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On fixed point free $SO(3)$-actions on homotopy 7-spheres

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§ 0. Introduction.

Let $SO(3)$ be the rotation group (see § 1). In this paper, we shall study smooth $SO(3)$-actions on homotopy 7-spheres without fixed points. Our category is the smooth category. In [5], we have studied some $SO(3)$-actions on homotopy 7-spheres, mainly in the case with two or three orbit types. In that case, the actions have fixed points (see [5]). Our present paper is concerned with the case without fixed points.

Let α and β be the real irreducible representations of $SO(3)$ of dimension 3 and 5 respectively (see § 1). Then $\alpha\oplus\beta$ induces a linear action of $SO(3)$ on the 7-sphere S^7. A simple observation shows that this is the only linear action on S^7 which has no fixed points. Let (Σ^7, φ) be a smooth $SO(3)$-action on a homotopy 7-sphere Σ^7 (here φ; $SO(3)\times\Sigma^7\to\Sigma^7$ is a smooth map defining the action). For $g\in SO(3)$ and $x\in\Sigma^7$, gx denotes $\varphi(g, x)$. The isotropy subgroup of x, G_x, is defined by $G_x=\{g\in SO(3)\mid gx=x\}$. Then the set of the conjugacy classes $\{(G_x)\mid x\in\Sigma^7\}$ is called as the isotropy subgroup type of (Σ^7, φ). Now we assume that (Σ^7, φ) is fixed point free, that is, for each $x\in\Sigma^7$, G_x is a proper subgroup of $SO(3)$. Then we ask if the isotropy subgroup type of (Σ^7, φ) coincides with that of the linear action $\alpha\oplus\beta$. The answer is given by the following two theorems.

THEOREM I. *Let (Σ^7, φ) be a smooth $SO(3)$-action on a homotopy 7-sphere Σ^7 without fixed points. Then the isotropy subgroup type of (Σ^7, φ) is one of the following two types,*

(a) $\{(e), (Z_2), (D_2), (SO(2)), (N)\}$ *and*

(b) $\{(e), (Z_2), (D_2), (SO(2)), (N), (Z_{2k+1}), (D_{2k+1})\}$ (*k is a positive integer*),

(*For the notations see* § 1).

The type (a) in the above theorem is that of the linear action $\alpha\oplus\beta$ (§ 2). There is no linear action having (b) as its isotropy subgroup type.

THEOREM II. *For each positive integer k, there is a smooth $SO(3)$-action on the standard 7-sphere S^7 with isotropy subgroup type* (b) *of Theorem* I.

Theorem I will be proved in § 3 and Theorem II in § 4.

It can be seen that if (Σ^7, φ) has a fixed point, its isotropy subgroup type coincides with one of those realized by linear $SO(3)$-actions on S^7 ([5]). Hence the two isotropy subgroup types of Theorem I together with those of linear actions give a complete list of the isotropy subgroup types occurring in smooth $SO(3)$-actions on homotopy 7-spheres.

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§ 1. Notations and definitions.

A) $SO(3)$ and its closed subgroups.

Let $SO(3)$ be the group of those 3×3 real matrices $\{g=(a_{ij})_{1\leqq i,j\leqq 3}\}$ such that ${}^tgg$ is the identity matrix and $|g|=1$ where ${}^tg$ is the transpose of g and $|g|$ is the determinant of g. We denote the identity matrix by e. The closed subgroups of $SO(3)$ are denoted as follows,

$SO(2)$: the subgroup of the matrices of the form

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 0\leqq\theta<2\pi\,,$$

N: the subgroup generated by $SO(2)$ and

$$c=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

for a positive integer k,

Z_k: the cyclic subgroup of $SO(2)$ of order k,

D_k: the subgroup generated by Z_k and c,

T (the tetrahedral group): the subgroup of the matrices

$$\left\{\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}, \begin{bmatrix} 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \\ \varepsilon_3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \varepsilon_1 \\ \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \end{bmatrix}, \ \begin{matrix} \Pi\varepsilon_i=1 \\ \varepsilon_i=\pm 1 \end{matrix}\right\}$$

O (the octahedral group): the subgroup of the matrices

$$\left\{\begin{matrix}\begin{bmatrix}\varepsilon_1 & 0 & 0\\ 0 & \varepsilon_2 & 0\\ 0 & 0 & \varepsilon_3\end{bmatrix} & \begin{bmatrix}0 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_2\\ \varepsilon_3 & 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0 & \varepsilon_1\\ \varepsilon_2 & 0 & 0\\ 0 & \varepsilon_3 & 0\end{bmatrix} & \\ \begin{bmatrix}0 & \varepsilon_1 & 0\\ \varepsilon_2 & 0 & 0\\ 0 & 0 & -\varepsilon_3\end{bmatrix} & \begin{bmatrix}0 & 0 & \varepsilon_1\\ 0 & \varepsilon_2 & 0\\ -\varepsilon_3 & 0 & 0\end{bmatrix} & \begin{bmatrix}\varepsilon_1 & 0 & 0\\ 0 & 0 & \varepsilon_2\\ 0 & -\varepsilon_3 & 0\end{bmatrix} & \Pi\varepsilon_i=1,\ \varepsilon_i=\pm 1\end{matrix}\right\},$$

I: the icosahedral group.

We note that N is the normalizer of $SO(2)$ in $SO(3)$ and isomorphic to $O(2)$, the orthogonal group. Any closed subgroup of $SO(3)$ is conjugate to one of those listed above (Wolf [**6**]).

B) Real irreducible representations of $SO(3)$, α and β.

α: Let R_α^3 be the 3-dimensional real vector space consisting of vectors $\{v=(v_1, v_2, v_3),\ v_i: \text{real number}\}$. For $v=(v_i)\in R_\alpha^3$ and $g=(a_{ij})\in SO(3)$, we define $gv\in R_\alpha^3$ by

$$gv={}^t\begin{bmatrix}(a_{ij}) & \widehat{v_1}\\ & v_3\\ & \underset{\smile}{v_2}\end{bmatrix}\quad \text{(matrix multiplication)}.$$

This is a 3-dimensional real irreducible representation of $SO(3)$ and denoted by α. We define the norm of $v=(v_i)$ by $\|v\|^2=\sum v_i^2$.

β: Let R_β^5 be the 5-dimensional real vector space consisting of those 3×3 real symmetric matrices $\{s=(s_{ij})\}$ such that the trace of $s=\sum s_{ii}=0$. For $s\in R_\beta^5$ and $g\in SO(3)$, $gs\in R_\beta^5$ is defined by $gs=gsg^{-1}$ (matrix multiplication). This is a 5-dimensional real irreducible representation of $SO(3)$ and denoted by β. The norm of $s\in R_\beta^5$ is defined by $\|s\|^2=$ trace of ss. This norm is $SO(3)$-invariant.

§ 2. Linear action $\alpha\oplus\beta$.

Let R_α^3 and R_β^5 be as in § 1. Let S_α and S_β be the unit sphere in R_α^3 and R_β^5 respectively. Then S_α and S_β are $SO(3)$-manifolds. The isotropy subgroup type of S_α is $\{(SO(2))\}$ and that of S_β is $\{(D_2), (N)\}$ ([**2**] p. 43). The orbit space $S_\alpha/SO(3)$ is a point and $S_\beta/SO(3)$ is an arc whose end points correspond to the orbits of type $(SO(3)/N)$.

Now let $R_{\alpha\oplus\beta}^8$ be the direct sum of R_α^3 and R_β^5, $R_{\alpha\oplus\beta}^8=R_\alpha^3\oplus R_\beta^5$. Let $x=x_1+x_2$ be a point of $R_{\alpha\oplus\beta}^8$ (here $x_1\in R_\alpha^3$ and $x_2\in R_\beta^5$). The action of $SO(3)$ on $R_{\alpha\oplus\beta}^8$ is defined as $gx=gx_1+gx_2$ for $g\in SO(3)$. The norm of x, $\|x\|$ is defined by $\|x\|^2=\|x_1\|^2+\|x_2\|^2$. Let S^7 be the unit sphere of $R_{\alpha\oplus\beta}^8$. $SO(3)$ acts on S^7 and this is the linear action $\alpha \oplus \beta$.

LEMMA 2.1. *The isotropy subgroup structure of the linear action* $\alpha \oplus \beta$ *is type* (a) *of Theorem I.*

PROOF. S^7 is equivalent to the equivariant join $S_\alpha * S_\beta$ as $SO(3)$-spaces. A simple calculation gives the result. Q. E. D.

§ 3. Isotropy subgroup type.

In this section, we shall prove Theorem I (see § 0).

Let (Σ^7, φ) be a smooth $SO(3)$-action on a homotopy 7-sphere without fixed points. For a closed subgroup H of $SO(3)$, $F(H)$ denotes the subset of Σ^7 pointwisely fixed by H; $F(H)=\{x \in \Sigma^7 \mid H \subset G_x\}$. It is well known that each connected component of $F(H)$ is a smooth submanifold of Σ^7. If H and K are two closed subgroups such that $H \subset K$, then $F(K) \subset F(H)$.

First we note that D_2 is isomorphic to $Z_2 \times Z_2$ and the all elements of order 2 in $SO(3)$ are mutually conjugate. Hence by a theorem of A. Borel concerning elementary-abelian-group actions on spheres ([**1**] XIII) we have

$$7-\dim F(D_2)=3 \dim F(Z_2)-3 \dim F(D_2) .$$

It follows that $\dim F(Z_2)=5$ and $\dim F(D_2)=4$ or $\dim F(Z_2)=3$ and $\dim F(D_2)=1$. Now we have shown in [**5**] that the action has fixed points if $\dim F(Z_2)=5$ or if $\dim F(Z_2)=3$ and $F(Z_2)=F(SO(2))$ (Theorem III [**5**]). Therefore if (Σ^7, φ) has no fixed point we have $\dim F(Z_2)=3$, $\dim F(D_2)=1$ and $F(Z_2) \neq F(SO(2))$. By P. A. Smith's theorem ([**1**] III), $F(D_2)$ is a Z_2-homology sphere, hence a circle.

Now $SO(2)$ acts on $F(Z_2)$ and its fixed point set is $F(SO(2))$. By the dimension parity, $\dim F(SO(2))=1$ or -1. But $F(SO(2))$ is not empty by Theorem 4 of [**4**] (this theorem is proved for actions on the standard 7-sphere in [**4**]. But as the proof uses only the differentiability and the homology properties, it holds also for actions on homotopy 7-spheres). Hence $\dim F(SO(2))=1$. By P. A. Smith's theorem, $F(SO(2))$ is a Z-homology sphere. Therefore $F(SO(2))$ is a circle.

Let Y be the orbit space $F(Z_2)/SO(2)$. Y is an orientable 2-manifold with boundary $\partial Y=F(SO(2))$. Let $p: F(Z_2) \rightarrow Y$ be the projection. Then p_*: $H_1(F(Z_2); \boldsymbol{Z}_2) \rightarrow H_1(Y; \boldsymbol{Z}_2)$ is onto. As $F(Z_2)$ is a $\boldsymbol{Z}_2$-homology 3-sphere, $H_1(F(Z_2); \boldsymbol{Z}_2)=0$. Hence $H_1(Y; \boldsymbol{Z}_2)=0$. It follows that Y is the 2-disc D^2.

The quotient group $N/SO(2)=Z_2$ acts on Y and its fixed point set is $p(F(D_2))$. It is 1-dimensional. By P. A. Smith's theorem, it is $\boldsymbol{Z}_2$-acyclic (note that $Y=D^2$ is $\boldsymbol{Z}_2$-acyclic). Therefore $p(F(D_2))$ is an arc with two endpoints in ∂Y. It follows that $F(SO(2)) \cap F(D_2)=F(N)$ consists of 2 points.

Now the octahedral group O (see § 1) is the normalizer of D_2. The quotient group O/D_2 acts on $F(D_2)$. O/D_2 is isomorphic to the symmetric group of 3 letters. The subgroup of O/D_2 generated by the class of $d=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is a cyclic group of order 3. As $F(D_2)$ is a circle, this subgroup acts on $F(D_2)$ freely or trivially. If it acts on $F(D_2)$ trivially, then for $x \in F(N)$, G_x contains N and d. But as N is maximal in $SO(3)$, G_x must be $SO(3)$ and x is a fixed point. This is a contradiction. Hence the above group acts on $F(D_2)$ freely, that is, it acts by the rotation of $2\pi k/3$ angles $(k=1, 2, 3)$. This group is the only normal subgroup of O/D_2, hence O/D_2 acts on $F(D_2)$ effectively. The class of $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ in O/D_2 is of order 2 and leaves $F(N)$ pointwisely fixed, and acts on $F(D_2)$ by the reflection through the diameter whose endpoints are $F(N)$.

We put $N_1 = dNd^{-1}$ and $N_2 = d^2Nd^{-2}$. N_1 and N_2 contain D_2. $F(N_1) = dF(N)$ and $F(N_2) = d^2F(N)$. They are contained in $F(D_2)$ and consist of two points.

Now the $SO(2)$-action on $F(Z_2)$ induces naturally an action of $SO(2)/Z_2$ on $F(Z_2)$.

LEMMA 3.1. *$SO(2)/Z_2$ acts on $F(Z_2)$ semifreely, and its fixed point set is $F(SO(2))$.*

PROOF. Since $F(SO(2))$ is not empty and $\pi_1(D^2)$ is trivial, $\pi_1(F(Z_2))$ is trivial. $F(Z_2)$ is a simply connected 3-manifold. Hence it is a $\boldsymbol{Z}$-homology 3-sphere. Now let p^r be a power of a prime p such that $p^r \geqq 3$. Z_{p^r} acts on $F(Z_2)$ orientation preservingly. As $F(Z_2)$ is a $\boldsymbol{Z}$-homology 3-sphere, $F(Z_{p^r}) \cap F(Z_2)$ (this is the fixed point set of the above Z_{p^r}-action on $F(Z_2)$) is a $\boldsymbol{Z}_p$-homology sphere. By the dimension parity, the dimension of it is 1 or 3 (note that $F(Z_{p^r}) \cap F(Z_2) \supset F(SO(2))$ is not empty). Hence it is connected. If $F(Z_{p^r}) \cap F(Z_2)$ is 3-dimensional, then it coincides with $F(Z_2)$ and for $x \in F(N_1)$ $(\subset F(Z_2))$, G_x contains N_1 and Z_{p^r}. As N_1 is maximal in $SO(3)$, G_x must be $SO(3)$. This is a contradiction. Therefore $F(Z_{p^r}) \cap F(Z_2)$ is 1-dimensional, hence coincides with $F(SO(2))$. It follows that the quotient group $SO(2)/Z_2$ acts on $F(Z_2)$ semifreely with fixed point set $F(SO(2))$. Q. E. D.

LEMMA 3.2. *If $x \in F(D_2)$, then $G_x = D_2$ or N or N_1 or N_2.*

PROOF. Since N, N_1 and N_2 are maximal in $SO(3)$, for $x \in F(N) \cup F(N_1) \cup F(N_2)$, G_x is N or N_1 or N_2. Now N, N_1 and N_2 are the all of the proper infinite subgroups which contain D_2. Hence for $x \in F(D_2) - (F(N) \cup F(N_1) \cup F(N_2))$, G_x is a finite subgroup containing D_2. By the argument before Lemma 3.1, O/D_2 acts on $F(D_2)$ effectively and $G_x \cap O = D_2$ if $x \notin F(N) \cup F(N_1) \cup F(N_2)$.

Hence G_x must be D_{2k} for some positive integer k. If $k \geqq 2$, then $G_x = D_{2k}$ contains a cyclic subgroup Z_{p^r} for some positive prime power $p^r \geqq 3$. By Lemma 3.1 $F(Z_{p^r}) \cap F(Z_2) = F(SO(2))$. Hence $x \in F(SO(2)) \cap F(D_2) = F(N)$. This is a contradiction. It follows that $k = 1$. Q. E. D.

LEMMA 3.3. *Let S_β be the unit sphere in R^5_β as in* § 1. *The orbit of $F(D_2)$, $SO(3)F(D_2)$, is a smooth $SO(3)$-manifold and is equivariantly diffeomorphic to S_β.*

PROOF. First we show that if $g \notin O$ and $gF(D_2) \cap F(D_2) \neq \emptyset$, then $gF(D_2) \cap F(D_2) = F(N)$ or $F(N_1)$ or $F(N_2)$. Let x be a point of $F(D_2) - (F(N) \cup F(N_1) \cup F(N_2))$. Then G_x is D_2 by Lemma 3.2. If $gx \in F(D_2)$, then $gG_xg^{-1} = gD_2g^{-1} = D_2$, hence $g \in O$. Consequently if $g \notin O$, then $gF(D_2) \cap F(D_2) = \emptyset$ or $F(N)$ or $F(N_1)$ or $F(N_2)$. Now the two points of $F(N)$ are not in a same orbit. $F(N)$, $F(N_1)$ and $F(N_2)$ are translated onto one another by the action of O. Therefore we have $SO(3)F(D_2)/SO(3) = F(D_2)/O$. Hence $SO(3)F(D_2)/SO(3)$ is an arc. The interior points of the arc are the image of the orbits of type $(SO(3)/D_2)$ and the two endpoints are the image of the orbits of type $(SO(3)/N)$. Since the projection $(SO(3)/D_2 \to SO(3)/N)$ is a circle bundle, $SO(3)F(D_2)$ is a smooth $SO(3)$-manifold. Now there are just two equivariant diffeomorphism classes of such $SO(3)$-manifolds, and the fixed point set of D_2 of the class of S_β is a circle and that of the other class is disconnected (see [2] and Lemma 2.1 in § 2 of [5]). The Lemma follows. Q. E. D.

LEMMA 3.4. *Let x be a point of $F(SO(2))$. For a positive integer i, let t^i be the 2-dimensional real representation of $SO(2)$, t^i; $SO(2) \to SO(2)$ with kernel Z_i. Then the tangential representation of $SO(2)$ at x is $t^{2k+1} + t^2 + t + 1$, where* 1 *denotes the* 1*-dimensional trivial representation and k is a positive integer.*

PROOF. Since $F(SO(2))$ is connected, the representations of $SO(2)$ at x and y are equivalent for any two points x and $y \in F(SO(2))$. Hence we may take as x a point of $F(N) \subset F(SO(2))$. Now $F(N) \subset F(D_2) \subset S_\beta$. The tangent space at x is decomposed as $V_0 + V_1 + V_2$, where V_0 is the normal subspace to S_β, V_1 is the tangent space of the orbit $SO(3)x$ ($= P^2$ the real projective plane), and V_2 is the normal subspace to $SO(3)x$ in S_β. The dimensions of V_0, V_1 and V_2 are 3, 2 and 2 respectively. V_0, V_1 and V_2 are all N-invariant. N acts on V_1 by the homomorphism $N \to O(2)$ with trivial kernel, and acts on V_2 by the homomorphism $N \to O(2)$ with kernel $Z_2 (\subset SO(2))$. Since $\dim F(Z_2) = 3$ and $\dim F(SO(2)) = 1$, the representation of N in V_0 is given by the homomorphism $N \to N/Z_{2k+1} = N \subset SO(3)$ for some integer $k \geqq 0$ (the first map is the quotient map). Now the representation of $SO(2)$ at x is $(t^{2k+1} + 1)$ in V_0, t in V_1 and t^2 in V_2. Hence the tangential representation of $SO(2)$ at x is $t^{2k+1} + t^2 + t + 1$.

Q. E. D.

PROOF OF THEOREM I. By Lemma 3.2, (D_2) and (N) appear as isotropy subgroup types. For a point $x \in F(SO(2)) - F(N)$, G_x is $SO(2)$. Thus we have shown that (D_2), $(SO(2))$ and (N) appear as isotropy subgroup types. Now let x be a point of Σ^7 such that G_x is a finite nontrivial subgroup. Let $H\subset G_x$ be a nontrivial cyclic subgroup of G_x. There is an element $g\in SO(3)$ such that $gHg^{-1}\in SO(2)$. Let $p^r\geqq 2$ be a prime power such that $Z_{p^r}\subset gHg^{-1}$. Then $F(Z_{p^r})$ is a $\boldsymbol{Z}_p$-homology sphere and $F(Z_{p^r})\supset F(SO(2))$. By Lemma 3.4, p^r divides 2 or $2k+1$. First, we assume that $k=0$. In this case, the only possibility of p^r is 2. Hence, all the nontrivial cyclic subgroup of G_x must be of order 2. G_x is conjugate to Z_2 or D_2. Therefore if $x\in F(Z_2)-SO(3)F(D_2)$ $(\neq\emptyset)$, then $G_x=Z_2$. Thus (Z_2) appears as an isotropy subgroup type. By the above argument, it can be seen that for each point $x\in\Sigma^7-SO(3)F(Z_2)$, G_x is the trivial group (e) (note that $\dim F(Z_2)=3$ and $\dim SO(3)F(Z_2)=5$). We obtain type (a) of Theorem I in this case.

Nextly, we assume that $k\geqq 1$. As $F(I)$, $F(O)$ and $F(T)$ are contained in $F(D_2)$, $F(I)$, $F(O)$ and $F(T)$ are empty by Lemma 3.2. Hence (I), (O) and (T) cannot occur. For a prime power p^r such that $p^r|2k+1$ we have $F(Z_{p^r})\supset F(Z_{2k+1})$. By Lemma 3.4, the dimensions of $F(Z_{p^r})$ and $F(Z_{2k+1})$ are both 3. Since $F(Z_{p^r})$ is a $\boldsymbol{Z}_p$-homology sphere, it is connected. It follows that $F(Z_{p^r})=F(Z_{2k+1})$. Hence if $x\in\Sigma^7$ and G_x is a nontrivial finite groups, then each maximal cyclic subgroup of G_x is of order 2 or $2k+1$. Therefore G_x is conjugate to Z_2 or D_2 or Z_{2k+1} or D_{2k+1}. Now if $x\in F(Z_2)-(SO(3)F(D_2)\cup SO(3)F(Z_{2k+1}))$, then $G_x=Z_2$. Thus (Z_2) appear as an isotropy subgroup type. By the proof of Lemma 3.4, it can be seen that (Z_{2k+1}) and (D_{2k+1}) appear as isotropy subgroup types. If $x\in\Sigma^7-(SO(3)F(Z_2)\cup SO(3)F(Z_{2k+1}))$, then G_x must be the trivial group (e). Hence we obtain type (b) of Theorem I in this case.

§4. Actions with exotic isotropy subgroup type.

In this section, we shall prove Theorem II (see §0).

Let $R^8_{\alpha\oplus\beta}$ be the direct sum of R^3_α and R^5_β as in §2. In this section, S^7 denotes the 7-sphere with the linear $SO(3)$-action $\alpha\oplus\beta$, that is, the unit sphere in $R^8_{\alpha\oplus\beta}$.

Let v_1, v_2 and v_3 $(\in R^3_\alpha)$ be as follows; $v_1=(1,0,0)$, $v_2=(0,1,0)$, $v_3=(0,0,1)$. Let y_1, y_2 and y_3 $(\in R^5_\beta)$ be as follows;

$$y_1=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad y_2=\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix},\quad y_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}.$$

Now we put

$$w_1=v_3,\quad w_2=v_1+y_1,\quad w_3=v_2+y_2\quad\text{and}\quad w_4=y_3$$

considered as elements of $R^8_{\alpha\oplus\beta}$.

LEMMA 4.1. *The isotropy subgroups of w_i are as follows;*

$$G_{w_1}=SO(2)\,, \qquad G_{w_4}=N\,,$$

$$G_{w_2}=\left\{e,\ \begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\right\},\quad G_{w_3}=\left\{e,\ \begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\right\}.$$

PROOF. We note that if $x_1\in R^3_\alpha$, $x_2\in R^5_\beta$ and $x=x_1+x_2\in R^8_{\alpha\oplus\beta}$, then $G_x=G_{x_1}\cap G_{x_2}$. A simple calculation gives the result. Q.E.D.

Let W be the 4-dimensional subspace of $R^8_{\alpha\oplus\beta}$ spanned by $\{w_i\}_{i=1,2,3,4}$. Let S be the unit sphere in W. Then $S\subset S^7$.

LEMMA 4.2. *For $g\in SO(3)$, $S\cap gS$ is not empty if and only if $g\in N$.*

PROOF. First, we prove the following two sublemmas.

SUBLEMMA 1. *Let U be the 2-dimensional subspace of R^5_β spanned by y_1, y_3. Let $g=(a_{ij})$ be an element of $SO(3)$. If $gY\in U$ for some $Y(\neq 0)\in U$, then g belongs to N or has the form* $\begin{bmatrix}*&0&0\\0&&\\0&&*\end{bmatrix}$.

PROOF. Put $Y=ty_1+ry_3$, where t, r are real numbers. As $gY\in U$, the (1, 2) and (1, 3) components of the matrix $gY=gYg^{-1}$ is 0 and the (1, 1) component is equal to the (2, 2) component. Hence we have the following equations

$$t(a_{13}a_{22}+a_{12}a_{23})+r(-3a_{13}a_{23})=0$$

$$t(a_{13}a_{32}+a_{12}a_{33})+r(-3a_{13}a_{33})=0$$

$$t(2a_{13}a_{12}-2a_{22}a_{23})+r(3a_{23}^2-3a_{13}^2)=0$$

As $(t, r)\neq(0, 0)$, we have

1) $a_{13}(a_{22}a_{33}-a_{23}a_{32})=0$

2) $(a_{13}^2+a_{23}^2)(a_{12}a_{23}-a_{13}a_{22})=0$

and if $a_{22}=a_{23}=0$, then

3) $a_{13}(a_{12}a_{33}-a_{13}a_{32})=0$.

The equations 1), 2) and 3) shows that $a_{13}=a_{23}=0$ or $a_{12}=a_{13}=0$. Q.E.D.

SUBLEMMA 2. *Let W_0 be the 3-dimensional subspace of $R^8_{\alpha\oplus\beta}$ spanned by $\{w_1, w_2, w_4\}$. Let g be an element of $SO(3)$. If $gZ\in W_0$ for some $Z(\neq 0)\in W_0$, then $g\in N$.*

PROOF. Put $Z=tw_1+rw_2+sw_4$, where t, r and s are real numbers. Now $Z=(tv_3+rv_1)+(ry_1+sy_3)$. If $(r, s)=(0, 0)$, then $Z=tv_3$ and gZ must be $\pm tv_3$. Hence $g\in N$. We assume that $(r, s)\neq(0, 0)$. Let $Y=ry_1+sy_3$. Then $Y\in U$. As $gZ\in W_0$, $gY\in U$. By Sublemma 1, $g\in N$ or g can be written as $\begin{bmatrix}\varepsilon&0&0\\0&\varepsilon a&\varepsilon b\\0&-b&a\end{bmatrix}$

where $\varepsilon=\pm 1$ and $a^2+b^2=1$. In the latter case $gv_1=\varepsilon v_1$, $gv_3=\varepsilon bv_2+av_3$ and the (1, 1) component of the matrix $gy_3=gy_3g^{-1}$ is $+1$. As $gZ\in W_0$, we see that $g(rw_2+sw_4)=r\varepsilon w_2+sw_4$. Hence $gY=g(ry_1+sy_3)=r\varepsilon y_1+sy_3$. Calculating (2, 2) and (2, 3) component of the matrix gY, we obtain the following two equations, $r(a^2-b^2)+s(-3ab)=r$ and $r(2ab)+s(a^2-2b^2)=s$. From these equations, $b=0$ follows. Hence $g\in N$. Q.E.D.

Now we proceed to the proof of Lemma 4.2. Put $S_0=S\cap W_0$. Then $S=NS_0$ and S is N-invariant. Let $g\in SO(3)$ be such an element as $S\cap gS\neq\emptyset$. Then $Y=gX$ for some X, $Y\in S$. Since $S=NS_0$, $Y=n_1Y_0$ and $X=n_2X_0$ for some X_0, $Y_0\in S_0$ and n_1, $n_2\in N$. Now $Y_0=(n_1^{-1}gn_2)X_0$. By Sublemma 2, $n_1^{-1}gn_2\in N$, hence $g\in N$. Q.E.D.

Now by Lemma 4.2, $SO(3)S$ is equivariantly diffeomorphic to $SO(3)\times_N S$. Let ν be the equivariant normal bundle of $SO(3)S$ in S^7. Let ν_0 be the restriction $\nu|S$. Then ν_0 is N-equivariant bundle over S and ν is equivalent to $SO(3)\times_N\nu_0$. Let $R^2_{\delta_i}$ be the 2-dimensional real vector space on which N acts by the homomorphism $\delta_i: N\to O(2)$ with kernel Z_i ($Z_1=\{e\}$). Then as an N space, $R^8_{\alpha\oplus\beta}=W+R^2_{\delta_1}+R^2_{\delta_2}$. Hence the normal bundle of S in S^7 is N-equivalent to $S\times(R^2_{\delta_1}+R^2_{\delta_2})$. Let $p: SO(3)\times_N S\to SO(3)/N=P^2$ be the projection. Let x be the point of P^2 such that $G_x=N$. Then the normal bundle of S, $\bar{\nu}$, in $SO(3)\times_N S$ is N-equivalent to $(p|S)^*TP^2_x=S\times TP^2_x$, where $p|S$ is the restriction of p to S and TP^2_x denotes the tangent space of P^2 at x. Now N acts on TP^2_x by the homomorphism $\delta_1: N\to O(2)$ with trivial kernel. Hence $\bar{\nu}$ is N-equivalent to $S\times R^2_{\delta_1}$. Therefore ν_0 is N-equivalent to $S\times R^2_{\delta_2}$ and ν is equivalent to $SO(3)\times_N(S\times R^2_{\delta_2})$.

Let D^2 be the unit disk in $R^2_{\delta_2}$. Then by the above argument, there is an equivariant embedding $\mu: SO(3)\times_N(S\times D^2)\to S^7$ such that $\mu(SO(3)\times_N(S\times\{0\}))=SO(3)S$.

Let W_k be the 4-dimensional real vector space on which N acts by the homomorphism $\psi_k: N\to N/Z_{2k+1}=N\to SO(3)\to SO(4)$, where the first map is the quotient map, and the second and the last are the canonical inclusion. Let S_k be the unit sphere in W_k. Then N acts on S_k with isotropy subgroup type $\{(Z_{2k+1}), (D_{2k+1}), (SO(2)), (N)\}$. Now let S^1 be the unit sphere in $R^2_{\delta_2}$, that is $\partial D^2=S^1$.

LEMMA 4.3. *There is an N-equivariant diffeomorphism* $\tilde{H}: W\times S^1\to W_k\times S^1$.

PROOF. Let R^1 and R^1_τ be the 1-dimensional real vector spaces on which N acts trivially and by the homomorphism $\tau: N\to O(1)$ with kernel $SO(2)$ respectively. Then as an N-space, W is decomposed as $R^2_{\delta_1}+R^1_\tau+R^1$. Similarly, W_k is decomposed as $R^2_{\delta_{2k+1}}+R^1_\tau+R^1$.

We identify $SO(2)$ with the complex numbers $\{z; |z|=1\}$. Put $c=$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Then, by choosing a suitable complex structure on $R^2_{\delta_1}$ and $R^2_{\delta_{2k+1}}$, we can write down the actions of N on them as follows; for $z \in SO(2)$ and $w \in R^2_{\delta_i}$, z acts on w by the complex multiplication by z^i $(i=1, 2k+1)$ and c acts on w by the complex conjugation, that is $cw=\bar{w}$. Similarly, by identifying S^1 suitably with the complex numbers $\{w; |w|=1\}$, we can write down the action of N on S^1 as follows; for $z \in SO(2)$ and $w \in S^1$, z acts on w by the complex multiplication by z^2 and $cw=\bar{w}$.

Now we define $\tilde{H}: W \times S^1 \to W_k \times S^1$ by $\tilde{H}(w+x+y, w_0)=(w_0^k w+x+y, w_0)$ where $w \in R^2_{\delta_1}$, $x \in R^1_{\tau}$, $y \in R^1$ and $w_0 \in S^1$ and $w_0^k w$ denotes the complex multiplication (considered as an element of $R^2_{\delta_{2k+1}}$). $\tilde{H}$ is a diffeomorphism. We show that $\tilde{H}$ is an N-equivariant map. For $z \in SO(2) \subset N$, $\tilde{H}(z(w+x+y, w_0))$ and $z\tilde{H}(w+x+y, w_0)$ are both equal to $(z^{2k+1}w_0^k w+x+y, z^2 w_0)$. For c, $\tilde{H}(c(w+x+y, w_0))$ and $c\tilde{H}(w+x+y, w_0)$ are both equal to $(\bar{w}_0^k \bar{w}+(-x)+y, \bar{w}_0)$. Hence $\tilde{H}$ is an N-equivariant map. Q.E.D.

If we restrict the above map to $S \times S^1 \subset W \times S^1$, we obtain an N-equivariant diffeomorphism $\tilde{H}: S \times S^1 \to S_k \times S^1$. Hence we obtain an $SO(3)$ equivariant diffeomorphism

$$H=1 \times_N \tilde{H}: SO(3) \times_N (S \times S^1) \longrightarrow SO(3) \times_N (S_k \times S^1).$$

Now as before, let $\mu: SO(3) \times_N (S \times D^2) \to S^7$ be an equivariant embedding. Let $\mathring{D}^2$ be the interior of D^2. Put

$$G=H \circ \mu^{-1}: \mu(SO(3) \times_N (S \times S^1)) \longrightarrow SO(3) \times_N (S_k \times S^1).$$

Let

$$\Sigma^7_k=(S^7-\mu(SO(3) \times_N (S \times \mathring{D}^2))) \cup_G SO(3) \times_N (S_k \times D^2)$$

be the manifold obtained from the disjoint union $S^7-\mu(SO(3) \times_N (S \times \mathring{D}^2)) \cup SO(3) \times_N (S_k \times D^2)$ by identifying their boundaries by G. This manifold is a differentiable $SO(3)$-manifold with isotropy subgroup type $\{(e), (Z_2), (D_2), (SO(2)), (N), (Z_{2k+1}), (D_{2k+1})\}$.

LEMMA 4.4. *Σ^7_k is a homotopy sphere.*

PROOF. Put $L_0=S^7-\mu(SO(3) \times_N (S \times \mathring{D}^2))$. L_0 is an $SO(3)$-manifold with boundary $\partial L_0=\mu(SO(3) \times_N (S \times S^1))$. Then,

$$\pi_1(\Sigma^7_k)=\pi_1(L_0) * \pi_1(SO(3) \times_N (S_k \times D^2))/\pi_1(SO(3) \times_N (S \times S^1))$$

where $*$ denotes the amalgamated product and the two inclusions of $\pi_1(SO(3) \times_N (S \times S^1))$ into the two factors are induced by μ and H respectively. Now $\pi_1(SO(3) \times_N (S_k \times D^2))=\boldsymbol{Z}_2=\pi_1(SO(3) \times_N (S \times D^2))$, and the diagram

$$\begin{tikzcd} \pi_1(SO(3)\times_N(S\times S^1)) \arrow[r, "H_*"] \arrow[dr, "j_*"'] & \pi_1(SO(3)\times_N(S_k\times D^2)) \arrow[d, equal] \\ & \pi_1(SO(3)\times_N(S\times D^2)) \end{tikzcd}$$

(j is the inclusion)

is commutative. Hence, $\pi_1(\Sigma_k^7)=\pi_1(S^7)=1$. Now $H_*(SO(3)\times_N(S_k\times D^2)\,;\boldsymbol{Z})$ and $H_*(SO(3)\times_N(S\times D^2)\,;\boldsymbol{Z})$ are both isomorphic to $H_*(P^2\,;\boldsymbol{Z})\otimes H_*(S^3\,;\boldsymbol{Z})$, where P^2 denotes the real projective plane. The diagram

$$\begin{tikzcd} H_*(SO(3)\times_N(S\times S^1)\,;\boldsymbol{Z}) \arrow[r, "H_*"] \arrow[dr, "j_*"'] & H_*(SO(3)\times_N(S_k\times D^2)\,;\boldsymbol{Z}) \arrow[d, equal] \\ & H_*(SO(3)\times_N(S\times D^2)\,;\boldsymbol{Z}) \end{tikzcd}$$

is commutative. Therefore, the Mayer-Vietoris sequence for the triple $(\Sigma_k^7, L_0, SO(3)\times_N(S_k\times D^2))$ shows that $H_*(\Sigma_k^7\,;\boldsymbol{Z})$ is isomorphic to $H_*(S^7\,;\boldsymbol{Z})$. Consequently Σ_k^7 is a homotopy sphere. Q. E. D.

LEMMA 4.5. *Σ_k^7 is diffeomorphic to the standard 7-sphere.*

PROOF. Let D and D_k be the unit 4-discs in W and W_k respectively. Then $\partial D=S$ and $\partial D_k=S_k$. Let D^8 be the unit disc in $R^8_{\alpha\oplus\beta}$. Then $\partial D^8=S^7$. Let $X=D^8\cup SO(3)\times_N(D\times D^2)$ be the disjoint union, where D^2 is the unit disc in $R^2_{\delta_2}$ as before. Let $\sim$ be an equivalence relation on X such that for $x, y\in X$, $x\sim y$ if and only if $x=y$ or $x\in SO(3)\times_N(S\times D^2)$ and $y=\mu(x)\in S^7$. Then, we have a manifold $K_1=X/\sim$ which has a differentiable structure by corner rounding. Similarly, let K_2 be a manifold obtaining from the disjoint union $\Sigma_k^7\times[0, 1]\cup SO(3)\times_N(D_k\times D^2)$ by identifying $x\in SO(3)\times_N(S_k\times D^2)$ and the corresponding point $y\in SO(3)\times_N(S_k\times D^2)\subset\Sigma_k^7\times\{1\}$. Then we have

$$\partial K_1=(S^7-\mu(SO(3)\times_N(S\times \mathring{D}^2)))\cup_\mu SO(3)\times_N(D\times S^1)$$

where $\mu\colon SO(3)\times_N(S\times S^1)\to\mu((SO(3)\times_N(D\times S^1))$, and

$$\partial K_2=(S^7-\mu(SO(3)\times_N(S\times \mathring{D}^2)))\cup_{\mu\circ H^{-1}} SO(3)\times_N(D_k\times S^1)$$
$$\cup \text{ disjoint union } \Sigma_k^7\times\{0\}\,,$$

where $\mu\circ H^{-1}\colon SO(3)\times_N(S_k\times S^1)\to\mu(SO(3)\times_N(S\times S^1))$. By Lemma 4.3, H^{-1} can be extended to a diffeomorphism $H^{-1}\colon SO(3)\times_N(D_k\times S^1)\to SO(3)\times_N(D\times S^1)$. Hence we have a diffeomorphism, $F\colon\partial K_1\to(\partial K_2-\Sigma_k^7\times\{0\})$. Now we define a manifold K by $K=K_1\cup_F K_2$. Then ∂K is diffeomorphic to Σ_k^7. Let $[\Sigma_k^7]$ be the orientation class of Σ_k^7. We determine an orientation class of K, $[K]$ by $\partial[K]=[\Sigma_k^7]$.

SUBLEMMA. *The integral cohomology groups of K, $H^*(K)$, are as follows;*

$H^0=\boldsymbol{Z}$, $H^3=\boldsymbol{Z}_2$, $H^4=\boldsymbol{Z}+\boldsymbol{Z}$, $H^6=\boldsymbol{Z}_2$ *and* $H^j=0$, j *otherwise.*

PROOF OF SUBLEMMA. K_1 is homotopically equivalent to the quotient space $SO(3)\times_N D/SO(3)\times_N S$. Hence, $H^*(K_1)$ are as follows; $H^0=H^4=\boldsymbol{Z}$, $H^6=\boldsymbol{Z}_2$ and $H^j=0$, j otherwise. As CW complexes, $K_1=K_2\cup$ one 8-cell, and $H^*(K_2)$ are as follows; $H^0=H^4=H^7=\boldsymbol{Z}$, $H^6=\boldsymbol{Z}_2$ and $H^j=0$, j otherwise. Now let $L=K_1\cap K_2$. $L=L_0\cup L_1$, where $L_0=S^7-\mu(SO(3)\times_N(S\times \mathring{D}^2))$ and $L_1=SO(3)\times_N(D\times S^1)$. Then $L_0\cap L_1=SO(3)\times_N(S\times S^1)$. By the Mayer-Vietoris sequence for the triple $(L_0, L_1, L_0\cap L_1)$, we have $H^*(L)$ as follows; $H^0=H^3=H^4=H^7=\boldsymbol{Z}$, $H^2=H^6=\boldsymbol{Z}_2$ and $H^j=0$, j otherwise. Now by the Mayer-Vietoris sequence for the triple (K_1, K_2, L), we obtain the result. Q.E.D.

We continue the proof of Lemma 4.5. Let SO denote the inductive limit $\lim SO(n)$. The homotopy groups $\pi_*(SO)$ are as follows; $\pi_i=\boldsymbol{Z}_2$ if $i\equiv 1, 0 \pmod 8$, $\pi_i=\boldsymbol{Z}$ if $i\equiv 3, 7 \pmod 8$, and $\pi_i=0$ otherwise. Hence by Sublemma, the only obstruction for the parallelizability of K lies in $H^4(K;\boldsymbol{Z})=\boldsymbol{Z}+\boldsymbol{Z}$. Let D^4 be the 4-disc $\{e\}\times(D\times\{0\})\subset SO(3)\times_N(D\times D^2)\subset K_1$. Then $\partial D^4=S=S^7\cap W$. As S bounds the 4-disc $D^8\cap W$, we obtain an embedded 4-sphere S^4 in K_1. The normal bundle of S^4 is trivial. The 4-cycle $[S^4]$ and its dual 4-cycle generate $H_4(K;\boldsymbol{Z})=\boldsymbol{Z}+\boldsymbol{Z}$. If we carry a surgery at S^4, we obtain a manifold $\tilde{K}$ such that $H^4(\tilde{K};\boldsymbol{Z})=0$ and $H^j(\tilde{K};\boldsymbol{Z})=H^j(K;\boldsymbol{Z})$ for $j\neq 4$. Hence, $\tilde{K}$ is parallelizable and its index is 0. As $\partial\tilde{K}=\Sigma_k^7$, Σ_k^7 is diffeomorphic to the standard sphere ([3]). Q.E.D.

This completes the proof of Theorem II.

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