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On flat over-rings of a Krull domain

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Introduction.

Let A be an integral domain and let K be the quotient field of A. In this paper we are mainly concerned with a subring B of K containing A. For the sake of simplicity we shall call such an intermediate ring an over ring of A hereafter. The purpose of this paper is to study the relationship between an over ring B and subsets $F_A(B)$ and $F_A^*(B)$ of Spec A defined by

$$F_A(B)=\{\mathfrak{p}\in \operatorname{Spec} A\,;\ A_{\mathfrak{p}}\subsetneqq B\otimes_A A_{\mathfrak{p}}=B_{\mathfrak{p}}\}$$

and

$$F_A^*(B)=\{\mathfrak{p}\in F_A(B)\,;\ \text{height}\,\mathfrak{p}=1\}$$

respectively. Among others it will be shown that if A is a Krull domain and B is a flat over-domain of A, then B is determined uniquely by $F_A^*(B)$. Moreover if B is a flat over-domain of A, B is finitely generated over A if and only if $F_A^*(B)$ is a finite set.

Following the usual terminology, rings are always understood to be commutative and to have the identity elements. For a ring A, Spec A stands for the set of all prime ideals of A and $\mathrm{Ht}_1(A)$ is the set of all prime ideals of A with height 1.

§ 1. On $F_A(B)$.

The following well-known fact will be used frequently in this paper, so we write down it as a lemma without proof (cf. [3]).

(1.1) LEMMA. *Let A be a ring and B an A-algebra contained in the total quotient ring of A. Then the following four conditions are equivalent to each other:*

(1) *B is flat over A.*

(2) *$B_{\mathfrak{p}}=B\otimes_A A_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for any $\mathfrak{p}\in \operatorname{Spec} A$.*

(3) *$A_{A\cap\mathfrak{P}}=B_{\mathfrak{P}}$ for any $\mathfrak{P}\in \operatorname{Spec} B$.*

(4) *For every $\mathfrak{p}\in \operatorname{Spec} A$, either $\mathfrak{p}B=B$ or $A_{\mathfrak{p}}=B_{\mathfrak{p}}$.*

Let A be an integral domain and let B be an over-ring of A. We shall introduce the sets:

$$F_A(B)=\{\mathfrak{p}\in\operatorname{Spec} A\ ;\ A_\mathfrak{p}\subsetneqq B_\mathfrak{p}\}\ ,$$

$$F_A^*(B)=F_A(B)\cap \operatorname{Ht}_1(A)\ .$$

General properties of $F_A(B)$ and $F_A^*(B)$ are summarized in the following three lemmas.

(1.2) LEMMA. *Let A be an integral domain and let B be an over-ring. Then $F_A(B)$ is closed under specializations. We have $F_A(B)=\emptyset$ if and only if $A=B$*[1].

PROOF. Let $\mathfrak{p}$ and q be prime ideals of A such that $\mathfrak{p}\subseteqq q$. If q is not an element of $F_A(B)$, $A_q=B_q\supseteqq B$. Therefore $A_\mathfrak{p}=(A_q)_{\mathfrak{p}A_q}\supseteqq B$. Hence $A_\mathfrak{p}=B_\mathfrak{p}$ namely $\mathfrak{p}\notin F_A(B)$ proving the first half of the lemma. If $F_A(B)=\emptyset$, then $A=\bigcap_{\mathfrak{p}\in\operatorname{Spec} A} A_\mathfrak{p}=\bigcap_{\mathfrak{p}\in\operatorname{Spec} A} B_\mathfrak{p}\supseteqq B$. Hence we have $A=B$. It is trivially seen that $F_A(A)=\emptyset$.

A maximal point of $F_A(B)$ is, by definition, a prime ideal of $F_A(B)$ which is minimal under inclusion.

(1.3) LEMMA. *If A is a Krull domain, any maximal point of $F_A(B)$ has height* 1.

PROOF. Let q be a maximal point of $F_A(B)$. We shall show that height $q=1$. Assuming the contrary, i.e., height $q>1$, we see that prime ideals which are properly contained in q are not in $F_A(B)$. Therefore $A_q=\bigcap_{\substack{\mathfrak{p}\in\operatorname{Ht}_1(A)\\ \mathfrak{p}\subseteqq q}} A_\mathfrak{p}=\bigcap_{\substack{\mathfrak{p}\in\operatorname{Ht}_1(A)\\ \mathfrak{p}\subseteqq q}} B_\mathfrak{p}\supseteqq B$. Hence we have $A_q=B_q$. This is a contradiction.

(1.4) LEMMA. *Let A be an integral domain and let B_1 and B_2 be over-rings of A such that $B_2\supseteqq B_1$. Then $F_A(B_2)\supseteqq F_A(B_1)$.*

PROOF. Let $\mathfrak{p}\notin F_A(B_2)$. Then $A_\mathfrak{p}=(B_2)_\mathfrak{p}\supseteqq(B_1)_\mathfrak{p}$. Hence $A_\mathfrak{p}=(B_1)_\mathfrak{p}$, i.e., $\mathfrak{p}\notin F_A(B_1)$.

(1.5) THEOREM. *Let A be an integral domain ond let B_1 and B_2 be over-rings of A. Assume that B_2 is flat over A. Then $F_A(B_2)\supseteqq F_A(B_1)$ if and only if $B_2\supseteqq B_1$.*

PROOF. By (1.4) it suffices to prove the "only if" part. Let $\mathfrak{P}\in\operatorname{Spec} B_2$ and $\mathfrak{p}=\mathfrak{P}\cap A$. Since B_2 is flat over A, $A_\mathfrak{p}=(B_2)_\mathfrak{P}$ by (1.1). Hence $A_\mathfrak{p}\supseteqq B_2$ and we see that $\mathfrak{p}\notin F_A(B_2)$. From the assumption it follows that $\mathfrak{p}\notin F_A(B_1)$, hence $A_\mathfrak{p}=(B_1)_\mathfrak{p}$. Since $B_1\subseteqq(B_1)_\mathfrak{p}=A_\mathfrak{p}=(B_2)_\mathfrak{P}$, we have $B_1\subseteqq\bigcap_{\mathfrak{P}\in\operatorname{Spec} B_2}(B_2)_\mathfrak{P}=B_2$.

(1.6) COROLLARY. *Let A be an integral domain and let B_1 and B_2 be flat over-rings of A. Then $F_A(B_1)=F_A(B_2)$ if and only if $B_1=B_2$.*

(1.7) LEMMA. *Let A be a Krull domain and let Δ be a subset of* $\operatorname{Ht}_1(A)$. *Let $C=\bigcap_{\mathfrak{p}\in\Delta} A_\mathfrak{p}$. Then we have $F_A^*(C)=\operatorname{Ht}_1(A)-\Delta$.*

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1) We denote by $\emptyset$ the empty set.

PROOF. As is well known, $\mathrm{Ht}_1(C)=\{C\cap\mathfrak{p}A_\mathfrak{p} \mid \mathfrak{p}\in\Delta\}$, from which the assertion follows easily.

From now on we shall mainly be concerned with flat over-rings B and we shall show how they are determined by $F_A(B)$.

(1.8) LEMMA. *Let A be a Krull domain and B a flat over-ring of A. Then* $B=\bigcap_{\mathfrak{p}\in\Delta} A_\mathfrak{p}$, *where* $\Delta=\mathrm{Ht}_1(A)-F_A^*(B)$.

PROOF. Obvious by virtue of (1.1), (1.4).

(1.9) THEOREM. *Let A be a Krull domain and let B be an over-ring of A. Then B is flat over A if and only if either $B_\mathfrak{p}=A_\mathfrak{p}$ or $\mathfrak{p}B=B$ holds for any $\mathfrak{p}$ in* $\mathrm{Ht}_1(A)$.

PROOF. From (1.1) it suffices to prove the "if part" of the theorem. If q is a prime ideal of A not in $F_A(B)$, then by definition $A_q=B_q$. Hence to prove the theorem it is sufficient to show that for any $q\in F_A(B)$ we have $qB=B$ (cf. [**1**]). From (1.3) there exists a prime ideal $\mathfrak{p}$ in $F_A^*(B)$ with $\mathfrak{p}\subseteqq q$. Since $A_\mathfrak{p}\neq B_\mathfrak{p}$ the assumption implies that, we have $\mathfrak{p}B=B$, a fortiori, $qB=B$.

(1.10) THEOREM. *Let A be a Krull domain and let B be an over-ring of A. If B is finitely generated over A, then $F_A^*(B)$ is a finite set. If we impose an additional assumption that B is flat over A, the converse also holds.*

PROOF. Suppose B is finitely generated over A, then there exists an element $a\in A$ such that we have $B\subseteqq A\left[\frac{1}{a}\right]$. Whence we see immediately that $F_A^*(B)$ is a finite set.

Conversely assume that B is a flat over-ring and $F_A^*(B)$ is a finite set, say, $F_A^*(B)=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_t\}$. Then $A_{\mathfrak{p}_i}\neq B_{\mathfrak{p}_i}$ and we must have $\mathfrak{p}_iB=B$ for $i=1,\cdots,t$ by (1.1). Hence we can find elements $a_k\in\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_t$ and $\alpha_k\in B$ such that $\sum_{k=1}^{n} a_k\alpha_k=1$. Let $C=A[\alpha_1,\cdots,\alpha_n]$. Then we have $\mathfrak{p}_iC=C$ for $i=1,\cdots,t$, and $F_A^*(C)\supseteqq\{\mathfrak{p}_1,\cdots,\mathfrak{p}_t\}$. On the other hand C is contained in B, hence we have the inclusion relation $F_A^*(C)\subseteqq F_A^*(B)=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_t\}$. Therefore we have $F_A^*(C)=F_A^*(B)=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_t\}$. For any prime ideal $\mathfrak{p}$ of height 1 other than $\mathfrak{p}_1,\cdots,\mathfrak{p}_t$, $\mathfrak{p}$ is not contained in $F_A^*(C)$, whence we have $A_\mathfrak{p}=C_\mathfrak{p}$. Then (1.9) implies that C is flat over A. Now $B=C$ follows from (1.6).

§ 2. Relations between epimorphic over-rings and flat over rings.

In this section A and B are not necessarily integral domains. Let A be a ring and let B be an A-algebra with the structure homomorphism $f: A\to B$. A ring homomorphism $f: A\to B$ is called an epimorphism, if for any ring C and any two homomorphisms $g, g': B\to C$, the relation $g\circ f=g'\circ f$ implies $g=g'$.

(2.1) LEMMA. *Let A be a ring and B an epimorphic A-algebra. Let M*

be a B-module which admits a direct sum decomposition $M=M_1\oplus M_2$ as A-modules. Then A-modules M_1 and M_2 have natural B-module structures and $M=M_1\oplus M_2$ as B-modules. In particular if $B=B_1\oplus B_2$ as A-modules, then B is a direct product of subrings B_1 and B_2.

PROOF. Let b be an element of B. Then it is known that there are elements $b_1, b_2, \cdots, b_r \in B$, $c_1, c_2, \cdots, c_s \in B$ and $\beta_{ij} \in A$ ($1 \leqq i \leqq r$ and $1 \leqq j \leqq s$) such that $b=\sum_{i,j}\beta_{ij}b_i c_j$ and both $\sum_i \beta_{ij}b_i$ and $\sum_j \beta_{ij}c_j$ are in A (cf. [3]). Then for any $m\in M$ we have $b\otimes m=1\otimes bm$. Define a B-module homomorphism ϕ: $B\otimes_A M \to M$ by $\phi(b\otimes m)=bm$ and a B-module homomorphism $\psi: M\to B\otimes_A M$ by $\psi(m)=1\otimes m$. Then the above consideration implies that $\psi\circ\phi=1_{B\otimes M}$ and $\phi\circ\psi=1_M$. Therefore $M\cong B\otimes_A M$ as B-modules. Now assume that M (regarded as A-module) is a direct sum of A-modules M_1 and M_2. Then we have $B\otimes_A M = B\otimes_A M_1 \oplus B\otimes_A M_2$. Let m be any element of M_1 and let b be an element of B. Write $bm=m_1+m_2$, where $m_1\in M_1$ and $m_2\in M_2$. Then $b\otimes m=\psi\circ\phi(b\otimes m) =\psi(bm)=\psi(m_1+m_2)=1\otimes m_1+1\otimes m_2$. Hence $b\otimes m-1\otimes m_1=1\otimes m_2\in B\otimes_A M_1 \cap B\otimes_A M_2=(0)$. Hence $1\otimes bm=1\otimes m_1$. Therefore $bm=m_1\in M_1$. Thus M_1 has a B-module structure and similarly M_2 has a B-module structure. It is now immediate to see that $M=M_1\oplus M_2$ as B-module.

(2.2) COROLLARY. *Let A be a ring and B an epimorphic A-algebra. Let M be a B-module. Then M is an irreducible B-module if and only if M is an irreducible A-module.*

The next lemma is proved in [3].

(2.3) LEMMA. *Let A be a Noetherian local ring and let B be a local A-algebra. If $f: A\to B$ is a local epimorphism, B is A-isomorphic to a localization of a finite A-algebra.*

Making use of (3.3), we can give a relationship between flat over-rings and epimorphic over-rings.

(2.4) THEOREM. *Let A be a Noetherian normal domain and B an over-ring of A. Then B is epimorphic over A if and only if B is flat over A.*

PROOF. The "if" part was proved in [3] in a more general setting. Hence we shall give here a proof of the "only if" part of the theorem. Assume that B is epimorphic over A. Let $\mathfrak{P}$ be any prime ideal in B and let $\mathfrak{p}=\mathfrak{P}\cap A$. Then $A_\mathfrak{p}\to B_\mathfrak{P}$ is a local epimorphism and $A_\mathfrak{p}$ is a Noetherian normal local domain. Hence by (3.3), $B_\mathfrak{P}$ is $A_\mathfrak{p}$-isomorphic to a localization C_Q of a finite A-algebra $C\subseteqq K$, where K is the quotient field of A. Indeed there is a finite $A_\mathfrak{p}$-algebra C' and a prime ideal Q' such that we have $B_P=C'_{Q'}$. Then we can take C, Q as the images of C', Q' in K. Since $A_\mathfrak{p}$ is normal, $C=A_\mathfrak{p}$, so $B_\mathfrak{P}=A_\mathfrak{p}$. Therefore B is flat over A.

In the next theorem we shall determine the structure of epimorphic A-algebras.

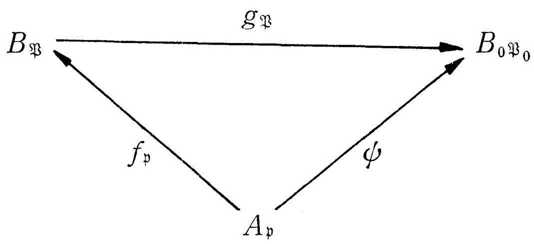
(2.5) THEOREM. *Let A be a Noetherian normal domain and B an epimorphic Noetherian A-algebra. Let I be the torsion A-submodule of B. Then the following exact sequence of A-modules*

$$0 \longrightarrow I \longrightarrow B \overset{g}{\longrightarrow} B/I \longrightarrow 0$$

splits as A-module and B is isomorphic to $I \times B/I$ as B-algebra.

PROOF. First of all, we shall show that I is a prime ideal in B. Let $B_0 = B/I$. Since B is epimorphic over A, B_0 is also epimorphic over A with a ring homomorphism $gf: A \overset{f}{\to} B \overset{g}{\to} B_0$ where f is a structure homomorphism of B and g is the natural homomorphism. Then $f \otimes 1: A \otimes_A K \to B_0 \otimes_A K$ is also an epimorphism. Since $A \otimes_A K = K$ is a field $f \otimes 1$ must be an isomorphism and we see that $B_0 \otimes_A K$ is a field. Being a subdomain of $B_0 \otimes_A K$, B_0 is also an integral domain.

Next we shall show that g is a flat homomorphism. Let $\mathfrak{P}_0$ be an arbitrary prime ideal in B_0, and let $\mathfrak{P} = g^{-1}(\mathfrak{P}_0)$ and $\mathfrak{p} = \mathfrak{P} \cap A$.



In the above diagram, ϕ is a local epimorphism, $A_{\mathfrak{p}}$ is a Noetherian normal local domain, and $B_{0\mathfrak{P}_0}$ is an over-ring of $A_{\mathfrak{p}}$ (cf. [3]). Therefore by the proof of (3.4), ϕ is an isomorphism, so $g_{\mathfrak{P}} \cdot f_{\mathfrak{p}} \cdot \phi^{-1} = 1_{B_{0\mathfrak{P}_0}}$. Hence $B_{\mathfrak{P}}$ is a direct sum of $B_{0\mathfrak{P}_0}$ and $\ker g_{\mathfrak{P}}$ as $A_{\mathfrak{p}}$-modules. By (3.1), $B_{\mathfrak{P}}$ is a direct product of $B_{0\mathfrak{P}_0}$ and $\ker g_{\mathfrak{P}}$ as rings because $B_{\mathfrak{P}}$ is epimorphic over $A_{\mathfrak{p}}$. On the other hand Spec $B_{\mathfrak{P}}$ is connected since $B_{\mathfrak{P}}$ is a local ring. Hence $\ker g_{\mathfrak{P}} = 0$ and $g_{\mathfrak{P}}$ is an isomorphism. Therefore g is flat.

Since g is a flat surjective homomorphism, the morphism Spec $B_0 \to$ Spec B is an open and closed immersion. Therefore Spec $B = V \amalg$ Spec B_0 for a closed subscheme V of Spec B. Since closed subscheme of an affine scheme are also affine ones, Spec $B =$ Spec $B/J \amalg$ Spec B/I for an ideal J in B. It is easy to show by using the Noetherian property of B that $B = B/I \times B/J$ and $B/J \cong I$.

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