as follows: If the tangent plane at a boundary point of the domain in which the solutions are defined is non-characteristic *for any Cauchy data*, then every *bounded* solution can be holomorphic near that point. Our argument depends on the quantitative property of the domain in which the solutions of the Cauchy problem become holomorphic. In § 2, following carefully the well-known proof ([1], [2]) of the Cauchy-Kowalewsky theorem, we obtain the desired result. Then in § 3 we find the continuation theorem. In the last section, § 4, we study the single equation of the first order with two independent variables for which the assumptions of the results in section 3 are not satisfied. In this case we can construct a solution which cannot be prolonged under some conditions on the boundary of the domain and the characteristic curve.

The results of this paper was already announced in [4] without proofs.

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## § 2. The Cauchy-Kowalewsky theorem.

We consider the following system of quasi-linear first order equations for $N$ unknown functions $w_1(z), \cdots, w_N(z)$ in some domain in the complex $n$ dimensional space $C^n$ with the coordinates $z=(z_1, \cdots, z_n)$.

$$\frac{\partial w_j}{\partial z_1}=\sum_{k=1}^{N}\sum_{l=2}^{n} g_{jkl}(z', w)\frac{\partial w_k}{\partial z_l}+h_j(z', w), \qquad j=1, \cdots, N \tag{1}$$

where $z'=(z_2, \cdots, z_n)$ and $w=(w_1, \cdots, w_N)$. For the regularity of the functions $g_{jkl}(z', w)$ and $h_j(z', w)$, we assume that

(i) $g_{jkl}(z', w)$ and $h_j(z', w)$ are holomorphic on a closed polydisc $|z_\nu|\leqq r$ $(\nu=2, \cdots, n)$, $|w_\mu|\leqq r$ $(\mu=1, \cdots, N)$.

Then we set

$$M=\max_{\substack{j,k,l\\ z',w}}\{|g_{jkl}(z', w)|, \ |h_j(z', w)|\}.$$

Under these situations we have the next lemma.

LEMMA 1. *A unique solution of* (1) *satisfying the next initial conditions*

$$w_j(0, z')=0, \qquad j=1, \cdots, N, \tag{2}$$

*exists in*

$$|z_1|<r/(4MNn), \qquad |z_2|+\cdots+|z_n|<r/4. \tag{3}$$

PROOF. By the equation (1) and the initial conditions (2), the formal solution is uniquely determined. The convergence of this formal solution is proved by the method of majorants. (See for details John [2], pp. 81-85.)

By Cauchy's integral formula, $\dfrac{Mr}{r-(z_2+\cdots+z_n+w_1+\cdots+w_N)}$ becomes a majorant of all functions $g_{jkl}(z', w)$ and $h_j(z', w)$. Then, if we set $\zeta=z_1$ and $\xi=z_2+\cdots+z_n$, the solutions $W(\zeta, \xi)$ of the next initial value problem

$$\begin{cases} \dfrac{\partial W}{\partial\zeta}=\dfrac{Mr}{r-\xi-NW}\Big(N(n-1)\dfrac{\partial W}{\partial\xi}+1\Big) \\ W(0, \xi)=0 \end{cases} \tag{4}$$

becomes a majorant of all components of the formal solution $w=(w_1, \cdots, w_N)$ of (1) and (2). Therefore the formal solution is holomorphic where the solution $W(\zeta, \xi)$ of (4) is holomorphic. Now we can solve (4) explicitly as follows.

$$W(\zeta, \xi)=\frac{r-\xi}{Nn}-\frac{\sqrt{(r-\xi)^2-2MNrn\zeta}}{Nn}$$

where the branch of $\sqrt{(r-\xi)^2-2MNrn\zeta}$ is taken so that it becomes $r-\xi$ when $\zeta=0$. If we suppose that

$$|\zeta|<r/(4MNn)\,, \qquad |\xi|<r/4\,,$$

then $(r-\xi)^2-2MNrn\zeta$ does not vanish. Since the above domain is simply connected, $W(\zeta, \xi)$ becomes holomorphic (and single-valued) there. This completes the proof.

We next study the more general system of equations for $N$ unknown functions $u_1(z), \cdots, u_N(z)$, which is of the *normal form* (Courant-Hilbert [**1**], p. 39),

$$\frac{\partial^m u_j}{\partial z_1^m}=f_j\Big(z, u_1, \cdots, \Big(\frac{\partial}{\partial z}\Big)^\alpha u_k, \cdots\Big) \qquad j=1, \cdots, N, \tag{5}$$

where $f_j$ depends on the variables $z=(z_1, \cdots, z_n)$ and $(\partial/\partial z)^\alpha u_k$ $(k=1, \cdots, N)$ with multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n\leqq m$ and $\alpha_1\leqq m-1$. We impose the initial conditions on $u_j(z)$ on the complex hyperplane $z_1=0$ as follows.

$$\begin{cases} u_j(0, z')=\phi_{j,0}(z') \\ \cdots\cdots \\ \dfrac{\partial^{m-1}u_j}{\partial z_1^{m-1}}(0, z')=\phi_{j,m-1}(z') \end{cases} \qquad j=1, \cdots, N, \tag{6}$$

where $\phi_{j,k}(z')$ are arbitrary given functions. Now we assume that

(i) $f_j(z, \cdots, p_{k,\alpha}, \cdots)$, where the variables $p_{k,\alpha}$ correspond to the terms $(\partial/\partial z)^\alpha u_k$, are holomorphic on a closed polydisc $|z_\nu|\leqq r$, $|p_{k,\alpha}|<\infty$,

(ii) $\phi_{j,k}(z')$ are holomorphic on $|z_\nu|\leqq r$,

and set

$$C=\max_{j,k,\alpha'}\Big\{\Big|\Big(\frac{\partial}{\partial z'}\Big)^{\alpha'}\phi_{j,k}(z')\Big| \;\Big|\; |z_\nu|\leqq r,\ |\alpha'|+k\leqq m+1\Big\},$$

$$M=\max_{j,k,\alpha}\Big\{1,\ |f_j|,\ \Big|\frac{\partial f_j}{\partial z_k}\Big|,\ \Big|\frac{\partial f_j}{\partial p_{k,\alpha}}\Big| \;\Big|\; |z_\nu|\leqq r,\ |p_{k,\alpha}|\leqq C+r\Big\}$$

and $\hat{N}=(m+n)!/(m!\,n!)$, the cardinal number of the set of all multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$ with $|\alpha|\leqq m$.

Then we have the next theorem.

THEOREM 1. *There exists a unique solution* $u(z)=(u_1(z), \cdots, u_N(z))$ *of the initial value problem* (5) *and* (6) *in the following domain:*

$$|z_1|<r/\{4\hat{M}(N\hat{N}+1)n\}\,, \qquad |z_2|+\cdots+|z_n|<r/4$$

*where* $\hat{M}=3(1+r+C)(N\hat{N}M)^2$.

PROOF. We will prove this theorem by showing that the initial value problem (5) and (6) can be reduced to the equivalent problem for a system of quasi-linear differential equations of the first order, which is considered in Lemma 1. The method is an adaptation of that employed in Courant-Hilbert

[1], pp. 43-48 (see also F. John [2], pp. 78-82). For the reduction of the equations (5) to a quasilinear system, we introduce new unknown functions $v_{j,\alpha}(z)$ ($j=1, \cdots, N$, $\alpha=(\alpha_1, \cdots, \alpha_n)$ such that $|\alpha|\leqq m$) which correspond to $(\partial/\partial z)^{\alpha}u_j(z)$. The equations for $v_{j,\alpha}(z)$ are then as follows.

$$\frac{\partial}{\partial z_1} v_{j,\alpha}=v_{j,\alpha+1} \tag{7}$$

if $|\alpha|\leqq m-1$, where $\alpha+1=(\alpha_1+1, \alpha_2, \cdots, \alpha_n)$, and

$$\frac{\partial}{\partial z_1} v_{j,\alpha}(z)=\frac{\partial}{\partial z_l} v_{j,\alpha+1-1_l} \tag{8}$$

if $|\alpha|=m$, $\alpha_1\leqq m-1$ and for some $l=l(\alpha)\neq 1$, $\alpha_l\geqq 1$, where $\alpha+1-1_l=(\alpha_1+1, \alpha_2, \cdots, \alpha_l-1, \cdots, \alpha_n)$, and if $\alpha_1=m$,

$$\frac{\partial}{\partial z_1} v_{j,\alpha}=\frac{\partial}{\partial z_1}[f_j(z, v(z))], \tag{9}$$

where the terms $(\partial/\partial z_1)v_{k,\beta}$ arising in the right hand side of (9) should be replaced by $v_{k,\beta+1}$ or $(\partial/\partial z_l)v_{k,\beta+1-1_l}$ according as (7) or (8) holds. The initial conditions for $v_{j,\alpha}$ are determined by

$$\begin{cases} v_{j,\alpha}(0, z')=\left(\dfrac{\partial}{\partial z'}\right)^{\alpha'} \phi_{j,\alpha_1}(z'), & \text{if } \alpha_1\leqq m-1, \\ v_{j,\alpha}(0, z')=f_j\left(0, z', \cdots, \left(\dfrac{\partial}{\partial z'}\right)^{\beta'} \phi_{k,\beta_1}(z'), \cdots\right), & \text{if } \alpha_1=m, \end{cases} \tag{10}$$

where $\alpha'=(\alpha_2, \cdots, \alpha_n)$, $\beta'=(\beta_2, \cdots, \beta_n)$.

To make the initial condition homogeneous, we introduce the unknown functions $w_{j,\alpha}(z)$ defined by

$$\begin{cases} w_{j,\alpha}(z)=v_{j,\alpha}(z)-\left(\dfrac{\partial}{\partial z'}\right)^{\alpha'} \phi_{j,\alpha_1}(z'), & \alpha_1\leqq m-1, \\ w_{j,\alpha}(z)=v_{j,\alpha}(z)-f_j\left(0, z', \cdots, \left(\dfrac{\partial}{\partial z'}\right)^{\beta'} \phi_{k,\beta_1}(z'), \cdots\right), & \alpha_1=m. \end{cases} \tag{11}$$

Then, we obtain the following equations for these functions $w_{j,\alpha}(z)$:

(i) if $|\alpha|\leqq m-1$ and $\alpha_1\leqq m-2$,

$$\frac{\partial}{\partial z_1} w_{j,\alpha}=w_{j,\alpha+1}+\left(\frac{\partial}{\partial z'}\right)^{\alpha'} \phi_{j,\alpha_1+1}(z'), \tag{12}$$

(ii) if $|\alpha|\leqq m-1$ and $\alpha_1=m-1$ (i.e. $\alpha=(m-1, 0, \cdots, 0)$),

$$\frac{\partial}{\partial z_1} w_{j,\alpha}=w_{j,\alpha+1}+f_j\left(0, z', \cdots, \left(\frac{\partial}{\partial z'}\right)^{\beta'} \phi_{k,\beta_1}(z'), \cdots\right), \tag{13}$$

(iii) if $|\alpha|=m$ and $\alpha_1\leqq m-2$,

(14)
$$\frac{\partial}{\partial z_1} w_{j,\alpha} = \frac{\partial}{\partial z_l} w_{j,\alpha+1-1_l} + \left(\frac{\partial}{\partial z'}\right)^{\alpha'} \phi_{j,\alpha_1+1}(z'),$$

(iv) if $|\alpha| = m$ and $\alpha_1 = m-1$,

(15)
$$\frac{\partial}{\partial z_1} w_{j,\alpha} = \frac{\partial}{\partial z_l} w_{j,\alpha+1-1_l} + \frac{\partial}{\partial z_l}\Big[f_j\Big(0, z', \cdots, \Big(\frac{\partial}{\partial z'}\Big)^{\beta'} \phi_{k,\beta_1}(z'), \cdots\Big)\Big]$$

and

(v) if $\alpha_1 = m$,

(16)
$$\begin{aligned}\frac{\partial}{\partial z_1} w_{j,\alpha} &= \frac{\partial}{\partial z_1}\Big[f_j\Big(z, \cdots, w_{k,\beta} + \Big(\frac{\partial}{\partial z'}\Big)^{\beta'} \phi_{k,\beta_1}(z'), \cdots\Big)\Big] \\ &= \frac{\partial f_j}{\partial z_1} + \sum_{k=1}^{N} \sum_{\substack{|\beta| \leqq m-1 \\ \beta_1 \leqq m-2}} \frac{\partial f_j}{\partial p_{k,\beta}}\Big(w_{k,\beta+1} + \Big(\frac{\partial}{\partial z'}\Big)^{\beta'} \phi_{k,\beta_1+1}(z')\Big) \\ &\quad + \sum_{\substack{k=1 \\ \beta=(m-1,0,\cdots,0)}}^{N} \frac{\partial f_j}{\partial p_{k,\beta}}\Big(w_{k,\beta+1} + f_k\Big(0, z', \cdots, \Big(\frac{\partial}{\partial z'}\Big)^{\gamma'} \phi_{\mu,\gamma_1}(z'), \cdots\Big)\Big) \\ &\quad + \sum_{k=1}^{N} \sum_{\substack{|\beta| = m \\ \beta_1 \leqq m-2}} \frac{\partial f_j}{\partial p_{k,\beta}}\Big(\frac{\partial w_{k,\beta+1-1_l}}{\partial z_l} + \Big(\frac{\partial}{\partial z'}\Big)^{\beta'} \phi_{k,\beta_1+1}(z')\Big) \\ &\quad + \sum_{k=1}^{N} \sum_{\substack{|\beta| = m \\ \beta_1 \leqq m-1}} \frac{\partial f_j}{\partial p_{k,\beta}}\Big(\frac{\partial w_{k,\beta+1-1_l}}{\partial z_l} + \frac{\partial}{\partial z_l}\Big[f_k\Big(0, z', \cdots \\ &\qquad\qquad \cdots, \Big(\frac{\partial}{\partial z'}\Big)^{\gamma'} \phi_{\mu,\gamma_1}(z'), \cdots\Big)\Big]\Big).\end{aligned}$$

Lastly we set $w_{N+1}(z) = z_1$ and obtain the following additional equation and the initial condition.

(17)
$$\begin{cases} \dfrac{\partial}{\partial z_1} w_{N+1}(z) = 1 \\ w_{N+1}(0, z') = 0. \end{cases}$$

Then we can substitute $w_{N+1}$ for the variable $z_1$ which appears in the right hand side of (16). Consider the Cauchy problem consisting of the equations (12)-(17) and the homogeneous initial conditions. Since the equations (12)-(17) do not depend explicitly on the variable $z_1$, if we can take a constant $\hat{M} > 0$ so large that all the absolute values of the coefficients of the equations (12)-(17) de not exceed $\hat{M}$ in $|z_\nu| \leqq r$ $(\nu = 2, \cdots, n)$, $|w_{k,\beta}| \leqq r$ and $|w_{N+1}| \leqq r$, then, by applying Lemma 1, we obtain the conclusion of the theorem. In view of the equations (12)-(17), it is sufficient for $\hat{M}$ to be larger than the following numbers:

$$r+C,\ r+M,\ \max\{1, C\},\ \max\{1, M+N\hat{N}MC\},$$
$$\max\{N\hat{N}M, M+N\hat{N}M(r+C+r+M+C+M+N\hat{N}MC)\}.$$

It is easy to see that the constant $\hat{M} = 3(1+r+C)(N\hat{N}M)^2$ has the desired property. This completes the theorem.

## §3. Holomorphic continuation.

Throughout the rest of this paper, we denote by $U$ some neighborhood of 0 in $\boldsymbol{C}^n$ and by $\Omega$ some domain in $U$ with regular boundary $\partial\Omega$ containing the origin, that is, there exists a real-valued $C^1$ function $\phi$ in $U$ such that $\phi(0)=0$, $d\phi\neq 0$ in $U$ and

$$\Omega=\{z\in U \mid \phi(z)<0\}\,.$$

Then, under the suitable coordinates $(z_1, \cdots, z)$ with $z_j=x_j+\sqrt{-1}\,y_j$ $(j=1, \cdots, n)$, we can assume that

$$(18)\qquad \begin{cases} \dfrac{\partial\phi}{\partial x_1}(0)=1\,, & \dfrac{\partial\phi}{\partial x_j}(0)=0\,, \qquad j=2, \cdots, n\,, \\ \dfrac{\partial\phi}{\partial y_k}(0)=0\,, & k=1, \cdots, n\,. \end{cases}$$

In this case, we have the next lemma due to Zerner [5].

LEMMA 2 (Zerner). *There exist sequences* $\{\alpha_\nu\}$, $\nu=1, 2, \cdots$ $(\alpha_\nu<0,\ \alpha_\nu\to 0)$ *and* $\{\rho_\nu\}$, $\nu=1, 2, \cdots$ $(\rho_\nu>0)$ *such that*

(i) $(\alpha_\nu, 0, \cdots, 0)\in\Omega$,

(ii) $\{z_1=\alpha_\nu\}\cap\Omega\supset\{z_1=\alpha_\nu,\ |z_2|\leqq\rho_\nu, \cdots, |z_n|\leqq\rho_\nu\}$,

*and*

(iii) $\lim_{\nu\to\infty}\alpha_\nu/\rho_\nu=0$.

Here we introduce some notion to be used later.

DEFINITION. A holomorphic function $u(z)$ in $\Omega$ is said to be *bounded of order* $m$ if $u(z)$ and all its derivatives of order less than or equal to $m$ are bounded in $\Omega$.

We first consider the system of quasi-linear equations for $N$ unknown functions $u_1(z), \cdots, u_N(z)$.

$$(19)\qquad \sum_{|\alpha|=m}\sum_{k=1}^{N} a_\alpha^{j,k}(z, (\partial/\partial z)^\beta u_l)(\partial/\partial z)^\alpha u_k(z)=f_j(z, (\partial/\partial z)^\beta u_l)\,, \qquad j=1, \cdots, N\,,$$

where $a_\alpha^{j,k}$ and $f_j$ depend on the variables $z_1, \cdots, z_n$ and $p_{l,\beta}=(\partial/\partial z)^\beta u_l$ $(l=1, \cdots, N,\ |\beta|\leqq m-1)$. We suppose that $a_\alpha^{j,k}$ and $f_j$ are holomorphic in $z\in U$ and $|p_{l,\beta}|<\infty$. Since the complex tangent plane at 0 of the surface $\partial\Omega$ is $\{z_1=0\}$, we make the following condition (A).

*Condition* (A): *for all* $z\in U$ *and* $p_{l,\beta}$,

$$\det(a_{(m,0,\cdots,0)}^{j,k}(z, p))\neq 0\,.$$

Then we have

THEOREM 2. *Under the condition* (A), *every solution of* (19) *in* $\Omega$ *which is bounded of order* $m+1$ *becomes holomorphic near the origin.*

PROOF. The idea of this proof is due to Zerner [**5**]. By the condition (A), we can solve (19) with respect to $\partial^m u_j/\partial z_1^m$ $(j=1, \cdots, N)$ and obtain the following equations:

$$(20) \qquad \left(\frac{\partial}{\partial z_1}\right)^m u_j(z)=F_j\left(z, \cdots, \left(\frac{\partial}{\partial z}\right)^\alpha u_k, \cdots\right), \qquad j=1, \cdots, N$$

where $F_j$ depends on the variables $z_1, \cdots, z_n$ and $(\partial/\partial z)^\alpha u_k$, $|\alpha|\leqq m$, $\alpha_1\leqq m-1$, $k=1, \cdots, N$. And this type of the problem we have just studied in Theorem 1. Let $\{\alpha_\nu\}$, $\{\rho_\nu\}$ be the sequences guaranteed by Lemma 2. We may assume that $\rho_\nu$ is bounded by some $\rho>0$ and the set $\{z \mid |z_1-\alpha_\nu|\leqq\rho, |z_2|\leqq\rho, \cdots, |z_n| \leqq\rho\}$ is contained in $U$ for every $\nu$. Then we consider the initial value problem consisting of (20) and the initial conditions on the plane $\{z_1=\alpha_\nu\}$ with the restrictions of $u_j(z), \cdots, (\partial/\partial z_1)^{m-1}u_j(z)$. If we apply Theorem 1 to this situation, $u_1(z), \cdots, u_N(z)$ become holomorphic in

$$|z_1-\alpha_\nu|<\rho_\nu/\{4\hat{M}_\nu(N\hat{N}+1)n\}, \qquad |z_2|+\cdots+|z_n|<\rho_\nu/4$$

for every $\nu$. But since $u_1(z), \cdots, u_N(z)$ are bounded of order $m+1$ in $\Omega$, the sequence $\{\hat{M}_\nu\}$ is bounded. Therefore there exists a constant $c>0$ which is independent of $\nu$ such that $u_1(z), \cdots, u_N(z)$ are holomorphic in

$$|z_1-\alpha_\nu|<c\rho_\nu, \qquad |z_2|+\cdots+|z_n|<\rho_\nu/4.$$

Then it is sufficient to show that

$$(21) \qquad 0\in\bigcup_{\nu=1}^{\infty}\{z\in U \mid |z_1-\alpha_\nu|<c\rho_\nu, \ |z_2|+\cdots+|z_n|<\rho_\nu/4\}.$$

If (21) is not valid, then we have, for any $\nu$, $|\alpha_\nu|\geqq c\rho_\nu$. This contradicts the property of $\alpha_\nu$, $\rho_\nu$ in Lemma 2. Thus the proof is complete.

REMARK. When the equation (19) is a single linear differential equation $P(z, \partial/\partial z)u(z)=0$, the condition (A) means that the plane $\{z_1=\text{const.}\}$ is non-characteristic. In this case, Theorem 2 states that every solution of $P(z, \partial/\partial z)u(z)=0$ which is bounded of order $m+1$ can be holomorphically continued over the non-characteristic hypersurface. Therefore this may be regarded as a partial extension of Zerner's theorem [**5**].

We next study the system of general nonlinear equations of order $m$.

$$(22) \qquad F_j\left(z, \cdots, \left(\frac{\partial}{\partial z}\right)^\alpha u_k, \cdots\right)=0, \qquad j=1, \cdots, N,$$

where $F_j$ depend on $z_1, \cdots, z_n$ and $p_{k,\alpha}=(\partial/\partial z)^\alpha u_k$ with $k=1, \cdots, N$, $|\alpha|\leqq m$, and are holomorphic in $z \in U$ and $|p_{k,\alpha}|<\infty$. If we differentiate the equation (22) with respect to $z_1$, then (22) is reduced to a system of quasi-linear equations of order $m+1$. Therefore, applying Theorem 2, we have the next corollary.

COROLLARY. *Under the condition* (B) *below, every solution of* (22) *in* $\Omega$ *which is bounded of order* $m+2$ *becomes holomorphic near the origin.*

*Condition* (B): *for all* $z \in U$ *and* $p_{k,\alpha}$

$$\det\left(\frac{\partial F_j}{\partial p_{k,(m,0,\cdots,0)}}(z, p)\right) \neq 0 .$$

We remark that the boundedness in the assumptions in Theorem 2 and Corollary seems to be necessary because of the following example.

EXAMPLE. The equation

$$\exp\left(\frac{\partial u}{\partial z_1}\right)=z_1 \quad \text{in} \quad \boldsymbol{C}^n$$

has a solution $u(z)=z_1 \log z_1-z_1$, which is bounded (of order 0) in any simply connected bounded domain not containing $z_1=0$.

## § 4. Solution with singularities.

We study, in this last section, the single equation of the first order with two independent variables for which the condition (B) in the preceding section is not satisfied. We denote the variables by $(x, y)$ instead of $(z_1, z_2)$ and write $p=u_x=\partial u/\partial x$, $q=u_y=\partial u/\partial y$. The equation is given by

$$F(x, y, u, u_x, u_y)=0 \tag{23}$$

where $F(x, y, u, p, q)$ is holomorphic in $U \times \boldsymbol{C}^3$. We now suppose that $F(0, 0, u_0, p_0, q_0)=0$ and $F_p(0, 0, u_0, p_0, q_0)=0$ for some $u_0, p_0, q_0$. We may assume, without loss of generality, that $u_0=p_0=q_0=0$, since otherwise we could introduce $u(x, y)-u_0-p_0x-q_0y$ as a new unknown function. Then our last theorem is the following.

THEOREM 3. *For the equation* (23) *we assume that* $F(0, \cdots, 0)=0$, $F_p(0, \cdots, 0)=0$ *and* $F_q(0, \cdots, 0)=1$. *Then under the condition* (C) *below, there exists a solution of* (23) *which is holomorphic in* $\{(x, y) \in V \mid \phi(x, y)<0\}$ *but cannot be holomorphic at* 0, *where* $V$ *is a small neighborhood of* 0 *and* $\phi(x, y)$ *is of class* $C^2$.

*Condition* (C): *Let* $(x_0(t), y_0(t), u_0(t), p_0(t), q_0(t))$ *be a characteristic strip of* (23) *through* $(0, \cdots, 0)$ *at* $t=0$, *then for every* $t_0 \neq 0$ $(t_0 \in \boldsymbol{C})$ *and a real parameter* $\tau$,

$$\frac{d^2}{d\tau^2}\phi(x_0(\tau t_0), y_0(\tau t_0))\Big|_{\tau=0}>0 .$$

Before the proof of this theorem, we prepare some lemmas. Let

$$f(z)=\begin{cases} z^{k+1}\log z & z\neq 0 \\ 0 & z=0 \end{cases}$$

for $z\in \boldsymbol{C}$ where $k$ is any positive integer and for any $\theta_0$ and $\theta_1$ such that $0<\theta_0<\theta_1<\pi/2$,

$$\psi(\theta)=\begin{cases} 1 & |\theta|\geqq\theta_1 \\ 0 & |\theta|\leqq\theta_0 \end{cases}$$

for $-\pi<\theta\leqq\pi$ and suppose that $\psi(\theta)$ is infinitely differentiable. And set

$$g(z)=\begin{cases} f(z)\psi(\theta) & z\neq 0 \\ 0 & z=0 \end{cases}$$

where $\theta=\arg z$ $(-\pi<\theta\leqq\pi)$. Then we have the next

LEMMA 3. *$g(z)$ is a $C^k$ function with respect to the real two variables* $\mathrm{Re}\, z$ *and* $\mathrm{Im}\, z$ *and is holomorphic in the sector* $|\theta|>\theta_1$.

PROOF. If we denote $z=re^{i\theta}$, then $\partial/\partial z=(e^{-i\theta}/2)(\partial/\partial r)+(e^{-i\theta}/2ir)(\partial/\partial\theta)$ and $\partial/\partial\bar{z}=(e^{i\theta}/2)(\partial/\partial r)-(e^{i\theta}/2ir)(\partial/\partial\theta)$. Therefore if $D^m$ is any differentiation of order $m$ $(m\leqq k)$ with respect to $z$ and $\bar{z}$,

$$|D^m f|\leqq \text{const.}|z^{k+1-m}|(|\log z|+1)$$

and

$$|D^m\psi|\leqq \text{const.}\, r^{-m}$$

for $r$ sufficiently small. Then

$$|D^m g|\leqq \text{const.}\sum_l |D^{m-l}f|\,|D^l\psi|$$

$$\leqq \text{const.}\, r^{k+1-m}(|\log r|+1)\,.$$

Thus $D^m g$ tends to zero if $r$ converges to 0, which shows that $g(z)$ is a $C^k$ function. It is trivial that $g(z)$ is holomorphic where $\psi(\theta)=1$, thus the proof is complete.

LEMMA 4. *Let $F(x, y, u, p, q)$ be the function given in Theorem* 3 *and $g(z)$ be the function defined in Lemma* 3 *with $k\geqq 2$. Then the equation for $h(z)$*

$$F\left(z, 0, g(z), \frac{\partial g}{\partial z}, h(z)\right)=0 \tag{24}$$

*where* $\dfrac{\partial g}{\partial z}=\dfrac{1}{2}\left(\dfrac{\partial g}{\partial z_1}-i\dfrac{\partial g}{\partial z_2}\right)$, $z=z_1+iz_2$ $(z_1, z_2\in \boldsymbol{R})$, *has a unique $C^{k-1}$ solution in a neighborhood of* 0 *such that $h(0)=0$ and $h(z)$ is holomorphic where $g(z)$ is so.*

PROOF. We denote the real and imaginary parts of $h$ and $F$ by $h_1$, $h_2$ and

$F_1$, $F_2$, respectively. By the assumption that $g(z)$ is of class $C^k$, $F_1$ and $F_2$ are $C^{k-1}$ functions. Since $F=F_1+iF_2$ is holomorphic with respect to $h=h_1+ih_2$ and $(\partial F/\partial h)(0, \cdots, 0)=1$, we have at 0,

$$\frac{\partial F_1}{\partial h_1}=\frac{\partial F_2}{\partial h_2}=1, \qquad \frac{\partial F_1}{\partial h_2}=-\frac{\partial F_2}{\partial h_1}=0.$$

Thus

$$\frac{\partial(F_1, F_2)}{\partial(h_1, h_2)}=1,$$

which implies that, by the implicit function theorem, there exist unique $C^{k-1}$ functions $h_1(z)$ and $h_2(z)$ with $h_1(0)=h_2(0)=0$ such that $h=h_1+ih_2$ satisfies (24) in a neighborhood of 0. If $g(z)$ is holomorphic at some point, then so is $F$. Therefore, applying the implicit function theorem for holomorphic functions, $h(z)$ obtained above is holomorphic at that point. This completes the proof.

Now we prove Theorem 3.

PROOF OF THEOREM 3. Consider the following characteristic differential equations related to the equation (23).

$$\text{(25)} \qquad \begin{cases} \dfrac{dx}{dt}=F_p, & \dfrac{dy}{dt}=F_q, \quad \dfrac{du}{dt}=pF_p+qF_q, \\[2mm] \dfrac{dp}{dt}=-F_x-pF_u, & \dfrac{dq}{dt}=-F_y-qF_u. \end{cases}$$

Since all the right hand sides of equations (25) are holomorphic, the solutions $x(t)$, $y(t)$, $u(t)$, $p(t)$, $q(t)$ are holomorphic in $t$. For the initial conditions of these functions, we set

$$\text{(26)} \qquad \begin{cases} x(0)=z, & y(0)=0, \quad u(0)=g(z), \\[2mm] p(0)=\dfrac{\partial g}{\partial z}, & q(0)=h(z), \end{cases}$$

where $g(z)$ and $h(z)$ are the functions in Lemma 3 and 4, respectively, with $k=3$. Since the initial condition is of class $C^2$, the solution of (25) and (26) is $C^2$ in $z$ and $\bar{z}$ and holomorphic in $t$. But the solution becomes holomorphic in $z$ and $t$ where $g(z)$ is holomorphic.

Let

$$\text{(27)} \qquad \begin{cases} x=X(z, t), & y=Y(z, t), \quad u=U(z, t) \\ p=P(z, t), & q=Q(z, t) \end{cases}$$

be the solutions of the initial problem (25)-(26). If we can solve for $z$ and $t$ in terms of $x$ and $y$ in the first two equations, then, substituting in the third, we obtain the function

$$u=u(x, y)=U(z(x, y),\ t(x, y))$$

which is obviously the solution of (23) taking the initial values $g(z)$ on the hyperplane $\{y=0\}$. Therefore, to prove this theorem, it is sufficient to show that

1) the inverse mapping theorem is valid in a neighborhood of 0 for the functions $x=X(z, t)$ and $y=Y(z, t)$ in (27),

and that

2) the functions $z=z(x, y)$ and $t=t(x, y)$ obtained above are holomorphic in $\{(x, y)\in V\,|\,\phi(x, y)<0\}$ where $V$ is a small neighborhood of 0.

If these assertions are shown to be true, then the solution $u(x, y)$ becomes holomorphic in $\{(x, y)\in V\,|\,\phi(x, y)<0\}$, but on the plane $\{y=0\}$, $u(x, 0)=g(x)$ cannot be holomorphic at 0. Thus this function $u(x, y)$ is the desired one in Theorem 3.

We now prove the assertions 1) and 2).

1) Inverse mapping theorem. We denote $X(z, t)=X_1(z, t)+iX_2(z, t)$, $Y=Y_1+iY_2$, $z=z_1+iz_2$ and $t=t_1+it_2$ where $X_1, X_2, \cdots, t_1, t_2$ are real valued. Since $X(z, 0)=z$, $Y(z, 0)=0$, $(dX/dt)(0)=F_p(0, \cdots, 0)=0$ and $(dY/dt)(0)=F_q(0, \cdots, 0)=1$, we have

$$\left.\frac{\partial(X_1, X_2, Y_1, Y_2)}{\partial(z_1, z_2, t_1, t_2)}\right|_{z=t=0}=\begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix}=1\,.$$

Therefore in a neighborhood of 0 there exist unique inverse $C^2$ functions $z=z(x, y)$ and $t=t(x, y)$. We remark that the above functions are holomorphic if $X(z, t)$ and $Y(z, t)$ are holomorphic by the inverse mapping theorem for holomorphic functions.

2) Holomorphy of $z(x, y)$ and $t(x, y)$. To show that $X(z, t)$ and $Y(z, t)$ are holomorphic, it suffices to prove that $g(z)$ is holomorphic in $\{z\,|\,\phi(X(z, t), Y(z, t))<0,\ |t| \text{ small enough}\}$. We now expand $\phi$ to the second order in the variables $z$ and $t$. In view of (18) and $(dX/dt)(0)=F_p(0, \cdots, 0)=0$, $(dY/dt)(0)=F_q(0, \cdots, 0)=1$, we have

$$\begin{aligned}
(28)\qquad & \phi(X(z, t),\ Y(z, t)) \\
&=\frac{1}{2}(z+\bar{z}) \\
&\quad+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(0)z^2+\frac{\partial^2\phi}{\partial x\partial\bar{x}}(0)z\bar{z}+\frac{1}{2}\frac{\partial^2\phi}{\partial\bar{x}^2}(0)\bar{z}^2 \\
&\quad+\left\{\frac{\partial^2\phi}{\partial x\partial y}(0)+\frac{1}{2}\frac{\partial^2 X}{\partial z\partial t}(0)\right\}zt+\left\{\frac{\partial^2\phi}{\partial\bar{x}\partial\bar{y}}(0)+\frac{1}{2}\frac{\partial^2\bar{X}}{\partial\bar{z}\partial\bar{t}}(0)\right\}\bar{z}\bar{t} \\
&\quad+\frac{\partial^2\phi}{\partial x\partial\bar{y}}(0)z\bar{t}+\frac{\partial^2\phi}{\partial\bar{x}\partial y}(0)\bar{z}t+\frac{1}{2}\frac{\partial^2\bar{X}}{\partial z\partial\bar{t}}(0)z\bar{t}+\frac{1}{2}\frac{\partial^2 X}{\partial\bar{z}\partial t}(0)\bar{z}t
\end{aligned}$$

$$+\frac{1}{2}\left\{\frac{1}{2}\frac{\partial^2 X}{\partial t^2}(0)+\frac{\partial^2\phi}{\partial y^2}(0)\right\}t^2+\frac{\partial^2\phi}{\partial y\partial\bar{y}}(0)t\bar{t}$$

$$+\frac{1}{2}\left\{\frac{1}{2}\frac{\partial^2\bar{X}}{\partial\bar{t}^2}(0)+\frac{\partial^2\phi}{\partial\bar{y}^2}(0)\right\}\bar{t}^2+o(|z|^2+|t|^2).$$

If $z$ is in the sector $\Gamma$: $|\arg z|\leqq\theta_1$ $(0<\theta_1<\pi/2)$, then

$$z_1\geqq 0 \quad\text{and}\quad |z_2|\leqq\alpha z_1 \tag{29}$$

where $0<\alpha=\tan\theta_1$. Since $x_0(t)=X(0, t)$, $y_0=Y(0, t)$, the condition (C) in Theorem 3 is given by the following

$$\frac{1}{2}\frac{\partial^2 X}{\partial t^2}t^2+\frac{1}{2}\frac{\partial^2\bar{X}}{\partial\bar{t}^2}\bar{t}^2+\frac{\partial^2\phi}{\partial y^2}t^2+2\frac{\partial^2\phi}{\partial y\partial\bar{y}}t\bar{t}+\frac{\partial^2\phi}{\partial\bar{y}^2}\bar{t}^2\geqq\gamma|t|^2 \tag{30}$$

for some constant $\gamma>0$. Then by (29) and (30), (28) can be written as follows:

$$\begin{aligned}&\phi(X(z, t),\ Y(z, t))\\&\geqq z_1-C_1|z|^2-C_2(1+\alpha)|t|z_1+\frac{1}{2}\gamma|t|^2+o(|z|^2+|t|^2)\\&\geqq\{1-C_1(1+\alpha^2)z_1-C_2(1+\alpha)|t|\}z_1+\frac{1}{2}\gamma|t|^2+o(|z|^2+|t|^2),\end{aligned}$$

where

$$C_1=\frac{1}{2}\left|\frac{\partial^2\phi}{\partial x^2}(0)\right|+\left|\frac{\partial^2\phi}{\partial x\partial\bar{x}}(0)\right|+\frac{1}{2}\left|\frac{\partial^2\phi}{\partial\bar{x}^2}(0)\right|$$

and

$$C_2=2\left|\frac{\partial^2\phi}{\partial x\partial y}(0)\right|+2\left|\frac{\partial^2\phi}{\partial x\partial\bar{y}}(0)\right|+\left|\frac{\partial^2 X}{\partial z\partial t}(0)\right|+\left|\frac{\partial^2\bar{X}}{\partial z\partial\bar{t}}(0)\right|.$$

Thus if $z$ and $t$ are sufficiently small and $z$ is in the sector $\Gamma$, then $\phi\geqq 0$. This means that if $z$ and $t$ are sufficiently small and $\phi<0$, then $g(z)$ becomes holomorphic. This proves 2) and the proof of the theorem is complete.

REMARK. By (30) which is the condition (C), we have $(\partial^2\phi/\partial y\partial\bar{y})(0)>0$. Consider the Levi form of $\phi$ at 0. Since the complex vector $\lambda=(\lambda_1, \lambda_2)$ satisfying $\lambda_1(\partial\phi/\partial x)+\lambda_2(\partial\phi/\partial y)=0$ is of the form $(0, \lambda_2)$, the Hessian of $\phi$ for this $\lambda$ is equal to

$$\frac{\partial^2\phi}{\partial y\partial\bar{y}}\lambda_2\bar{\lambda}_2>0 \qquad (\lambda_2\neq 0).$$

This means that the domain $\{(x, y)\in V\,|\,\phi(x, y)<0\}$ is strictly pseudoconvex at 0.

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