J. Math. Soc. Japan
Vol. 27, No. 2, 1975

Curvature and metric in Riemannian 3-manifolds

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(Received Feb. 9, 1973)

§ 1. Introduction.

Let (M, g) and $(\bar{M}, \bar{g})$ be two Riemannian n-manifolds $(n \geqq 3)$ and f a diffeomorphism of (M, g) to $(\bar{M}, \bar{g})$. f is called a *curvature-preserving* diffeomorphism if for every point $p \in M$ and for every 2-plane section σ of the tangent space $T_p(M)$

$$\bar{K}(f_* \sigma) = K(\sigma)$$

holds, where K and $\bar{K}$ denote the sectional curvatures of (M, g) and $(\bar{M}, \bar{g})$, respectively. A point $p \in M$ is said to be *isotropic* if $K(\sigma) = \text{const.}$ for every 2-plane section σ of $T_p(M)$, and is said to be *non-isotropic* otherwise.

Recently, R. S. Kulkarni considered in [**3**] the converse of the *theorema egregium* of Gauss, which asserts that the curvature is a metric invariant, and proved that the curvature, in general, determines a conformal class of metric, that is, a curvature-preserving diffeomorphism $f: (M, g) \to (\bar{M}, \bar{g})$ is conformal if the set of non-isotropic points is dense in M (cf. Theorem 1 in [**3**]). It is natural to ask furthermore whether f is isometric or not. He showed in [**3**] that the answer to this question is affirmative if $n \geqq 4$ (cf. Fundamental Theorem in [**3**]), but he obtained only partial results for 3-manifolds assuming compactness and restricting sign of curvature (cf. § 6 in [**3**]). The purpose of this note is to give some affirmative answers to the above question for 3-manifolds.

In § 2 we shall prepare some general formulas on the conformal change of metric. In § 3, starting with Kulkarni's results, we shall obtain several lemmas on the curvature-preserving diffeomorphism f for later use. In § 4, after constructing a useful *constant associated with* f whose vanishing gives a necessary and sufficient condition for f to be isometric (cf. Theorem 1), we shall show as a corollary to Theorem 1 that the answer to the above question is also affirmative for conformally flat or compact 3-manifolds (cf. Corollary 1 and Corollary 2). Furthermore, as an application of Theorem 2 we shall give a partial result for complete manifolds with non-vanishing scalar curvatures (cf. Theorem 3). The hypothesis $n = 3$ is essential in § 4.

We shall assume, throughout this paper, that Riemannian manifolds under consideration are connected and of dimension $n \geqq 3$, their metrics are positive definite, and all manifolds and all diffeomorphisms are of class C^∞. For the terminology and notation, we generally follow [3].

§ 2. Notation and conformal diffeomorphism.

In this section, we shall summarize general transformation formulas of some geometric objects under the conformal change of metric (for details see [3] or [5]).

Let (M, g) and $(\bar{M}, \bar{g})$ be Riemannian n-manifolds with metrics g and $\bar{g}$, respectively. A diffeomorphism $f: (M, g) \to (\bar{M}, \bar{g})$ is said to be conformal if the induced metric $g^* = f^*\bar{g}$ is related to g by

$$g^* = e^{2\varphi} g\,, \tag{2.1}$$

where the function φ is necessarily differentiable and is called the associated function of f. φ will be sometimes denoted by φ_f. If φ is constant, then f is homothetic, and if φ is identically zero, then f is an isometry.

Let $\mathfrak{F}(M)$ be the ring of differentiable real-valued functions on M and $\mathfrak{X}(M)$ the Lie algebra of differentiable vector fields on M. Let ∇ be the Riemannian connection with respect to the metric g and $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ $(X, Y \in \mathfrak{X}(M))$ the curvature operator of ∇. The Ricci tensor field and the scalar curvature will be denoted by Ric and Sc, respectively. And also we indicate the corresponding quantities with respect to the metric g^* or $\bar{g}$ by asterisking or by bar overhead, respectively. Then it is known that the above quantities with respect to g^* coincide with the induced ones of the corresponding quantities with respect to $\bar{g}$ by f and we have the following formulas. For any X, $Y \in \mathfrak{X}(M)$, we have

$$\nabla^*_X Y = \nabla_X Y + S(X, Y) \tag{2.2}$$

with

$$S(X, Y) = (X\varphi)Y + (Y\varphi)X - \langle X, Y \rangle G\,, \tag{2.3}$$

where $\langle X, Y \rangle = g(X, Y)$ and $G = \operatorname{grad}\varphi$, the gradient of φ with respect to the metric g. Using the hessian of φ

$$\begin{aligned} \operatorname{hess}_\varphi(X, Y) &= (\nabla_X d\varphi)Y \\ &= \langle \nabla_X G, Y \rangle\,, \end{aligned} \tag{2.4}$$

we define the symmetric (0, 2)-tensor field

$$P(X, Y) = \operatorname{hess}_\varphi(X, Y) - (X\varphi)(Y\varphi) + \frac{1}{2}\|G\|^2 \langle X, Y \rangle\,, \tag{2.5}$$

where $\|G\| = \langle G, G \rangle^{\frac{1}{2}}$. In general, for a given symmetric (0, 2)-tensor field H, we denote by H_0 the canonical endomorphism of the tangent bundle $\mathfrak{T}(M)$

induced by H, that is, $\langle H_0(X), Y\rangle = H(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. Then by (2.4) and (2.5) we have

$$P_0(X) = \nabla_X G - (X\varphi)G + \frac{1}{2}\|G\|^2 X. \tag{2.6}$$

The following transformation formulas of the various tensor fields under the conformal change of metric (2.1) are known:

$$R^*(X, Y)Z = R(X, Y)Z + \tilde{T}(X, Y)Z, \tag{2.7}$$

where

$$\tilde{T}(X, Y)Z = P(Y, Z)X - P(X, Z)Y + \langle Y, Z\rangle P_0(X) - \langle X, Z\rangle P_0(Y);$$

$$\mathrm{Ric}^*(X, Y) = \mathrm{Ric}\,(X, Y) + \mathfrak{R}(X, Y), \tag{2.8}$$

where

$$\mathfrak{R}(X, Y) = -(n-2)P(X, Y) - \langle X, Y\rangle \,\mathrm{Trace}\, P_0;$$

$$e^{2\varphi}\,\mathrm{Ric}_0^*(X) = \mathrm{Ric}_0(X) + \mathfrak{R}_0(X), \tag{2.9}$$

where Ric_0^* is defined by $g^*(\mathrm{Ric}_0^*(X), Y) = \mathrm{Ric}^*(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$; and

$$e^{2\varphi}\,\mathrm{Sc}^* = \mathrm{Sc} - 2(n-1)\,\mathrm{Trace}\, P_0. \tag{2.10}$$

Weyl's conformal curvature tensor on M is a tensor field C of type (1, 3) defined by

$$\begin{aligned} C(X, Y)Z = R(X, Y)Z + \frac{1}{n-2}\{&L(Y, Z)X - L(X, Z)Y \\ &+ \langle Y, Z\rangle L_0(X) - \langle X, Z\rangle L_0(Y)\} \end{aligned} \tag{2.11}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where we have put

$$L = \mathrm{Ric} - \frac{\mathrm{Sc}}{2(n-1)} g. \tag{2.12}$$

The following Weyl's 3-index tensor D of type (0, 3) will also be useful:

$$D(X, Y, Z) = (\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z). \tag{2.13}$$

The tensor field C is invariant under any conformal change of metric, and vanishes identically for $n=3$. As is well-known, a necessary and sufficient condition for (M, g) to be conformally flat is that

$$C = 0 \quad \text{for} \quad n > 3$$

and

$$D = 0 \quad \text{for} \quad n = 3.$$

We recall the following well-known facts (cf. Yano [5]):

LEMMA 1. *The tensor fields C and D satisfy the following identities:*

(a) $$D^*(X, Y, Z) = D(X, Y, Z) - (n-2)\langle C(X, Y)Z, G\rangle,$$

(b) $$\mathrm{Trace}\,\{X \to D_0(X, Y)\} = 0$$

for all $X, Y, Z \in \mathfrak{X}(M)$, *where* D_0 *is the tensor field of type* (1, 2) *defined by* $\langle D_0(X, Y), Z\rangle = D(X, Y, Z)$.

Finally we remark that

$$\mathrm{Trace}\,\{X \to (\nabla_X \mathrm{Ric}_0)(Y)\} = \frac{1}{2} Y(\mathrm{Sc}) \tag{2.14}$$

for all $X, Y \in \mathfrak{X}(M)$.

§ 3. Curvature-preserving diffeomorphism.

The following theorem due to Kulkarni is a starting point of this paper:

THEOREM K ([3]). *Let* $f: (M, g) \to (\bar{M}, \bar{g})$ *be a curvature-preserving diffeomorphism of two Riemannian n-manifolds* ($n \geqq 3$). *Suppose that the set of non-isotropic points is dense in* M. *Then* f *is conformal, that is, there exists a function* $\varphi \in \mathfrak{F}(M)$ *such that*

$$g^* = e^{2\varphi} g \tag{3.1}$$

and furthermore we have

$$R^* = e^{2\varphi} R\,. \tag{3.2}$$

From the equations (3.1) and (3.2) it follows immediately

$$\mathrm{Ric}^* = e^{2\varphi}\,\mathrm{Ric} \quad \text{and} \quad \mathrm{Sc}^* = \mathrm{Sc}\,. \tag{3.3}$$

In this section, we shall prepare, for later use, some basic formulas for the curvature-preserving diffeomorphism f. We shall use only the equations (3.1) and (3.3), so that all results in the following are valid also for the Ricci-curvature-preserving conformal diffeomorphism $f: (M, g) \to (\bar{M}, \bar{g})$.

From the equations (2.12) and (3.3) we get $L^* = e^{2\varphi} L$, which gives

$$\begin{aligned} e^{-2\varphi} D^*(X, Y, Z) - D(X, Y, Z) = {} & (X\varphi) L(Y, Z) - (Y\varphi) L(X, Z) \\ & + \langle X, Z\rangle L(Y, G) - \langle Y, Z\rangle L(X, G)\,. \end{aligned} \tag{3.4}$$

In fact, we get

$$\begin{aligned} D^*(X, Y, Z) &= (\nabla^*_X L^*)(Y, Z) - (\nabla^*_Y L^*)(X, Z) \\ &= e^{2\varphi}\{2(X\varphi) L(Y, Z) + (\nabla^*_X L)(Y, Z) - 2(Y\varphi) L(X, Z) - (\nabla^*_Y L)(X, Z)\} \\ &= e^{2\varphi}[\{2(X\varphi) L(Y, Z) + (\nabla_X L)(Y, Z) - L(S(X, Y), Z) - L(Y, S(X, Z))\} \\ &\qquad - \{\text{replace } X \text{ by } Y \text{ in the above expression}\}] \qquad \text{(by (2.2))} \\ &= e^{2\varphi}[D(X, Y, Z) + 2\{(X\varphi) L(Y, Z) - (Y\varphi) L(X, Z)\} \\ &\qquad - \{L(Y, S(X, Z)) - L(X, S(Y, Z))\}]\,. \end{aligned}$$

On the other hand, using (2.3), we obtain

$$L(Y, S(X, Z))-L(X, S(Y, Z))$$
$$=(X\varphi)L(Y, Z)+(Z\varphi)L(Y, X)-\langle X, Z\rangle L(Y, G)$$
$$-\{\text{replace } X \text{ by } Y \text{ in the above expression}\}$$
$$=(X\varphi)L(Y, Z)-(Y\varphi)L(X, Z)-\{\langle X, Z\rangle L(Y, G)-\langle Y, Z\rangle L(X, G)\},$$

which implies (3.4). The equation (3.4) is equivalent to

$$D_0^*(X, Y)-D_0(X, Y)=(X\varphi)L_0(Y)-(Y\varphi)L_0(X)+L(Y, G)X-L(X, G)Y, \tag{3.5}$$

where $g^*(D_0^*(X, Y), Z)=D^*(X, Y, Z)$ for all $X, Y, Z\in\mathfrak{X}(M)$. In (3.5), we take the trace of the linear map $\{X\to(D_0^*(X, Y)-D_0(X, Y))\}$, where Y is fixed. Then by virtue of (b) in Lemma 1 and

$$\text{Trace } L_0=\frac{n-2}{2(n-1)}\text{Sc}$$

we have

$$L(Y, G)-\frac{n-2}{2n(n-1)}\text{Sc}\langle Y, G\rangle=0 \tag{3.6}$$

because of

$$\text{Trace}\{X\to(X\varphi)L_0(Y)\}=L_0(Y)\varphi$$

and

$$\text{Trace}\{X\to L(X, G)Y\}=L(Y, G).$$

For convenience, let us define another symmetric tensor field T of type (0, 2) by

$$T=\text{Ric}-\frac{1}{n}\text{Sc}\, g. \tag{3.7}$$

Then we have

LEMMA 2. *The tensors T and T^* satisfy on M the following;*

(a) $\text{Trace } T_0=0$,

(b) $T^*=T-(n-2)P+\frac{n-2}{n}(\text{Trace } P_0)g$,

(c) $T^*=e^{2\varphi}T$,

(d) $T(X, G)=0$, *or equivalently* $T_0(G)=0$,

(e) $e^{-2\varphi}D^*(X, Y, Z)-D(X, Y, Z)=(X\varphi)T(Y, Z)-(Y\varphi)T(X, Z)$

for all $X, Y, Z\in\mathfrak{X}(M)$.

PROOF. The equations (a), (b) and (c) follow immediately. The equation (3.6) implies (d) because of

$$T=L-\frac{n-2}{2n(n-1)}\text{Sc}\, g, \tag{3.8}$$

which is a consequence of (2.12) and (3.7). And also the equation (3.4) implies (e) because of (d) and (3.8). q. e. d.

Eliminating T^* from (b) and (c) in Lemma 2, we have by (2.6)

$$\nabla_X G = \frac{1-e^{2\varphi}}{n-2} T_0(X) + (X\varphi)G + \frac{1}{n}(\Delta\varphi - \|G\|^2)X \tag{3.9}$$

for all $X \in \mathfrak{X}(M)$, where $\Delta\varphi$ is the Laplacian of φ defined by

$$\Delta\varphi = \mathrm{Trace}\,\{X \to \nabla_X G\}\,.$$

The equation (3.9) implies

LEMMA 3. *The associated function φ of f has the following properties;*

(a) *the trajectories of the gradient vector field G of φ are geodesic arcs in a neighborhood of an ordinary point of φ,*

(b) $d(\|G\|^2) = \frac{2}{n}\{\Delta\varphi + (n-1)\|G\|^2\}d\varphi.$

PROOF. Putting $X = G$ in (3.9), we get by (d) in Lemma 2

$$\nabla_G G = \frac{1}{n}\{\Delta\varphi + (n-1)\|G\|^2\}G\,,$$

which implies (a) in Lemma 3. Take the inner product of the both sides of (3.9) with G, we have

$$\frac{1}{2}X\langle G, G\rangle = \frac{1}{n}\{\Delta\varphi + (n-1)\|G\|^2\}X\varphi$$

for any $X \in \mathfrak{X}(M)$, which implies (b) in Lemma 3. q. e. d.

Let M' be an open subset of M defined by

$$M' = \{p \in M\,;\ p \text{ is the ordinary point of } \varphi,\ (d\varphi)_p \neq 0\}\,.$$

Then we have

LEMMA 4. *There exist two smooth functions ρ and ϕ on M' such that*

(a) $d(\mathrm{Sc}) = \rho d\varphi$, *and*

(b) $d\rho = \phi d\varphi$.

The function ρ is given explicitly by

$$\rho = 2n(e^{2\varphi}-1)(n-2)^{-2}\|G\|^{-2}\,\mathrm{Trace}\,(T_0^3)\,. \tag{3.10}$$

PROOF. Putting $Z = G$ in the equations (a) in Lemma 1 and (e) in Lemma 2, we have by (d) in Lemma 2

$$D^*(X, Y, G) = D(X, Y, G) \quad \text{and} \quad e^{-2\varphi}D^*(X, Y, G) = D(X, Y, G)\,,$$

respectively, and hence by eliminating $D^*(X, Y, G)$ from these equations

$$(e^{2\varphi}-1)D(X, Y, G) = 0\,.$$

Since the set of zeroes of the function φ is discrete in M', if there is any, we have by continuity of $D(X, Y, G)$

$$D(X, Y, G)=0. \tag{3.11}$$

On the other hand, we get by substituting (3.8) into (2.13)

$$\begin{aligned} D(X, Y, Z) &= (\nabla_X T)(Y, Z)-(\nabla_Y T)(X, Z) \\ &+\frac{n-2}{2n(n-1)}\{\langle Y, Z\rangle X(\mathrm{Sc})-\langle X, Z\rangle Y(\mathrm{Sc})\}. \end{aligned} \tag{3.12}$$

Putting $Z=G$ in the above, we get, on account of (3.11),

$$(X\varphi)Y(\mathrm{Sc})-(Y\varphi)X(\mathrm{Sc})=0,$$

because

$$\begin{aligned} (\nabla_X T)(Y, G)-(\nabla_Y T)(X, G) &= -T(Y, \nabla_X G)+T(X, \nabla_Y G) \quad \text{(by (d) in Lemma 2)} \\ &= \frac{1-e^{2\varphi}}{n-2}\{T(X, T_0(Y))-T(Y, T_0(X))\} \quad \text{(by (3.9))} \\ &= 0. \end{aligned}$$

Hence there exists a function ρ defined on M' such that

$$X(\mathrm{Sc})=\rho X\varphi \tag{3.13}$$

on M' for all $X\in\mathfrak{X}(M)$. Since ρ is independent of X, this implies (a) in Lemma 4.

The explicit form (3.10) of ρ is obtained as follows. Since $T_0(G)=0$ by (d) in Lemma 2, we have

$$(\nabla_X T_0)G=-T_0(\nabla_X G). \tag{3.14}$$

We now obtain

$$\begin{aligned} &\mathrm{Trace}\,\{X\to(\text{the left hand side of (3.14)})\} \\ &=\mathrm{Trace}\,\{X\to(\nabla_X \mathrm{Ric}_0)G\}-\frac{1}{n}\,\mathrm{Trace}\,\{X\to X(\mathrm{Sc})G\} && \text{(by (3.7))} \\ &=\frac{1}{2}G(\mathrm{Sc})-\frac{1}{n}G(\mathrm{Sc}) && \text{(by (2.14))} \\ &=\frac{n-2}{2n}\rho\|G\|^2 && \text{(by (3.13))} \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Trace}\,\{X\to(\text{the right hand side of (3.14)})\} \\ &=\frac{e^{2\varphi}-1}{n-2}\,\mathrm{Trace}\,(T_0^2) && \text{(by (3.9))} \end{aligned}$$

because of the equations (a) and (d) in Lemma 2, so that we obtain (3.10) by equating these two traces.

Finally taking exterior derivative of (a) in Lemma 4 we get (b) in Lemma 4 at once. q. e. d.

§ 4. Theorems.

In this section we shall assume $n=3$ throughout and define an associated constant of the curvature-preserving diffeomorphism.

First, we remark that the restriction "$n=3$" on the dimension of M implies two important relations as follows. Since Weyl's conformal curvature tensor C vanishes identically, we have by the equation (a) in Lemma 1

$$D^*(X, Y, Z)=D(X, Y, Z) \tag{4.1}$$

for all $X, Y, Z\in\mathfrak{X}(M)$. On the other hand, the equation (d) in Lemma 2 means that G is an eigen-vector of T_0 corresponding to an eigen-value zero at each point $p\in M'$. Hence the equation (a) in Lemma 2 and the assumption $n=3$ imply that the eigen-values of T_0 are 0, $\kappa(p)$ and $-\kappa(p)$ at each point $p\in M'$, so that we find

$$\mathrm{Trace}\,(T_0^3)=0 \tag{4.2}$$

on M'.

First we need the following two lemmas.

LEMMA 5. *We have on* M'

$$\psi\|G\|^2+\frac{4}{3}\rho\Delta\varphi+\frac{2}{3}\rho\|G\|^2=0\,.$$

PROOF. The equation (3.10) yields on M'

$$\rho\|G\|^2=6(e^{2\varphi}-1)\,\mathrm{Trace}\,(T_0^2)\,. \tag{4.3}$$

Applying ∇_G to (4.3) we obtain directly

$$\begin{aligned}&6(e^{2\varphi}-1)\|G\|^{-2}\nabla_G\,\mathrm{Trace}\,(T_0^2)\\&=\psi\|G\|^2+\frac{2}{3}\rho(\Delta\varphi+2\|G\|^2)-12e^{2\varphi}\,\mathrm{Trace}\,(T_0^2)\end{aligned} \tag{4.4}$$

because of the equations

$$\nabla_G\varphi=\|G\|^2\,,$$

$$\nabla_G(\|G\|^2)=\frac{2}{3}(\Delta\varphi+2\|G\|^2)\|G\|^2 \qquad \text{(by (b) in Lemma 3)}$$

and

$$\nabla_G\rho=\psi\|G\|^2 \qquad \text{(by (b) in Lemma 4)}.$$

On the other hand, we get by (e) in Lemma 2 and (4.1)

$$(e^{-2\varphi}-1)D(X, Y, Z)=(X\varphi)T(Y, Z)-(Y\varphi)T(X, Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$, so that we obtain by setting $Y=G$ and $Z=T_0(X)$

$$(e^{2\varphi}-1)D(X, G, T_0(X))=e^{2\varphi}\|G\|^2\langle T_0(X), T_0(X)\rangle . \tag{4.5}$$

Then we have by (3.12) and (a) in Lemma 4

$$\begin{aligned} D(X, G, T_0(X))=&(\nabla_X T)(G, T_0(X))-(\nabla_G T)(X, T_0(X)) \\ &+\frac{\rho}{12}\{(X\varphi)\langle G, T_0(X)\rangle-(G\varphi)\langle X, T_0(X)\rangle\} . \end{aligned} \tag{4.6}$$

Fix a point $p\in M'$ and let $E=\{E_1, E_2, E_3\}$ be a local orthonormal frame in a neighborhood of p such that $\nabla_{E_i}E_j=0$ at p for all i, j. Putting $X=E_i$ in (4.5) and summing up for $i=1, 2, 3$, we have

$$\begin{aligned} -(e^{2\varphi}-1)\Big\{\frac{1}{3}(\Delta\varphi-\|G\|^2)\operatorname{Trace}(T_0^2)+\frac{1}{2}\nabla_G \operatorname{Trace}(T_0^2)\Big\} \\ =e^{2\varphi}\|G\|^2 \operatorname{Trace}(T_0^2) , \end{aligned} \tag{4.7}$$

because we have at p

$$\begin{aligned} \sum_i (\nabla_{E_i}T)(G, T_0(E_i)) &=-\sum_i T(\nabla_{E_i}G, T_0(E_i)) && \text{(by (d) in Lemma 2)} \\ &=(e^{2\varphi}-1)\sum_i T(T_0(E_i), T_0(E_i)) \\ &\quad -\frac{1}{3}(\Delta\varphi-\|G\|^2)\sum_i T(E_i, T_0(E_i)) && \text{(by (3.9))} \\ &=-\frac{1}{3}(\Delta\varphi-\|G\|^2)\operatorname{Trace}(T_0^2) && \text{(by (4.2))} \end{aligned}$$

and

$$\begin{aligned} \sum_i(\nabla_G T)(E_i, T_0(E_i)) &=\frac{1}{2}\sum_i \nabla_G\langle T_0(E_i), T_0(E_i)\rangle-\sum_i T(\nabla_G E_i, T_0(E_i)) \\ &=\frac{1}{2}\nabla_G \operatorname{Trace}(T_0^2) . \end{aligned}$$

If we eliminate $\nabla_G(\operatorname{Trace}(T_0^2))$ from (4.4) and (4.7) and substitute (4.3) into the resulting equation, then the lemma follows. q. e. d.

LEMMA 6. *Let F be a function on M defined by*

$$F=(e^{-2\varphi}-1)\|G\|^2 \operatorname{Trace}(T_0^2) . \tag{4.8}$$

Then it is constant on M.

PROOF. We may assume that M' is not empty. Evidently the function F is smooth on M and given by

$$F=-\frac{1}{6}e^{-2\varphi}\rho\|G\|^4 \tag{4.9}$$

on M' by (4.3). Hence, from the equations (b) in Lemma 3 and (b) in Lemma 4 we have by direct calculation

$$-6dF=e^{-2\varphi}\|G\|^{2}\Big(\psi\|G\|^{2}+\frac{4}{3}\rho\Delta\varphi+\frac{2}{3}\rho\|G\|^{2}\Big)d\varphi$$
$$=0 \qquad \text{(by Lemma 5)}$$

on M'. Consequently, F is constant on each connected component of M'. Thus, because of (4.8) we find $F=0$ on M if $M\neq M'$, that is, if there exists at least one stationary point of φ. If $M=M'$, F is obviously constant on M by connectedness of M. q. e. d.

For the diffeomorphism f in Theorem K for $n=3$, we define

$$c_{f}=(e^{-2\varphi}-1)\|G\|^{2}\,\mathrm{Trace}\,(T_{0}^{2})\,.$$

Then owing to Lemma 6 we can call c_f the *associated constant* of the curvature-preserving diffeomorphism f.

THEOREM 1. *Under the circumstances of Theorem K, suppose $n=3$. Then a necessary and sufficient condition for f to be isometric is $c_f=0$.*

PROOF. The necessity is trivial, so we prove the sufficiency in the following. For the moment, suppose that M' is non-empty. Then, the set of zeroes of the function φ is closed in M', which is open. Thus we can choose a point and its open neighborhood $U\subset M'$, on which $\varphi\neq 0$. By the assumption $c_f=0$, we find $\mathrm{Trace}\,(T_{0}^{2})=0$ on U, from which $T=0$, i. e. $\mathrm{Ric}=\frac{1}{n}\mathrm{Sc}\,g$ on U, because we have

$$\mathrm{Trace}\,(T_{0}^{2})=\langle T_{0},\,T_{0}\rangle\,,$$

where $\langle\,,\rangle$ denotes the canonical inner product on tensor algebra induced by Riemannian metric g. Since $C=0$ on M by the assumption $n=3$, this implies by the equations (2.11) and (2.12)

$$R(X,\,Y)Z=\frac{\mathrm{Sc}}{n(n-1)}\,\{\langle X,\,Z\rangle Y-\langle Y,\,Z\rangle X\}$$

on U. Thus each point of U is isotropic. But this contradicts the assumption that the set of non-isotropic points is dense in M. Thus M' is empty, that is, $d\varphi=0$ on M. So f is homothetic. Then we have

$$\bar{K}(f_{*}\sigma)=e^{-2\varphi}K(\sigma)$$

by (2.7) for any 2-plane section $\sigma\subset T_{p}(M)$ at any point $p\in M$. Since f is curvature-preserving, we obtain

$$(e^{2\varphi}-1)K(\sigma)=0\,.$$

Since, by the assumption of Theorem 1, $K\neq 0$ for at least one σ at almost all points, it follows $\varphi=0$. Thus, f is isometric. q. e. d.

COROLLARY 1. *Under the assumptions of Theorem K, suppose that $n=3$ and (M,g) is conformally flat. Then f is an isometry.*

PROOF. Since (M, g) is conformally flat and $n=3$, we have $D^*=D=0$. So, it follows from (e) in Lemma 2

$$(X\varphi)T_0(Y)-(Y\varphi)T_0(X)=0$$

for all $X, Y\in\mathfrak{X}(M)$. Setting $Y=G$ in the above, we find easily $c_f=0$ by (d) in Lemma 2. Hence f is an isometry by Theorem 1. q. e. d.

This Corollary has been obtained independently in a different way by Kulkarni [4].

COROLLARY 2. *Under the assumptions of Theorem K, suppose that* $n=3$ *and* M *is compact. Then* f *is an isometry.*

PROOF. Since there exists at least one stationary point of φ by compactness of M, it follows $c_f=0$, from which f is isometric by Theorem 1. q. e. d.

Corollary 2 is an improvement of the results of Kulkarni (cf. Theorem 6 and Theorem 7 in [3]) in the sense that the additional assumptions on the sign of curvature have been removed in Corollary 2.

The author does not know as yet whether there exists a global non-isometric curvature-preserving diffeomorphism satisfying the assumptions of Theorem K in the case $n=3$. In this respect, it may be helpful to keep the next theorem in mind while constructing such an example, if there is.

THEOREM 2. *Under the circumstances of Theorem K, suppose* $n=3$. *A necessary and sufficient condition for* f *to be non-isometric is that the manifold* (M, g) *and the associated function* φ *of* f *satisfy simultaneously the following three conditions* (a), (b) *and* (c):

(a) φ *has no stationary point on* M,

(b) *there exists no isotropic point on* M,

(c) *the range of* φ *is either* $\varphi>0$ *or* $\varphi<0$,

or, equivalently, satisfy simultaneously the two conditions (a) *and*

(d) *the scalar curvature* Sc *has no stationary point on* M.

PROOF. The condition $c_f\neq 0$ is equivalent to the following:

$$\text{(i)}\quad \|G\|\neq 0, \qquad \text{(ii)}\quad \mathrm{Trace}(T_0^2)\neq 0 \qquad \text{and} \qquad \text{(iii)}\quad e^{2\varphi}\neq 1.$$

Evidently (i) $\Leftrightarrow$ (a). We have (ii) $\Leftrightarrow$ $T_0\neq 0$, which is equivalent to the condition (b) by the assumption $n=3$, as is easily verified by Lemma 1 in [3]. Since M is assumed to be connected and φ is continuous on M, the range of φ is a connected subset of $\boldsymbol{R}$, so that we see (iii) $\Leftrightarrow$ (c). Owing to another expression (4.9) of c_f, we find similarly $c_f\neq 0 \Leftrightarrow$ {(a) and (d)}. Thus, Theorem 2 follows from Theorem 1. q. e. d.

The technique developed in the proofs of Lemma 5 in [1] and Proposition 10.4 in [2] is applicable to the following

THEOREM 3. *Under the assumptions of Theorem K, suppose that* $n=3$ *and two metrics* g, $\bar{g}$ *are complete. If* f *is an onto diffeomorphism and* Sc *does not* *vanish, then f is an isometry.*

PROOF. On the contrary, assume that f is non-isometric. Then the function $\lambda=\|G\|$ vanishes nowhere on M by (a) of Theorem 2. The range of Sc is either Sc>0 or Sc<0, and hence one of two functions $(1-e^{2\varphi})$Sc and $(1-e^{-2\varphi})$Sc is positive-valued, because of (c) of Theorem 2. The diffeomorphism f is onto and the associated functions φ_f and $\bar{\varphi}_{f^{-1}}$ of conformal diffeomorphisms f and f^{-1}, respectively, are related by

$$\bar{\varphi}_{f^{-1}}=-\varphi_f\circ f^{-1},$$

so that we have by the equation (3.3)

$$\{(1-e^{2\bar{\varphi}_{f^{-1}}})\overline{\mathrm{Sc}}\}\circ f=(1-e^{-2\varphi_f})\mathrm{Sc}.$$

Thus, we may assume that

$$(1-e^{2\varphi})\mathrm{Sc}>0 \tag{4.10}$$

by considering f^{-1}, if necessary. The trajectory $x(t)$ of the vector field G passing through a point $p=x(0)$ of M is a geodesic by (a) of Lemma 3. We can assume that the parameter t is the arc-length. Let $X=\frac{1}{\lambda}G$ be the unit tangent vector field to $x(t)$. Then we have along $x(t)$

$$\begin{aligned}2\lambda\frac{d\lambda}{dt}&=\nabla_X\|G\|^2=\frac{1}{\lambda}\nabla_G\|G\|^2\\&=\frac{2}{3}\lambda(\Delta\varphi+2\lambda^2)\end{aligned} \tag{4.11}$$

by (b) in Lemma 3. On the other hand, we obtain by (2.10)

$$(1-e^{2\varphi})\mathrm{Sc}=4\,\mathrm{Trace}\,P_0=4\Big(\Delta\varphi+\frac{1}{2}\lambda^2\Big). \tag{4.12}$$

Eliminating $\Delta\varphi$ from the equations (4.11) and (4.12) we get

$$\frac{d\lambda}{dt}=\frac{1}{2}\lambda^2+\alpha(t) \tag{4.13}$$

along $x(t)$, where $\alpha=\frac{1}{12}(1-e^{2\varphi})$Sc is a smooth positive-valued function by (4.10).

We consider an auxiliary differential equation

$$\frac{d\lambda}{dt}=\frac{1}{2}\lambda^2 \tag{4.14}$$

on the (t, λ)-plane. The solution of (4.14) with initial condition $\mu(0)=\|G\|_p$ $(=\lambda(0))>0$ is given by

$$\mu(t)=-\frac{2}{t-a},$$

where $a=2\|G\|_p^{-1}>0$. It is easy to prove that for the solution $\lambda(t)$ of (4.13) and the continuous solution $\mu(t)$ of (4.14) it holds

$$\mu(t) \leqq \lambda(t) \qquad \text{for} \quad 0 \leqq t < a\,.$$

Hence the function $\lambda(t)=\|G\|(x(t))$ must have a singularity at finite positive time. But this is impossible, because $x(t)$ must be extended indefinitely with respect to the arc-length parameter t by the completeness of the metric g and the function $\lambda(t)$ must be defined for all t. Thus, f is isometric. q. e. d.

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Added in proof. Very recently, in the direction of Theorem 2, S. T. Yau has proved that there exist an open Riemannian 3-manifold (M, g) and a non-isometric diffeomorphism f satisfying the assumptions of Theorem K [cf. S. T. Yau: Curvature preserving diffeomorphisms, Ann. of Math., **100** (1974), 121–130].

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