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Some theorems of algebraicity for complex spaces

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Introduction.

This paper consists of a review of a certain collection of known theorems, which prove that, under suitable hypotheses, a given complex space is projective—or nearly projective. Many of the original proofs depended on use of the Riemann-Roch theorem, and thus were confined to low dimensions. Our technique is to find positive line bundles by patching together plurisubharmonic functions. Thus, a fundamental criterion for algebraicity is Kodaira's theorem that a Hodge manifold is projective [**10**], as generalized by Grauert in [**4**].

THEOREM. *A compact complex space which admits a positive line bundle is a projective variety.*

In the first section we collect the definitions and fundamental theorems in the positivity-pseudoconvexity-plurisubharmonicity cycle of ideas. Lemma 1.3 is fundamental, and we are indebted to J. E. Fornass for showing us the crucial trick. Section 2 is devoted to a new proof of Grauert's theorem on the extension of positivity of line bundles ([**4**], § 3 Satz 4), and in Section 3 this theorem is applied to give a new proof of Van de Ven's theorem that a compactification of $\boldsymbol{C}^2$ is algebraic [**16**]. In Section 4 we show that projective space is characterized by its hyperplane section (in dimensions greater than 2).

We turn next to Kodaira's theorem that a surface with one meromorphic function is elliptic [**11**]. This result was generalized to dimension 3 by Kawai [**9**], and in his review of Kawai's paper, Hironaka [**6**] indicated a proof of the appropriate generalization to arbitrary dimension. We prove the following generalization of Hironaka's argument: Let $\pi: X \to \varDelta$ be a holomorphic mapping of a compact complex space X of dimension n onto an algebraic variety $\varDelta$. Suppose $F \to X$ is a coherent sheaf whose restriction to the generic fiber is the sheaf of sections of a positive line bundle. Then X is *Moisheson*: X has n algebraically independent meromorphic functions. This result (as Hironaka has shown) easily gives the generalization: if X is of dimension n and the transcendence degree of its field of meromorphic functions is $n-1$, then X is bimeromorphically equivalent to an elliptic fibration over a projective variety. If, in the preceding result it is assumed that the sheaf F is a sheaf of sections of a line bundle which is positive on *every* fiber, then the conclusion is that X is algebraic.

Finally, in section 6 we give new proofs of the analogous theorems for surfaces, where the results are sharper than those obtainable in higher dimensions. We add, using the techniques of negativity, a proof that the Hopf σ-process preserves algebraicity in both directions.

§ 1. Preliminaries.

The basic techniques of this paper are those developed by Grauert in his fundamental paper on exceptional sets [4]. We shall need some basic definitions for complex spaces (see Narasimhan [13]).

1.1. Definitions. (a) Let X be a complex space, U an open subset of X such that U can be embedded as a closed subvariety of a domain D in $\boldsymbol{C}^n$. Define $C^\infty(U)=\{f|_U\,;\, f\in C^\infty_R(D)\}$. The association $U\to C^\infty(U)$ is a well-defined presheaf; we call C^∞_X the *associated sheaf of real-valued C^∞ functions on X.*

(b) For $x\in X$, let $T_x(X)$ be the (Zariski) tangent space of X at x. The set $T(X)=\bigcup_x T_x(X)$ has, in a natural way the structure of a linear space (complex space with linear structure on the fibers). This is the *tangent bundle to X.* Θ_X will denote the sheaf of holomorphic sections of $T(X)$.

(c) A *Hermitian form on X* is a C^∞ complex-valued function ω on $T(X)\oplus T(X)$ which is hermitian symmetric on the fibers. ω is an *hermitian metric* if it is positive definite.

(d) For $f\in C^\infty_X$, we define the *complex hessian of f at x* denoted $H_x(f)$, as the hermitian form on $T_x(X)$:

$$H_x(f)(X,Y)=\partial\bar\partial f(X,\bar Y)$$

(the differentiation is taken in the ambient $\boldsymbol{C}^n$, and is independent of the extension f). f is *(strictly) plurisubharmonic (spsh) at x* if $H_x(f)$ is positive semidefinite (definite) on $T_x(X)$.

(e) A domain $D\Subset X$ is *strongly pseudoconvex (spsc)* if there is a neighborhood U of ∂D and a spsh function $f\in C^\infty(U)$ such that

$$\text{(i)}\quad D\cap U=\{x\in U\,;\, f(x)<0\}$$

$$\text{(ii)}\quad \text{if}\quad f(x)=0\,,\qquad df(x)\neq 0\,.$$

It is easily verified that (d) is the same as requiring that f be locally the restriction of a function spsh in the ambient $\boldsymbol{C}^n$ (see Lemma 1.9 below). The fundamental theorem of Grauert and Narasimhan is the following result.

1.2. Theorem (See Grauert [4], p. 340). *Let D be a spsc domain in the* *complex space X. Then D has only finitely many connected nowhere discrete compact subvarieties, and the topological space obtained by identifying these to points carries a unique structure of normal Stein space such that the quotient map is holomorphic.*

A connected nowhere discrete compact subvariety which can be so blown down is said to be *exceptional.*

We shall need the following more precise description of a spsc space. (The crucial step in the proof was furnished to us by J. E. Fornass.)

1.3. LEMMA. *Let D be a spsc domain in a Stein space, and p the defining spsh function on the neighborhood U of ∂D. For $\varepsilon>0$, and sufficiently small, there is a spsh function $q\in C^{\infty}(D\cup U)$ such that $q(x)=p(x)$ if $p(x)\geqq-\varepsilon$.*

PROOF. Choose $\varepsilon_0>0$ so that $E=\{x\in U;\, |p(x)|\leqq\varepsilon_0\}$ is compact in U, $dp(x)\neq 0$, and $H_x(p)$ is positive definite for all $x\in E$. Choose $\varepsilon<\varepsilon_0/3$ and let $\varphi\in C_0^{\infty}(D)$, $1\geqq\varphi\geqq 0$, be chosen so that $\varphi=1$ on $D-\{x\in U;\, p(x)\geqq-2\varepsilon_0/3\}$ and $\varphi=0$ on $\{x\in U;\, p(x)\geqq-\varepsilon_0/3\}$.

Let $\psi\in C^{\infty}(R)$ have the following properties.

(i) $\psi'\geqq 0$, $\psi''\geqq 0$

(ii) $\psi'(t)>0$, for $(-2\varepsilon_0-\eta)/3<t<-\varepsilon$ where η is very small

(iii) $\psi(t)=t$, for $t\geqq-\varepsilon$

(iv) $\psi(t)=-2\varepsilon_0/3$ for $t\leqq-2\varepsilon_0/3-\eta$.

Since $\overline{D\cup U}$ has a Stein neighborhood, which can be realized as a subvariety of $\boldsymbol{C}^n$, there is an $h\in C^{\infty}(D\cup U)$ such that $H_x(h)$ is positive definite everywhere (take h as the restriction to $D\cup U$ of the square of the distance function to some point in $\boldsymbol{C}^n$, note in $\overline{D\cup U}$). Let $g=\varphi\cdot h+A(\psi\cdot p)$, $A>0$. Then $g\in C^{\infty}(D\cup U)$, and for $p(x)\geqq-\varepsilon$, $g(x)=A\,p(x)$. In $D\cup U$,

$$H(g)=hH(\varphi)+\varphi H(h)+\partial\varphi\wedge\bar\partial h-\bar\partial\varphi\wedge\partial h$$
$$+A\psi'(p)H(p)+A\psi''(p)\partial p\wedge\bar\partial p\,.$$

If $p(x)\leqq-2\varepsilon_0/3$, $X\in T_x(D\cup U)$, $X\neq 0$ then the first line applied to X is positive ($\varphi=1$ here), and the second line is non-negative. If $p(x)>-2\varepsilon_0/3$, the second line is positive. The lemma now follows by applying the following lemma to the bundle of unit tangent vectors to $\overline{D\cup U}$ in some fiber metric on the tangent bundle and letting $q=g/A$.

1.4. LEMMA. *Let K be a compact metric space, g, f continuous functions on K such that $f\geqq 0$ on K and $g(x)>0$ if $f(x)=0$. There is an $A>0$ such that $g+Af>0$ on K.*

Now let $L\xrightarrow{\pi}X$ be a line bundle over the complex space X. Then there is defined on the space L a natural $\boldsymbol{C}^*$ action $l\rightarrow t\cdot l$ which is fiber-preserving. A *fiber metric on L* is a non-negative real-valued C^{∞} function ρ defined on L such that $\rho(t\cdot l)=|t|^2\rho(l)$ and $\rho(l)=0$ only on the zero section Z_L of L. If U_α

is a coordinate neighborhood of L, $L|U_\alpha = U_\alpha \times \boldsymbol{C}$, then $\rho|U_\alpha$ is given by $\rho(x, \xi_\alpha) = \rho_\alpha(x)|\xi_\alpha|^2$ $(\rho_\alpha(x) = \rho(x, 1))$. If $\{f_\alpha^\beta\}$ and the transition functions for L, we obtain $\rho_\alpha |f_\alpha^\beta|^2 = \rho_\beta$, so $\partial\bar\partial ln\rho_\alpha = \partial\bar\partial ln\rho_\beta$ on the overlap. Thus the complex Hessians of $ln\rho_\alpha$, $ln\rho_\beta$ agree on $U_\alpha \cap U_\beta$ so define a global hermitian form on X, denoted Θ_ρ, and called *the curvature form of the metric* ρ. Note that $\pi^*\Theta_\rho = \partial\bar\partial ln\rho$. We shall need this fact, see Grauert [4], p. 341.

1.5. LEMMA. *Let X be a complex space, $L \to X$ a line bundle. The following are equivalent.*

(a) *L has a fiber metric ρ whose curvature is positive definite.*

(b) *There are an open set U in L, an $\varepsilon > 0$, a spsh function p defined in U such that*

(i) *for each x in X, p maps $U \cap \pi^{-1}(x)$ properly onto the interval* $(1-\varepsilon, 1+\varepsilon)$,

(ii) *if $p(l) = 1$, then $\partial p/\partial|\xi|(l) > 0$ and $e^{i\theta}l \in U$ for all θ (where $\partial/\partial\xi$ is the infinitesimal generator of the $\boldsymbol{C}^*$ action).*

PROOF. (a) trivially implies (b) by taking $U = \{l\,; 0 < \rho(l) < 2\}$, $\varepsilon = 1$ and $p = \rho$. Now suppose (b) holds. Let $\Lambda = \{e^{i\theta}l\,; p(l) = 1\}$. By hypothesis, $\Lambda \subset U$. For any $x \in X$, $\Lambda \cap \pi^{-1}(x)$ is a connected union of circles (i.e. an annulus Λ_x) so $L_x - \Lambda_x$ has two components, and by (ii), $p < 1$ on the inner boundary. Define $\hat{p}(l) = \frac{1}{2\pi}\int_0^{2\pi} p(e^{i\theta}l)d\theta$. If $p(l_0) = 1$, then $p(e^{i\theta}l)$ is spsh for all θ, l near l_0, so $\hat{p}$ is also spsh at all $e^{i\theta}l_0$. Suppose now that $\hat{p}(l_0) = 1$. Since $\hat{p}$ is the average of p, there is a θ such that $p(e^{i\theta}l_0) = 1$, thus $\hat{p}$ is spsh at l_0. Since $\hat{p}$ is invariant under the circle action, $\{\hat{p}(l) = 1\} \cap \pi^{-1}(x)$ is a union of circles, bounding annuli contained in Λ_x. Since $\hat{p}$ is strictly subharmonic on Λ_x, and $\hat{p} < 1$ on the inner boundary of Λ_x, there cannot be more than one circle where $\hat{p} = 1$. Define the metric ρ on L by taking the set $S = \{l\,; \hat{p}(l) = 1\}$ to be the unit vector bundle. Then $\{l \in L\,; ln\rho(l) < 0\} = \{l\,; \hat{p}(l) < 1\}$ is spsc at every boundary point above an $x \in X$, so the complex Hessian of $ln\rho$ is positive definite on the complex tangent space to S, which projects onto $T_x(X)$, so $\Theta\rho$ is positive definite.

We remark that if the metric ρ on the line bundle $L \xrightarrow{\pi} X$ has positive definite curvature, then ρ is, as a C^∞ function on L, strictly plurisubharmonic on the zero-section. For, in local coordinates for L, $\rho(x, \xi_\alpha) = \rho_\alpha(x)|\xi_\alpha|^2$, so $\partial\bar\partial ln\rho = \partial\bar\partial ln(\rho_\alpha \circ \pi)$, and

$$\partial\bar\partial\rho = \rho[\partial ln\rho \wedge \bar\partial ln\rho + \partial\bar\partial ln(\rho_\alpha \circ \pi)]\,.$$

If $v \in T_l(L)$, and we write $v = a\frac{\partial}{\partial\xi_\alpha} + v_0\,;\ v_0 \in \ker \partial ln\rho$, we have

$$\partial\bar\partial\rho(v, \bar v) = \rho(l)\Big[\frac{|a|^2}{|\xi_\alpha|^2} + \partial\bar\partial ln\rho_\alpha(\pi_*(v_0), \pi_*(\bar v_0))\Big]\,.$$

If $v \neq 0$, either $a \neq 0$, or the second factor is positive. Thus ρ is spsh.

1.6. DEFINITION. Let $L \xrightarrow{\pi} X$ be a line bundle on a complex space. L is *negative*, $L<0$, if L admits an spsh fiber metric. L is positive, $L>0$, if $L^{-1}<0$.

The following theorem of Grauert easily follows from Theorem 1.2 and Lemma 1.5 ([**4**], p. 341).

1.7. THEOREM. *Let $L \xrightarrow{\pi} X$ be a line bundle on a compact space. $L<0$ if and only if the zero-section, Z_L, is exceptional in L.*

The fundamental criterion for proving algebraicity is this generalization by Grauert of Kodaira's theorem [**10**].

1.8. THEOREM (Grauert, [**4**], p. 343). *If a compact complex space admits a positive line bundle, it is algebraic.*

We have a final preparatory lemma, which is easily verified (analogous results have already been used by Grauert [**4**], p. 350.)

1.9. LEMMA. *Let V be a closed subvariety of the complex space X and p an spsh function defined on V. Then there is a neighborhood (in X) U of V and an spsh function q defined in U such that $q|V=p$.*

PROOF. Let $\{U_\alpha\}$ be an open cover of V (by open sets in X) so that in U_α the ideal sheaf of V is generated by $f_\alpha^j \in (U_\alpha)$, $1 \leqq j \leqq k_\alpha$. In particular, for $x \in V \cap U_\alpha$, $T_x(V)=\{v \in T_x(V): df_\alpha^j(v)=0,\ 1 \leqq j \leqq k_\alpha\}$. Let $\rho_\alpha \in C_0^\infty(U_\alpha)$, $\rho_\alpha \geqq 0$ be a locally finite family of functions so that $\sum \rho_\alpha=1$ in a neighborhood U_0 of V. Let $\tilde{p}$ be any C^∞ extension of p to U_0 and set

$$q=\tilde{p}+\sum_\alpha C_\alpha \rho_\alpha \Big(\sum_j |f_\alpha^j|^2\Big),$$

where the constants $C_\alpha>0$ are yet to be chosen. For $x \in V$,

$$\partial\bar{\partial} q=\partial\bar{\partial}\tilde{p}+\sum_\alpha C_\alpha \rho_\alpha \Big(\sum \partial f_\alpha^j \wedge \bar{\partial} \bar{f}_\alpha^j\Big).$$

If $v \in T_x X$,

$$H_q(v, \bar{v})=H_{\tilde{p}}(v, \bar{v})+\sum_\alpha C_\alpha \rho_\alpha |df_\alpha^j(v)|^2.$$

Let K_α be the support of ρ_α (a compact set). The second form is nonnegative on $T_x(X)$ for $x \in V \cap K_\alpha$ and positive unless $v \in T_x(V)$, in which case the first term is positive. Since $V \cap K_\alpha$ is compact we may use Lemma 1.4 to get $C_\alpha>0$ so that q is spsh at all points of $V \cap K_\alpha$. The $\{K_\alpha\}$ cover V so q is spsh at all points of V and thus by continuity in a neighborhood U of V. Clearly $q|V \cap U=\tilde{p}=p$.

1.10. COROLLARY. *Let $L \xrightarrow{\pi} X$ be a line bundle over a complex space, and K a compact subvariety of X. Suppose $L|K<0$. Then there is a neighborhood N of K such that $L|N$ has metric ρ with positive definite curvature.*

PROOF. Let $V=L|K$, and ρ_0 a spsh metric on V. By the preceding lemma, there is a neighborhood (in L) W of $\{x \in V;\ |\rho_0(x)-1|<1/2\}$ and a spsh function p defined in W, $p|V\cap W=\rho_0$. Since $\{x; \pi(x)\in K; p(x)=1\}$ is compact in W, there is a neighborhood N of K, and an $\varepsilon>0$ such that $U=\{x; \pi(x)\in N; |p(x)-1|<\varepsilon\}$ has compact closure in W. Then, for $x\in N$, $p: U\cap\pi^{-1}(x)\to(1-\varepsilon, 1+\varepsilon)$ properly. We may take N so small so that this map is surjective for all x, and so that condition (b) (ii) of Lemma 1.5 is verified, since these are open conditions which hold on K. Then, by Lemma 1.5 $L|N$ has a spsh metric.

§ 2. Extension of positive bundles.

If X is an irreducible, reduced analytic space, let $\mathcal{M}$ be the sheaf of germs of meromorphic functions on X, $\mathcal{M}^*=\mathcal{M}-\{0\}$, $\mathcal{O}^*=\mathcal{O}-\{0\}$, and $\exp\mathcal{O}$ the sheaf of germs of invertible holomorphic functions. The *sheaf* $\mathcal{D}$ of *divisors on* X is defined via the (multiplicative) exact sequence:

$$1\longrightarrow\exp\mathcal{O}\longrightarrow\mathcal{M}^*\longrightarrow\mathcal{D}\longrightarrow 1\,.$$

2.1. DEFINITION. A (Cartier) *divisor on* X is an element $D\in H^0(X, \mathcal{D})$. D is a *holomorphic divisor* if it is in $H^0(X, \mathcal{O}^*/\exp\mathcal{O})$. The image of D under the coboundary map

$$H^0(X, \mathcal{D})\longrightarrow H^1(X, \exp\mathcal{O})$$

is denoted $[D]$, the *line bundle associated to the divisor D*.

Otherwise said, a holomorphic divisor is given by a covering $\{U_\alpha\}$ and functions $f_\alpha\in H^0(U_\alpha, \mathcal{O})$ not identically vanishing, such that the $f_\alpha^\beta=f_\alpha/f_\beta$ are invertible. The f_α^β are the transition functions for the line bundle $[D]$. Notice that, since $f_\alpha=f_\alpha^\beta f_\beta$, $[D]$ has a section σ_D defined by the $\{f_\alpha\}$ locally. We call this the *canonical section of the divisor*. Conversely, if L is a line bundle, and $\sigma\in H^0(X, L)$, $\sigma\neq 0$, σ defines a holomorphic divisor D with $[D]=L$.

The *support of the divisor* D is the set

$$|D|=\{x: \sigma_D(x)=0\}\,.$$

Notice that σ_D^{-1} defines a meromorphic section of $[D]^{-1}$ which has poles only on $|D|$, and no zeros.

2.2. DEFINITION. Let D be a Cartier divisor on X. The normal *bundle of* D is the bundle $[D]|_{|D|}\to|D|$.

2.3. DEFINITION. Let K be a compact subvariety of a complex space X, K is *negatively* (*positively*) *embedded* if there is a divisor D with $|D|=K$ with a negative (positive) normal bundle.

More specifically, suppose K is a subvariety whose sheaf of ideals I is invertible. Then, for some covering $\{U_\alpha\}$, $I=f_\alpha\mathcal{O}$ in U_α, and the $\{f_\alpha\}$ define a holomorphic divisor $\{K\}$. If $[\{K\}]$ has a negative (positive) normal bundle.

we shall say that K is *strongly negatively* (*positively*) embedded.

Grauert proved that if the normal bundle of a divisor D is negative, then $|D|$ is exceptional, as well as the following theorem. Our argument is a modification of his in the negative case and works as well in both cases.

2.4. THEOREM (Grauert) ([4], p. 347). *Let X be a compact complex space and D a holomorphic divisor with positive normal bundle. Suppose that $X-|D|$ has no exceptional varieties. Then $[D]>0$.*

PROOF. Let $L=[D]^{-1}$, and $K=|D|$. By hypothesis, $L|K<0$. By Corollary 1.10 there is a neighborhood N of K such that $L|N$ has a spsh metric ρ. Let $\tau=\sigma_D^{-1}$ be the canonical meromorphic section described above. Then $L|_{X-K}\cong (X-K)\times \boldsymbol{C}$ under the correspondence $(x, \xi)\to\xi\tau(x)$. Then, over $N-K$, the metric is given by an spsh function p on $N-K$ by

$$\rho(\xi\tau(x))=|\xi|^2p(x).$$

Since $p(x)=\rho(\tau(x))$, and $\tau(x)\to\infty$ as $x\to K$, $p(x)\to\infty$ also as $x\to K$. Thus there is an m such that the domain

$$D_m=X-K\cup\{x\in N-K\,;\ lnp(x)\geqq m\}$$

is strongly pseudoconvex. Since D_m has no exceptional subvarieties, it is Stein, so by Lemma 1.3 there is an spsh function q defined on $X-K$ such that $q=lnp$ on $X-D_{m-\varepsilon}$. In particular we can extend ρ to a spsh metric on all of L by defining $\rho(\xi\tau(x))=|\xi|^2e^{q(x)}$ throughout $X-K$. Thus L is negative.

2.5. COROLLARY. *Let X be a compact complex space, and D a holomorphic divisor with positive normal bundle. Suppose that every exceptional subvariety of $X-|D|$ is negatively embedded. Then X is algebraic.*

PROOF. Following through the above argument, we find that D_m is strongly pseudoconvex so contains a finite number of disjoint connected exceptional varieties $E_1, \cdots, E_k$. If these are identified to points, we obtain a Stein analytic space D_m, and we can extend lnp to a spsh function q. Lifting q back to D_m, it extends the metric ρ to a metric defined on all of L such that $\Theta_\rho(v, \bar{v})>0$ if $v\neq 0$, $v\in T_x(X)$, and $x\notin E_j$ for any j. Now each E_j is negatively embedded, so there are holomorphic divisors D_j such that $|D_j|=E_j$ and $[D_j]|_{E_j}<0$. By Lemma 1.9 we can find mutually disjoint neighborhoods N_j of the E_j, and spsh metrics ρ_j for $[D_j]|_{N_j}$. Let $N_j^0\subset N_j$ and let ψ_j, $0\leqq j\leqq k$ be a partition of unity subordinate to the cover $X-\cup N_j^0$, N_j such that $\psi_j=1$ in N_j^0.
Then $\hat{p}=\psi_0+\sum_j\psi_j\rho_j$ is a metric for the line bundle $[D_1\cdots D_k]$ and $\Theta_\rho(v, v)>0$ for $v\in T_x(X)$, $v\neq 0$, $x\in E_j$, $1\leqq j\leqq k$. By Lemma 1.4 there is a positive integer N such that $\Theta_\rho(v, \bar{v})+N\Theta_\rho(v, \bar{v})>0$ for any $v\neq 0$. Thus $\hat{\rho}\cdot\rho^N$ is a spsh metric for $[D^{-N}D_1\cdots D_k]$, so by Theorem 1.8, X is algebraic.

§ 3. Compactifications of C^2.

The results of the preceding section apply to give a direct proof of van de Ven's theorem that a compactification X of C^2 is algebraic; and that $X-C^2$ contains the support of a hyperplane section. First, we shall need to recall the machinery of curves on surfaces. Let X be a complex surface (X is a compact manifold of complex dimension 2). Let Γ be a nonsingular curve in X and let $[\Gamma]$ denote the bundle of the divisor $\{\Gamma\}$. Let Σ be another nonsingular curve in X. Then the intersection multiplicity $(\Gamma\cdot\Sigma)$ is defined to be

$$(\Gamma\cdot\Sigma)=\langle c([\Gamma])\cup c([\Sigma]), \zeta\rangle, \tag{3.1}$$

where $c([\Gamma])$ is the Chern class of $[\Gamma]$, $\cup$ denotes cup product, ζ denotes the generator of $H_4(X, Z)$ determined by the orientation of X as a complex manifold, and $\langle , \rangle$ denotes Kronecker product. If Γ and Σ have only transversal intersections then $(\Gamma\cdot\Sigma)$ is just the number of intersection points. We also note that $c([\Gamma])$ is the Poincare dual of the class in $H_2(X, Z)$ of the cycle Γ ([8], p. 72). Let $\frown$ denote cap product. Then we have

$$\langle c([\Sigma]), \zeta\frown c([\Gamma])\rangle=\langle c([\Gamma])\cup c([\Sigma]), \zeta\rangle \tag{3.2}$$

and therefore by Poincare duality

$$(\Gamma\cdot\Sigma)=\langle c([\Sigma]), \xi\rangle=c([\Sigma]|_{\Gamma})$$

where ξ is the class of Γ and $c([\Sigma]|_{\Gamma})\in H^2(\Gamma, \boldsymbol{Z})=\boldsymbol{Z}$ (canonically). We remark that a line bundle L on a curve Γ is positive (negative) if $c(L)>0$ (<0).

Let A be a one-dimensional analytic subset of X. We say that A has *normal crossings only* if $A=\bigcup_{i=1}^{m}\Gamma_i$ where the Γ_i are nonsingular and if Γ_i intersects Γ_j they intersect transversally (normally) in one point. We shall employ the following criterion for such an A to be exceptional.

3.3. THEOREM (Grauert-Mumford [4], p. 367). *Let $A=\bigcup_{j=1}^{m}\Gamma_j$ be a connected collection of curves with normal crossings only on a surface X. A is exceptional if and only if the matrix $(\Gamma_j\cdot\Gamma_k)$ is negative definite.*

Now we can give the proof of Van de Ven's theorem.

3.4. THEOREM (Van de Ven [16], p. 193). *Let X be a compactification of C^2. Then X is algebraic.*

REMARK. The proof shows also that a compactification of a complex homology cell is algebraic.

PROOF. By a *compactification* of C^2 we mean a nonsingular compact complex surface X such that $A=X-C^2$ is an analytic set. It follows easily that A is a connected one-dimensional complex space. We may resolve the singularities of A by quadratic transformations so that A becomes a space with normal crossings only. The new X thus created will be a compactification of $\boldsymbol{C}^2$ and will be algebraic if and only if the old X was algebraic (see Kodaira [**10**], p. 44 or Theorem 6.5). Consider the exact sequences

$$\text{(3.5)} \qquad \cdots \longrightarrow H_c^k(X-A) \longrightarrow H^k(X) \longrightarrow H^k(A) \longrightarrow H_c^{k+1}(X-A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{k+1}(X, A) \longrightarrow H_k(A) \longrightarrow H_k(X) \longrightarrow H_k(X, A) \longrightarrow \cdots .$$

Since $X-A$ is a cell, it follows that

$$H^2(X\,;\boldsymbol{Z}) = H^2(A\,;\boldsymbol{Z}) = \boldsymbol{Z}^p$$

$$H_2(X\,;\boldsymbol{Z}) = H_2(A\,;\boldsymbol{Z}) = \boldsymbol{Z}^p$$

where p is the number of curves in $A = \bigcup_{i=1}^{p} \Gamma_i$.

3.5. LEMMA. *The matrix $(\Gamma_j \cdot \Gamma_k)$ is nonsingular.*

PROOF. By duality and by (3.5),

$$0 = H^3(A, \boldsymbol{Z}) = H^3(X, \boldsymbol{Z}) = H_1(X, \boldsymbol{Z}) .$$

Thus the Kronecker pairing $\langle x, y\rangle$, $x \in H^2(X, \boldsymbol{Z})$, $y \in H_2(X, \boldsymbol{Z})$ is nondegenerate. Let $y_1, \cdots, y_p$ be the classes in $H_2(X\,;\boldsymbol{Z})$ defined by $\Gamma_1, \cdots, \Gamma_p$. Let $x^1, \cdots, x^p$ be their Poincare duals in $H^2(X, \boldsymbol{Z})$; $\zeta \frown x^j = y_j$. According to (3.1), (3.2),

$$(\Gamma_j \cdot \Gamma_k) = \langle x^j \cup x^k, \zeta\rangle = \langle x^j, \zeta \frown x^k\rangle = \langle x^j, y_k\rangle .$$

Since the x^j generate $H^2(X, \boldsymbol{Z})$ freely, and the y_j generate $H_2(X, \boldsymbol{Z})$ freely, this matrix is nonsingular.

We return to the proof of Van de Ven's theorem. The nonsingular matrix $(\Gamma_j \cdot \Gamma_k)$ is not negative definite. If it were, then by Theorem 3.3, we could collapse A to point p, getting a normal compact complex space $\tilde{X}$ with $\tilde{X} - \{p\} = \boldsymbol{C}^2$. But then, by Hartogs' theorem any nonconstant holomorphic function f on $\boldsymbol{C}^2$ would extend to a noncostant holomorphic function on X, which is impossible.

3.6. LEMMA. *Let (a_{ij}) be an integral symmetric $(p \times p)$ nonsingular, not negative definite matrix which has nonnegative nondiagonal entries. Then there are integers $n_i \geqq 0$, $1 \leqq i \leqq p$ not all zero such that $\sum n_i a_{ij} > 0$ for all j with $n_j > 0$.*

PROOF. Since the required property is an open homogeneous condition, it suffices to produce real numbers n_i with that property. (This lemma was suggested to us by a similar lemma in Laufer's book [**12**].) The proof is by induction on p; the case $p=1$ is trivial. If some $a_{ii} > 0$, take $n_i = 1$, $n_j = 0$ for $j \neq i$. If all a_{ii} are zero, choose j such that $a_{ij} > 0$. Let $n_i = n_j = 1$ and $n_k = 0$, $k \neq i, j$. Otherwise some $a_{kk} < 0$. Interchanging the *kth* and *pth* rows and columns leaves all the required properties intact, so we may assume $a_{pp} < 0$.

Multiplying by $-a_{pp}^{-1}>0$, nothing essential changes, and we obtain $a_{pp}=-1$. Let $S'=(\beta_{jk})=Q^tSQ$ where

$$Q^t=\begin{pmatrix} 1 & 0 & \cdots\cdots & a_{1p} \\ 0 & 1 & \cdots\cdots & a_{2p} \\ \vdots & \vdots & \cdots\cdots & \vdots \\ 0 & 0 \cdots & 1 & a_{p-1,p} \\ 0 & 0 \cdots & 0 & 1 \end{pmatrix}.$$

Then $\beta_{jk}=a_{jk}+a_{jp}a_{pk}$ for $j, k\leqq p-1$, $\beta_{pj}=\beta_{jp}=0$, $j<p$ and $\beta_{pp}=-1$, and the $(p-1)\times(p-1)$ matrix $B=(\beta_{jk})$, $j, k\leqq p-1$ has all the properties of the hypotheses. The induction hypotheses allows us to conclude that there are real numbers $R_i\geqq 0$, $1\leqq i\leqq p-1$, not all zero such that $\sum_{i=1}^{p-1}R_i\beta_{ij}>0$ if $R_j>0$. This then becomes

$$\sum_{i=1}^{p-1}R_ia_{ij}+a_{pj}\left(\sum_{i=1}^{p-1}R_ia_{ip}\right)>0 \qquad \text{if } R_j>0\,. \tag{3.7}$$

Let $R_p=-\varepsilon+\sum_{i=1}^{p-1}R_ia_{ip}$, if the second term in (3.7) is not positive, where $\varepsilon>0$ is to be chosen. Otherwise let $R_p=0$. In the latter case, we are finished. In the former case,

$$\sum_{i=1}^{p}R_ia_{ij}=\sum_{i=1}^{p-1}R_ia_{ij}+a_{pj}\left(\sum_{i=1}^{p-1}R_ia_{ip}\right)-\varepsilon a_{pj}\,.$$

Because of (3.7) we may choose $\varepsilon>0$ so that the sum is positive for those $j<p$ with $R_j>0$. For $j=p$ we get $(a_{pp}=-1)$

$$\sum_{i=1}^{p}R_ia_{ip}=\sum_{i=1}^{p-1}R_ia_{ip}+\varepsilon-\sum_{i=1}^{p-1}R_ia_{ip}=\varepsilon>0\,.$$

Now we can finish the proof of Theorem 3.4. Let D be the divisor $\{\Gamma_1\}^{n_1}\cdots\{\Gamma_p\}^{n_p}$ where the n_j are the integers found in Lemma 3.6 for the matrix $(\Gamma_i\cdot\Gamma_j)$, $|D|=A'=\bigcup_{n_j>0}\Gamma_j$, and for j such that $n_j>0$

$$c([D]|_{\Gamma_j})=\sum_{i=1}^{p}n_ic([\Gamma_i]|_{\Gamma_j})=\sum_{i=1}^{n}n_i(\Gamma_i\cdot\Gamma_j)>0\,.$$

Thus $[D]|_{\Gamma_j}$ is positive. Then by [**12**], p. 62, $[D]|_{A'}$ is positive, so D has a positive normal bundle. By Theorem 3.3 every exceptional subvariety of $X-A'$ is strongly negatively embedded, so Corollary 2.4 applies: X is algebraic.

§ 4. Characterizations of projective space.

In dimensions at least 2, Hartogs' theorem tells us that a normal projective variety is determined by a neighborhood of a hyperplane section. We would like that neighborhood to be an infinitesimal neighborhood. The following results tell us that for projective space, a first order neighborhood will suffice.

4.1. THEOREM. *Let X be a connected compact complex manifold of dimension $n \geqq 2$. Let K be a submanifold of codimension* 1 *such that there is a biholomorphic map $j: \boldsymbol{P}^{p-1} \to K$ with $j_*([K]|K)$ being the hyperplane section bundle of $\boldsymbol{P}^{n-1}$. Then if $X-K$ has no exceptional varieties, $X=\boldsymbol{P}^n$.*

Before we begin the proof we first prove a lemma.

4.2. LEMMA. *If X satisfies the hypothesis of Theorem* 4.1, *then $H^1(X, \mathcal{O})=0$.*

PROOF. We have assumed $\dim X=n \geqq 2$. We have the following exact sequence (where $H_c^*(-, \boldsymbol{Z})$ denotes cohomology with compact support).

$$\cdots \longrightarrow H_c^1(X-K, \boldsymbol{Z}) \longrightarrow H^1(X, \boldsymbol{Z}) \longrightarrow H^1(K, \boldsymbol{Z}) \longrightarrow H_c^2(X-K) \longrightarrow \cdots.$$

Now $X-K$ is holomorphically convex since $[K]|K>0$. Since $X-K$ has no compact subvarieties it is Stein, and thus has the homotopy type of a CW complex of (real) dimension n. Thus $H_c^1(X-K, \boldsymbol{Z})=H_{2n-1}(X-K, \boldsymbol{Z})=0$. Since $K=\boldsymbol{P}^{n-1}$, $H^1(K, \boldsymbol{Z})=0$. Then the exact sequence implies that $H^1(X, \boldsymbol{Z})=0$, so $b_1(X)=b_1=0$, where b_1 is the first Betti number of X. Now by Theorem 2.4, X is projective and hence $b_1=2h^{0,1}$ where $h^{0,1}=\dim H^1(X, \mathcal{O})$. Thus $H^1(X, \mathcal{O})=0$.

PROOF (of Theorem 4.1). By hypothesis $[K]|K$ has n sections $\sigma_1^0, \cdots, \sigma_n^0$ which define a biholomorphic map of K onto P^{n-1}. Let $\sigma_0 \in H^0(X, \underset{\sim}{[K]})$ be the canonical section, so that $K=\{x \in X: \sigma_0(x)=0\}$. From the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\sigma_0} \underset{\sim}{[K]} \xrightarrow{\pi} \underset{\sim}{[K]|K} \longrightarrow 0$$

and Lemma 4.2 we obtain the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}) \longrightarrow H^0(X, [K]) \xrightarrow{\pi_*} H^0(K, [K]|K) \longrightarrow 0.$$

Let $\sigma_i \in H^0(X, [K])$ be such that $\pi_*(\sigma_i)=\sigma_i^0$, $1 \leqq i \leqq n$, and let $F: X \to P^n$ be the meromorphic map $F(p)=[\sigma_0(p), \cdots, \sigma_n(p)]$, where we use homogeneous coordinates in P^n. F is actually holomorphic: if $p \notin K$, $\sigma_0(p) \neq 0$ and if $p \in K$ $\sigma_i(p) \neq 0$ for some $i>0$.

Let $V=\{x \in X; \text{rank } dF(x)<n\}$. Then V is compact and $V \cap K=\emptyset$ since $d\sigma_0(x)$ is independent of the $d\sigma_i(x)$ for $i>0$, $x \in K$ and $[\sigma_1(x), \cdots, \sigma_n(x)]$ gives a biholomorphic map onto $\boldsymbol{P}^{n-1}$. But V has codimension 1 in X (it is defined locally by the equation $\det dF(x)=0$, and $\det dF$ is not identically zero). Since $X-K$ has no compact subvarieties, $V=\emptyset$; and $F: X \to \boldsymbol{P}^n$ is an unbranched covering. P^n is simply connected so F is biholomorphic.

§ 5. Fiberings over projective varieties.

Let X be a compact complex space, reduced and irreducible, and $\mathcal{M}(X)$ its field of meromorphic functions. By $\operatorname{tr}(X)$ we mean the transcendence degree of $\mathcal{M}(X)$ over $\boldsymbol{C}$. If $\operatorname{tr}(X)=\dim X$, X is said to be a Moisheson space. It is clear that X is Moisheson if and only if there is a set of meromorphic functions, all holomorphic at some regular point of X, with spanning differentials there.

Now let $F\to X$ be a coherent analytic sheaf. As in [**15**], we see that there is a Zariski open set U in X on which F is locally free. If $F|U$ is locally free of rank k, we shall say that F is of *generic rank* k, writing $\operatorname{grk}(F)=k$. The construction in [**15**] demonstrates that any m sections of a coherent sheaf of generic rank 1 defines a meromorphic map into $\boldsymbol{P}^{m-1}$. Our next result generalizes (as we shall see) Kodaira's theorem on surfaces of transcendence degree 1 ([**11**], p. 131 and p. 134); the proof is not too different from, but more general than that of Hironaka [**6**].

5.1. THEOREM. *Let X be a compact complex space, Δ a projective variety and $\tau: X\to\Delta$ a surjective holomorphic map. Suppose there exists a coherent sheaf $F\to X$ of generic rank* 1, *and there is a $p\in\Delta$ such that $F=\underset{\sim}{L}$ over a neighborhood U of p, and $L|\tau^{-1}(p)>0$. Then X is Moisheson.*

PROOF. By Grauert's theorem [**3**], the 0th direct image F_0 of F is a coherent sheaf on Δ. By Corollary 1.10, $\{q\in U\,;\,L|\tau^{-1}(q)>0\}$ is open. Thus we may select our p so that $F_{0,p}$ is free, p is regular in Δ, and there is a regular point q for X on $\tau^{-1}(p)$. In this case $F_{0,p}^{\nu}$ is isomorphic to the sheaf of germs of holomorphic maps into $H^0(\tau^{-1}(p), L^{\nu}|\tau^{-1}(p))$ for every ν. Choose ν large enough so that the sections of L^{ν} on $\tau^{-1}(p)$ embed $\tau^{-1}(p)$ into $\boldsymbol{P}^k$. Now let D be a holomorphic divisor on Δ with $[D]>0$ and $p\notin|D|$ (such exist since Δ is projective). There is a $\mu>0$ such that the sections of $\underset{\sim}{[D]^{\mu}}\otimes F_0^{\nu}$ generate $F_{0,p}^{\nu}$ as $\mathcal{O}_{\Delta,p}$-module (see Grauert [**4**], p. 344). Thus there are sections $\sigma_0,\cdots,\sigma_k\in H^0(\Delta,[D]^{\mu}\otimes F_0^{\nu})$ such that $\sigma_0(p),\cdots,\sigma_k(p)$ span $H^0(\tau^{-1}(p), L^{\nu}|\tau^{-1}(p))$. $\tau^*(\sigma_0)$, $\cdots$, $\tau^*(\sigma_k)$ are thus sections of $\underset{\sim}{\tau^*([D])^{\mu}}\otimes F^{\nu}$ on X whose restrictions to $\tau^{-1}(p)$ define a projective embedding. If, say $\tau^*(\sigma_0)(q)\neq 0$, the meromorphic functions $f_i=\tau^*(\sigma_i)/\tau^*(\sigma_0)$, $1\leqq i\leqq k$ have differentials spanning $T_q(\tau^{-1}(p))$. Since Δ is algebraic, there are meromorphic functions on Δ, $f_{k+1},\cdots,f_n$, regular at p whose differentials span $T_p(\Delta)$. Thus $f_{k+1}\circ\tau,\cdots,f_n\circ\tau$ are also meromorphic on X, and constant on $\tau^{-1}(p)$, so the differentials $df_i(q)$, $1\leqq i\leqq k$, $d(f_j\circ\tau)(q)$, $k+1\leqq j\leqq n$ span $T_q(X)$. Thus X is Moisheson.

5.2. COROLLARY. *Let X be a compact complex space of dimension n, with* $\operatorname{tr}(X)=n-1$. *There is a proper modification $\pi: X'\to X$ of X for which there is a holomorphic map $\tau: X'\to\Delta^{n-1}$, inducing an isomorphism of function fields with Δ a projective variety and $\tau^{-1}(p)$ generically a curve of genus* 1.

PROOF. Let Δ be a projective variety with the function field $\mathcal{M}(X)$, then $\dim\Delta=n-1$ and there is a meromorphic map $F: X\to\Delta$. Let X' be the normalization of the closure of the graph of F, $\pi: X'\to X$, $\tau: X'\to\Delta$ the two projections. Let Θ be the sheaf of germs of holomorphic vector fields on X', $F_1=\{v\in\Theta\,;\,\tau_*(v)=0\}$, $F_2=\mathrm{Hom}(F_1, \mathcal{O})$. Let V be the singular locus of X'; since X' is normal, $\dim V\leqq n-2$, so $\tau(V)$ is a proper subvariety of Δ. Let $S=\{x\in X'-\tau^{-1}\tau(V)\,;\,\mathrm{rank}_x\tau_x<n-1\}$. By Bertini's Theorem, $\tau(S)$ is a proper subvariety of $\Delta-\tau(V)$. Let $\Delta_0=\Delta-[\tau(V)\cup\tau(S)]$. $X_0'=\tau^{-1}(\Delta_0)$. Then $\tau: X_0'\to\Delta_0$ is a fibration of connected manifolds, so for $p\in\Delta_0$, $\tau^{-1}(p)$ is a union of homeomorphic nonsingular curves. Let g be the common genus. Now, on X_0', $F_1=\underset{\sim}{L}^{-1}$, $F_2=\underset{\sim}{L}$ where $L|\tau^{-1}(p)$ is the canonical bundle of $\tau^{-1}(p)$. If $g\neq 1$, one of $L|\tau^{-1}(p)$, $L^{-1}|\tau^{-1}(p)$ is positive. Since X is not Moisheson, by the preceding theorem we must have $g=1$.

REMARK. In fact $\tau^{-1}(p)$ is generically a single nonsingular curve by the following reasoning (see [9]): if $\tau: X'\xrightarrow{\tau_1}\Delta'\xrightarrow{\tau_2}\Delta$ is the Stein factorization ($\tau_1^{-1}(p)$ is connected, $\tau_2^{-1}(p)$ is discrete), then Δ' is algebraic. For $f\in\mathcal{M}(\Delta')$; $\tau_1^*(f)=\tau^*(f_0)$ for some $f_0\in\mathcal{M}(\Delta)$, so $f=\tau_2^*(f_0)$. Thus $\tau_2^*: \mathcal{M}(\Delta)=\mathcal{M}(\Delta')$. Since Δ' is algebraic (see Corollary 5.4 below), we may use Δ' instead of Δ. (As Kawai points out since $\mathcal{M}(\Delta)=\mathcal{M}(\Delta')$, it follows that τ_2 is an isomorphism, so the fibers were connected to begin with).

Now, the Moisheson space of Theorem 5.1 is in general not algebraic. However, if the generic hypotheses are replaced by global ones X will be algebraic, for we can carry out the appropriate patching of spsh metrics.

5.3. THEOREM. *Let X be a compact complex space, Δ a projective variety and $\tau: X\to\Delta$ a holomorphic surjective map. Suppose that $L\xrightarrow{\pi}X$ is a line bundle such that $L|\tau^{-1}(p)<0$ for all $p\in\Delta$. Then X is algebraic.*

PROOF. By Corollary 1.9, there is, for each $p\in\Delta$, a neighborhood U_p and a spsh metric ρ_p for $L|\tau^{-1}(U_p)$. Let $\{U_1,\cdots,U_n\}$ be a cover of Δ by such neighborhoods and $\rho_1,\cdots,\rho_n$ the corresponding metrics. Let $\psi_1,\cdots,\psi_n$ be a partition of unity subordinate to the cover $\{U_1,\cdots,U_n\}$. Then $\rho=\sum_{i=1}^{n}(\psi_i\circ\tau\circ\pi)\rho_i$ is a metric on L. Now if $v\in T_x(L)$, $v\neq 0$ such that $(\tau\circ\pi)_*(v)=0$,

$$\partial\bar\partial\rho(v,\bar v)=\sum(\psi_i(\tau(\pi x))\partial\bar\partial\rho_i(v,\bar v)>0\,.$$

Now the forms $\partial\bar\partial ln\rho$ and $\partial\bar\partial\rho$ agree modulo a positive scalar on the complex tangent space to the surface of unit vectors in L. Since $\pi_{*,x}$ maps this space onto $T_{\pi(x)}(X)$, we conclude that $\Theta_\rho(v,\bar v)>0$ for $v\in T_{\pi(x)}(X)$, $v\neq 0$, $\tau_*(v)=0$. For if w is tangent to this surface at x, such that $\pi_x(w)=v$, then

$$\Theta_\rho(v,\bar v)=\pi^*\Theta_\rho(w,\bar w)=\partial\bar\partial ln\rho(w,\bar w)=c(x)\partial\bar\partial\rho(w,\bar w)>0\,.$$

Let H_0 be a negative line bundle on Δ, and σ_0 a spsh metric on H. Then $\tau^*\sigma=\tau^*\sigma_0$ is a metric on $H=\tau^*H_0$. The curvature of σ, Θ_σ, is positive semi-definite and if $\tau_*v\neq 0$, $\Theta_\sigma(v,\bar v)=\Theta_{\sigma_0}(\tau_*v,\overline{\tau_*v})>0$. Once again we use Lemma 1.4 to conclude that there is an integer N such that for any $v\in T_x(X)$, $v\neq 0$,

$$\Theta_\rho(v, \bar{v})+N\Theta_\sigma(v, \bar{v})>0 .$$

But $\Theta_\rho+N\Theta_\sigma=\Theta_{\rho\sigma}N$, where $\rho\sigma^N$ is a metric on $L\otimes H^N$. Thus this bundle is negative on X, so X is algebraic.

5.4. COROLLARY (see Wavrik [17]). *If $\tau: X\to\Delta$ is proper and light and Δ is algebraic, then so is X.*

PROOF. Take L to be the trivial bundle, and apply the proof of the above theorem.

5.5. COROLLARY. *Let X be a compact complex space of dimension n, and Δ a projective variety of dimension $n-1$. Let $\tau: X\to\Delta$ be a holomorphic map, and D a holomorphic divisor on X.*

(a) *If there is a $p\in\Delta$ such that* $\dim\tau^{-1}(p)=1$, $\dim|D|\cap\tau^{-1}(p)=0$ *and nonempty, then X is Moisheson.*

(b) *If for all $p\in\Delta$,* $\dim|D|\cap\tau^{-1}(p)=0$, *then X is algebraic.*

PROOF. In case (a) we could replace $|D|$ by a subvariety K of codimension 1. For we could choose p satisfying (a) so that the ideal sheaf of K is the sheaf of sections of a line bundle L near p. Clearly $L|\tau^{-1}(p)>0$ so theorem (5.1) applies: X is Moisheson. In case (b), $[D]|\tau^{-1}(p)>0$ for all $p\in\Delta$, so theorem (5.3) applies: X is algebraic.

§ 6. More on surfaces.

In the case of compact complex spaces of dimension 2 the preceeding theorems can be made sharper. Although the results in the section are well known we include them for purposes of comparison and completeness and because they are easy to prove.

6.1. THEOREM. *Let S be a 2-dimensional compact complex space with* $\mathrm{tr}(S)=2$. *Then there is a modification $\pi: S'\to S$ such that S' is algebraic.*

PROOF. Let Δ be a (non-singular) projective variety with function field $\mathcal{M}(S)$. Then there is a meromorphic map $\Phi: S\to\Delta$ with $\Phi^*: \mathcal{M}(\Delta)\cong\mathcal{M}(S)$. Let S' be the graph of the meromorphic map Φ and $\pi_1: S'\to S$, $\pi_2: S'\to\Delta$ the projection maps. Again, by resolution of singularities, we may assume that S' is a manifold. The map $\pi_1: S'\to S$ is a modification. By Stein factorization we can write π_2 as $S'\to\hat{\Delta}\to\Delta$. The map $S'\to\hat{\Delta}$ is a modification and $\hat{\Delta}\to\Delta$ is a branched covering. Now $\mathcal{M}(S')=\mathcal{M}(\hat{\Delta})=\mathcal{M}(\Delta)$. By 5.4 $\hat{\Delta}$ is algebraic and since $\mathcal{M}(\hat{\Delta})=\mathcal{M}(\Delta)$, it easily follows that $\hat{\Delta}=\Delta$. Thus π_2 is a modification and by Hopf [7] π_2^{-1} is given by a composition of σ-processes. Since Δ is algebraic so is S'.

6.2. COROLLARY (Kodaira-Chow [2]). *A Moisheson surface S is algebraic.*

PROOF. Let $\pi_1: S'\to S$ be as above. Then (since S, S' are manifolds) π_1^{-1} is given by a composition of σ-processes. By Kodaira [10], S is algebraic since S' is algebraic.

6.3. COROLLARY. *If S is an algebraic surface and if S' is a surface which has a holomorphic surjective map $\pi: S'\to S'$, then S' is algebraic.*

6.4. THEOREM. *Let S be a 2-dimensional normal compact complex space,* $\mathrm{tr}(S)=1$. *Then there is a curve Δ and a holomorphic map $\tau: S\to\Delta$, whose generic fiber is an elliptic curve.*

PROOF. We know there is a modification $\pi: S'\to S$ and a map $\tau: S'\to\Delta$ with the required properties. We shall show that $\tau\circ\pi^{-1}$ is well-defined on S. Let $p\in S$, so that $\pi^{-1}(p)$ is not a point. Then $\pi^{-1}(p)$ is a subvariety of S'. Let E be a branch of $\pi^{-1}(p)$, and $q\in\Delta$ such that $\tau^{-1}(q)\cap E\neq\emptyset$. By Corollary 5.5 (a), we must have $\dim\tau^{-1}(q)\cap E=1$, so $E\subset\tau^{-1}(q)$. Since $\pi^{-1}(p)$ is connected, that implies that $\pi^{-1}(p)\subset\tau^{-1}(q)$. Thus $\tau\circ\pi^{-1}$ has a continuous extension to p since $\tau\circ\pi^{-1}$ is already defined except at an isolated set of points. Since S is normal, $\tau\circ\pi^{-1}$ is holomorphic on S.

As a final application of these techniques we prove a generalization of the theorem in ([**10**], Appendix).

6.5. THEOREM. *Let X be a compact complex manifold of dimension n and $j: \boldsymbol{P}^{n-1}\to X$ an embedding whose normal bundle is the Hopf bundle H^{-1}. Then*

(a) *the space X_0 with $\boldsymbol{P}^{n-1}$ identified to a point p_0 is a manifold, and the projection $X\to X_0$ the quadratic transform,*

(b) *if X is algebraic, so is X_0.*

PROOF. Since $\boldsymbol{P}^{n-1}$ has a negative normal bundle, it has a strongly pseudoconvex neighborhood U. Let L be a line bundle on X. Then $L|_{\boldsymbol{P}^{n-1}}=H^{\nu(L)}$ for some integer $\nu(L)$.

6.6. LEMMA. *If $\nu(L)\geqq 0$, $H^1(U, L)=0$.*

PROOF. By [**4**], p. 355 or [**14**] there is an integer $\nu_0\geqq 0$ such that the projection $H^1(U, \underset{\sim}{L})\to H^1(U, \underset{\sim}{L}\otimes\mathcal{O}/I^{\nu_0})$, is injective, where I is the ideal sheaf of $\boldsymbol{P}^{n-1}$ in X. Now, from the exact sheaf sequence

$$0\longrightarrow \underset{\sim}{L}\otimes I^{\mu}/I^{\mu+1}\longrightarrow \underset{\sim}{L}\otimes\mathcal{O}/I^{\mu+1}\longrightarrow \underset{\sim}{L}\otimes\mathcal{O}/I^{\mu}\longrightarrow 0$$

we obtain the exact cohomology sequence

$$H^1(\boldsymbol{P}^{n-1}, \underset{\sim}{L}\otimes I^{\mu}/I^{\mu+1})\longrightarrow H^1(\boldsymbol{P}^{n-1}, \underset{\sim}{L}\otimes\mathcal{O}/I^{\mu+1}) \longrightarrow H^1(\boldsymbol{P}^{n-1}, \underset{\sim}{L}\otimes\mathcal{O}/I^{\mu}). \tag{6.7}$$

$I^{\mu}/I^{\mu+1}=H^{\mu}$ on $\boldsymbol{P}^{n-1}$, so $\underset{\sim}{L}\otimes I^{\mu}/I^{\mu+1}=H^{\nu(L)+\mu}$, and $H^1(\boldsymbol{P}^{n-1}, H^{\nu(L)+\mu})=0$ if $\nu(L)+\mu\geqq 0$. Thus, if $\nu(L)\geqq 0$, the second map of (6.7) is injective for all $\mu\geqq 0$, so, after composing all these maps, $0\leqq\mu\leqq\nu_0$, we obtain injectivity of the following map

$$H^1(U, \underset{\sim}{L})\longrightarrow H^1(\boldsymbol{P}^{n-1}, \underset{\sim}{L}\otimes\mathcal{O}/I) = H^1(\boldsymbol{P}^{n-1}, H^{\nu(L)})=0\,.$$

Thus $H^1(U, L)=0$.

We return now to the proof of (a). From the exact sequence

$$0 \longrightarrow I^2 \longrightarrow I \longrightarrow I/I^2 \longrightarrow 0$$

and the just proven fact that $H^1(X, I^2)=0$ ($I^2|_{P^{n-1}}=H^2$), we have that $H^0(U, I) \to H^0(U, I/I^2)=H^0(\boldsymbol{P}^{n-1}, H)$ is surjective. Thus there are $f_i \in H^0(U, I)$, $0 \leqq i \leqq n$ whose projections in I/I^2 generate the stalk at each point and send $\boldsymbol{P}^{n-1}$ to $\boldsymbol{P}^{n-1}$. Thus $F=(f_1, \cdots, f_n)$ defines a map of U into the quadratic transform $\boldsymbol{QC}^n$ of the origin in $\boldsymbol{C}^n$, which is one-one on $\boldsymbol{P}^{n-1}$ and an immersion there. Thus F is a biholomorphic map in some neighborhood of $\boldsymbol{P}^{n-1}$. (a) is proven.

(b). By part (a) we may take U to be the ball in $\boldsymbol{C}^n$ with the origin blown up. Since $H^1(U, \mathcal{O})=0$ and U is contractible to $\boldsymbol{P}^{n-1}$, the restriction map $H^1(U, \mathcal{O}^*) \to H^1(\boldsymbol{P}^{n-1}, \mathcal{O}^*)$ is injective. Assuming that X is algebraic, let L be a negative line bundle on X, and ρ a spsh metric for L. By the above we see that $L \cong [\boldsymbol{P}^{n-1}]^{\nu(L)}$ in U. Now the algebraicity of X_0 follows from this more general fact generalizing Grauert's Satz 8, p. 364 [**4**].

6.8. LEMMA. *Let X be a projective variety, D a holomorphic divisor with a negative normal bundle and X_0 be X with $|D|$ blown down to a point p_0 (normalized). If there is a negative line bundle $L \to X$ such that $L=[D]^\nu$ in some neighborhood of $|D|$, then X_0 is also algebraic.*

PROOF. We may assume that $L=[D]$ in the neighborhood U and that $H^0(X, L^{-1})$ projectively embeds X, by replacing L with L^k for some $k \geqq 0$, and D by $D^{\nu k}$ (notice that $\nu>0$ necessarily, since $L|_{|D|}$ must be negative). By hypothesis, $L^{-1}|_{(U-|D|)}$ is trivial (since $[D]|_{(X-|D|)}$ is trivial). Let K be the trivial line bundle extension of $L^{-1}|_{X_0-\{p_0\}}$ to X_0. If $\sigma \in H^0(X, L^{-1})$, $\sigma|_U$ corresponds, under the isomorphism $L^{-1}=[D]^{-1}$, to a function holomorphic in U, vanishing on $|D|$, so drops down to U_0 (which is U with $|D|$ replaced by p_0). Thus if $\sigma_1, \cdots, \sigma_N$ are a basis for $H^0(X, L^{-1})$, they drop down to $H^0(X_0, K)$, still denoted $\sigma_1, \cdots, \sigma_N$. Let $\rho=\sum|\sigma_i|^2$ considered as a function on K^{-1}. ρ is zero just on the fiber over p_0 and otherwise defines a metric on K^{-1} with $\Theta_\rho \gg 0$, since the σ_i embed X, $X_0-\{p_0\}$ into projective space. In U_0, ρ is a C^∞ function with $\partial\bar{\partial} ln\rho \gg 0$ and $ln\rho(p) \to -\infty$ as $p \to p_0$. By Lemma 1.3 we can find a $g \in C^\infty(U_0)$, $g=ln\rho$ near ∂U_0, with $\partial\bar{\partial} g \gg 0$ everywhere. Then

$$\rho=\begin{cases} \rho & \text{on } X-U_0 \\ e^g & \text{on } \bar{U}_0 \end{cases}$$

defines a global metric on K^{-1} with $\Theta_\rho \gg 0$, thus K^{-1} is negative, and X_0 is algebraic.

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