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On the prolongation of local holomorphic solutions of partial differential equations

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§ 1. Introduction.

Holomorphic continuation of solutions of partial differential equations with constant coefficients has been studied by several authors. C. O. Kiselman [10] showed that under the suitable conditions on the two convex domains $\Omega_1 \subset \Omega_2$ in C^n, every holomorphic solution u of $P(D)u=0$ in Ω_1 can be prolonged to the function holomorphic in Ω_2. He proved this theorem by the Fourier transformation of analytic functionals. On the other hand, M. Zerner [16] used more direct method based on the Cauchy-Kovalevsky theorem to prove the holomorphic continuation theorem over the non-characteristic surface, and G. Bengel [1] obtained a necessary and sufficient condition under which the above theorem was valid. For the system of differential equations, the same result was obtained by J. M. Bony and P. Schapira [2]. In [2] and [16], the case of variable coefficients was also studied. They dealt essentially with the continuation over the non-characteristic surface.

In this paper we study the holomorphic continuation of a solution $u(z)$ of $P(z, D)u=0$ over the simply characteristic surface. In § 3, we show that if the simply characteristic surface $\partial\Omega$ is in C^2 and the second directional derivative of $\phi(z)$, where $\Omega=\{z \mid \phi(z)<0\}$, along a certain direction in a complex bicharacteristic curve is negative at some point, then every holomorphic solution $u(z)$ of $P(z, D)u(z)=0$ in Ω becomes holomorphic near that point (Corollary 1). The proof of this theorem is motivated by the proof in E. C. Zachmanoglou [15] which states the uniqueness of the Cauchy problem. When the coefficients of the operator $P(D)$ are constant, F. Trèves [12] is also available. In § 4, we construct the solution of $P(z, D)u(z)=0$ with singularities in a characteristic variety. The method is employed from Y. Hamada [6] and C. Wagschal [13] in which the singular Cauchy problem is solved. In the last section, § 5, we construct the holomorphic characteristic function and, using the result in § 4, we find a necessary condition for the

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holomorphic continuation: If Ω is strictly pseudo-convex and the second directional derivative of $\phi(z)$, where $\Omega=\{z\,|\,\phi(z)<0\}$, along every direction in a complex bicharacteristic curve is positive, then, under some additional conditions, we can construct a solution $u(z)$ of $P(z, D)u(z)=0$ holomorphic in Ω which cannot be prolonged (Theorem 3).

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§2. Preliminaries.

Let $\boldsymbol{C}^n$ be the complex n-dimensional space with the coordinates $(z_1, \cdots, z_n)$. We set $z_j=x_j+iy_j$ $(j=1, \cdots, n)$ where x_j, y_j are real and $i=\sqrt{-1}$, then $\boldsymbol{C}^n$ may be regarded as the real $2n$-dimensional space $\boldsymbol{R}^{2n}$ with the coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. We denote $\frac{1}{2}\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right)$ by $\frac{\partial}{\partial z_j}$ and $\frac{1}{2}\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right)$ by $\frac{\partial}{\partial \bar{z}_j}$ and set $D=\frac{\partial}{\partial z}=\left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right)$. For any multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$, $D^\alpha=\left(\frac{\partial}{\partial z}\right)^\alpha=\left(\frac{\partial}{\partial z_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$. Let $P(z, D)$ be a differential operator of order m $(m\geqq 2)$ with holomorphic coefficients in an open set Ω in $\boldsymbol{C}^n$, that is

$$P(z, D)=\sum_{|\alpha|\leqq m} a_\alpha(z)\left(\frac{\partial}{\partial z}\right)^\alpha, \qquad m\geqq 2,$$

where $a_\alpha(z)$ is holomorphic in Ω. Its principal part $P_m(z, D)$ is then the homogeneous part of order m,

$$P_m(z, D)=\sum_{|\alpha|=m} a_\alpha(z)\left(\frac{\partial}{\partial z}\right)^\alpha.$$

DEFINITION 1 (Zerner [**16**]). A real hyperplane H through z_0 in $\boldsymbol{C}^n$ is said to be *characteristic at* z_0 *with respect to* $P(z, D)$ if the unique complex hyperplane through z_0 in H is characteristic at z_0.

REMARK. Let $H=\{(x, y)\,|\,\sum_{j=1}^n (x_j-x_j^{(0)})\xi_j+\sum_{j=1}^n (y_j-y_j^{(0)})\eta_j=0\}$, where $z_0=(z_1^{(0)}, \cdots, z_n^{(0)})$ and $z_j^{(0)}=x_j^{(0)}+iy_j^{(0)}$. Then H is characteristic at z_0 if and only if $P_m(z_0, \lambda)=0$, where $\lambda=\xi-i\eta$. Let $\phi(z)$ be a real-valued C^1 function near the point z_0 and

$$\operatorname{grad}_z\phi(z_0)=\left(\frac{1}{2}\left(\frac{\partial\phi}{\partial x_1}(z_0)-i\frac{\partial\phi}{\partial y_1}(z_0)\right), \cdots, \frac{1}{2}\left(\frac{\partial\phi}{\partial x_n}(z_0)-i\frac{\partial\phi}{\partial y_n}(z_0)\right)\right)\neq 0,$$

then the real tangent plane at z_0 of the hypersurface $\{\phi(z)=\phi(z_0)\}$ is characteristic if and only if $P_m(z_0, \operatorname{grad}_z\phi(z_0))=0$.

PROPOSITION 1 (Zerner [**16**], Proposition 1). *Let U be a neighborhood of z_0 in $\boldsymbol{C}^n$ and $\phi(z)$ be a real-valued C^1 function in U such that* $\operatorname{grad}_z\phi(z_0)\neq 0$. *We assume that the real tangent plane at z_0 of the surface $\{z\in U \mid \phi(z)=\phi(z_0)\}$ is non-characteristic with respect to $P(z, D)$. Then every function $u(z)$ which is holomorphic in $\{z\in U \mid \phi(z)<\phi(z_0)\}$ and satisfies $P(z, D)u(z)=0$ is also holomorphic in a neighborhood of z_0.*

DEFINITION 2. A complex hyperplane through z_0 $\{z \mid \langle z-z_0, \lambda\rangle=0\}$, where $\lambda\in\boldsymbol{C}^n$, is said to be *simply characteristic at z_0 with respect to $P(z, D)$* if $P_m(z_0, \lambda)=0$ and $P_m^{(j)}(z_0, \lambda)\neq 0$ for some j $(1\leqq j\leqq n)$, where $P_m^{(j)}(z, \xi)=\frac{\partial}{\partial\xi_j}P_m(z, \xi)$. We call that a real hyperplane is *simply characteristic* if it contains a simply characteristic complex hyperplane.

Here we quote some theorems which are used later.

2.1. Bicharacteristic curves.

Let $P(z, D)$ be a differential operator with holomorphic coefficients in an open set U and $P_m(z, D)$ be its principal part. We use the notation

$$P_m^{(j)}(z, \xi)=\frac{\partial}{\partial\xi_j}P_m(z, \xi), \qquad P_{m,j}(z, \xi)=\frac{\partial}{\partial z_j}P_m(z, \xi).$$

Now we choose a point z_0 in U and a vector $N\in\boldsymbol{C}^n$ such that $P_m(z_0, N)=0$ and $P_m^{(j)}(z_0, N)\neq 0$ for some j. Then a solution $(z(t), \xi(t))$ of the Hamilton equations

$$\text{(1)} \qquad \frac{dz_k}{dt}=P_m^{(k)}(z, \xi), \qquad \frac{d\xi_k}{dt}=-P_{m,k}(z, \xi), \qquad k=1, \cdots, n$$

with the initial conditions

$$z(0)=z_0, \qquad \xi(0)=N$$

is called a bicharacteristic strip through (z_0, N) and the curve described by $z(t)$ is called a bicharacteristic curve through (z_0, N), where t is a complex parameter. As for the relation between the bicharacteristic equations (1) and a holomorphic change of variables, we have the following proposition (Hörmander [**8**], p. 31, Remark 3, Duff [**3**], pp. 49-50).

PROPOSITION 2. *The equations* (1) *are invariant for coordinate transformations if ξ is transformed as a covariant vector.*

REMARK. From this proposition, we especially have that the t_0-direction $(t_0\in\boldsymbol{C},\ t_0\neq 0)$ in a complex bicharacteristic curve, $\{z(\tau t_0)\}$ $(\tau\in\boldsymbol{R})$, is also invariant for the change of coordinates.

2.2. Initial value problem for the characteristic equation.

PROPOSITION 3. *Let $P(z, D)$ be a differential operator with holomorphic coefficients in a neighborhood Ω of* 0 *in $\boldsymbol{C}^n$ and let $\psi(z)$ be a holomorphic function in Ω such that the equation*

$$P_m(0, \eta)=0,$$

where $\eta_j=\frac{\partial\psi}{\partial z_j}(0)$, $j=1, \cdots, n-1$, has a simple root η_n. In a neighborhood Ω' of 0 *there then exists a unique holomorphic solution ϕ of the initial value problem*

$$P_m(z, \operatorname{grad}\phi)=0,$$

$$\phi(z)=\psi(z) \text{ when } z_n=0 \text{ and } \operatorname{grad}\phi(0)=\eta.$$

See Hörmander [8], Theorem 1.8.2, p. 31, and the following Remark, p. 32. See also Y. Hamada [6], § 2.

2.3. Levi's condition and pluri-subharmonic functions.

Let ϕ be a real-valued C^2 function in a neighborhood of 0 in $\boldsymbol{C}^n$. The complex Hessian form defined by ϕ at 0 is denoted by $H_\phi(\lambda)$, where $\lambda\in\boldsymbol{C}^n$, that is

$$H_\phi(\lambda)=\sum_{j,k}\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(0)\lambda_j\bar{\lambda}_k.$$

Let Ω be a domain in $\boldsymbol{C}^n$ and $0\in\partial\Omega$. We say that Ω is pseudo-convex at 0 if there are a neighborhood U of 0 and a real-valued C^2 function ϕ defined in U such that

(i) $\Omega\cap U=\{z\in U \mid \phi(z)<0\}$,

(ii) if $\sum_{j=1}^n\frac{\partial\phi}{\partial z_j}(0)w_j=0$, then $H_\phi(w)\geqq 0$.

If (ii) holds with $H_\phi(w)>0$ whenever $w\neq 0$, Ω is said to be strictly pseudo-convex at 0.

A real-valued C^2 function ϕ in U is called strictly pluri-subharmonic if the Hessian $\left(\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(z)\right)$ of ϕ, is positive definite for all $z\in U$. Then we have the following

PROPOSITION 4 (Gunning-Rossi [5], p. 263, Proposition 4). *If Ω has a C^2 boundary and is strictly pseudo-convex at* 0, *there exists a strictly pluri-subharmonic function ϕ in a neighborhood U of* 0 *such that*

(i) $\Omega\cap U=\{z\in U \mid \phi(z)<0\}$,

(ii) $\left(\frac{\partial\phi}{\partial z_1}(z), \cdots, \frac{\partial\phi}{\partial z_n}(z)\right) \neq 0$ *in* U.

From this proposition, we may assume that the boundary of the strictly pseudo-convex domain is defined locally by a strictly pluri-subharmonic function.

§ 3. Sufficient condition for holomorphic continuation.

In this section we find a sufficient condition for the holomorphic continuation of local solutions of $P(z, D)u(z)=0$ over the simply characteristic surface. The coefficients of a differential operator $P(z, D)$ are supposed to be holomorphic in some open set.

LEMMA 1. *Let* U *be a neighborhood of* 0 *in* $\boldsymbol{C}^n$ *and* $\phi(z)$, $F(z)$ *be two real-valued* C^1 *functions in* U *such that* $\phi(0)=0$, $\mathrm{grad}_z F(z) \neq 0$ *in* U, *where* $\mathrm{grad}_z F(z) = \left(\frac{\partial F}{\partial z_1}, \cdots, \frac{\partial F}{\partial z_n}\right)$. *We assume the following conditions:*

(i) $P_m(z, \mathrm{grad}_z F(z)) \neq 0$ *in* U,

there exist constants $C_0 < C_1$ *such that*

(ii) $C_0 < F(0) < C_1$,

(iii) $\{z \in U \mid F(z) \leqq C_1\} \cap \{z \in U \mid \phi(z) \geqq 0)$ *is a compact set in* U,

(iv) $\{z \in U \mid F(z) \leqq C_0\} \cap \{z \in U \mid \phi(z) \geqq 0\} = \emptyset$,

(v) $\{z \in U \mid F(z) \leqq C_0\} \neq \emptyset$,

(vi) $\{z \in U \mid F(z) < C\}$ *is simply connected for all* C $(C_0 < C < C_1)$.

Then, every holomorphic function u *in* $\{z \in U \mid \phi(z) < 0\}$ *which satisfies the equation* $P(z, D)u(z)=0$ *can be prolonged to the function holomorphic in* $\{z \in U \mid F(z) < C_1\}$.

Remark that the hypersurface $\{z \mid \phi(z)=0\}$ may be characteristic at 0.

PROOF. Let $u(z)$ be a holomorphic function in $\{z \in U \mid \phi(z) < 0\}$ which satisfies the equation $P(z, D)u(z)=0$. Then we set

$$\alpha = \sup \{C \mid u(z) \text{ is holomorphic in } \{z \in U \mid F(z) < C\}\} .$$

From the conditions (iv) and (v), there exists such an $\alpha \geqq C_0$. It is sufficient to show that $\alpha \geqq C_1$. If we suppose that $\alpha < C_1$, then $u(z)$ is holomorphic in $\{z \in U \mid F(z) < \alpha\}$. Since the level surface $\{z \in U \mid F(z)=\alpha\}$ is non-characteristic, $u(z)$ becomes holomorphic at every boundary point by Proposition 1 in § 2. Since by the condition (iii), $\{z \in U \mid F(z)=\alpha\} \cap \{z \in U \mid \phi(z) \geqq 0\}$ is compact, we can choose a positive number ε such that $u(z)$ is holomorphic in $\{z \in U \mid F(z) < \alpha+\varepsilon\}$. Here we use the monodromy theorem (Fuks [4], p. 93) by the condition (vi). Then this is the contradiction to the definition of α, which proves Lemma 1.

Now we state the main theorem in this section which gives a sufficient condition for the holomorphic continuation of solutions of $P(z, D)u(z)=0$.

THEOREM 1. *Let V be a neighborhood of* 0 *in* $\boldsymbol{C}^n$ *and* $\phi(z)$ *be a real-valued* C^k *function* $(k\geqq 2)$ *in V such that* $\phi(0)=0$ *and* $\mathrm{grad}_z\phi(z)\neq 0$. *We suppose that the level surface* $\{z\in V\mid \phi(z)=0\}$ *is simply characteristic at* 0 *with respect to a differential operator $P(z, D)$ with holomorphic coefficients in V, that is*

$$P_m(0, N)=0 \quad and \quad P_m^{(k)}(0, N)\neq 0 \qquad for\ some\ k\,,$$

where $N=\mathrm{grad}_z\phi(0)$. *Then under the assumptions* (A1) *and* (A2) *below, every holomorphic solution $u(z)$ of $P(z, D)u(z)=0$ in* $\{z\in V\mid \phi(z)<0\}$ *becomes holomorphic near the origin.*

Assumptions: Let $(z(t), \xi(t))$ be the complex bicharacteristic strip of $P(z, D)$ through $(0, N)$. Then we assume that there exists some constant $t_0\neq 0$ such that for a real parameter τ,

$$\text{(A1)}\qquad \frac{d^j}{d\tau^j}\phi(z(\tau t_0))\,|_{\tau=0}=0 \qquad for\quad j<k\,,$$
$$\neq 0 \qquad for\quad j=k\ (k\ odd)\,,$$
$$<0 \qquad for\quad j=k\ (k\ even)\,,$$

$$\text{(A2)}\qquad \frac{d^j}{d\tau^j}[\mathrm{grad}_z\phi(z(\tau t_0))-\xi(\tau t_0)]\,|_{\tau=0}=0 \qquad for\quad j\leqq (k-1)/2\,.$$

Before the proof we remark that the conditions in this theorem are invariant for transformations of coordinates. In fact, that the level surface $\{z\in V\mid \phi(z)=0\}$ is simply characteristic at 0 is invariant (Hörmander [**8**], Definition 1.8.5) and the invariance of (A1) and (A2) follows from Proposition 2 and the following Remark in § 2. (See also Zachmanoglou [**15**], p. 520.)

PROOF. Our proof is an adaptation of Zachmanoglou's proof in [**15**] and also Trèves' proof of Theorem 6.9 in [**12**]. We first consider the following special case: in some neighborhood of 0 the function $\phi(z)$ has the following form

$$\text{(2)}\qquad \phi(z)=\psi(z', x_n)-y_n, \quad z'=(z_1, \cdots, z_{n-1}), \quad z_n=x_n+iy_n\,,$$

with $\psi(0, 0)=0$ and $\mathrm{grad}_z\phi(0)=N=(0, \cdots, 0, i/2)$, and the principal part of the differential operator $P(z, D)$ has the following form

$$\text{(3)}\qquad P_m(z, D)=c\Big(\frac{\partial}{\partial z_{n-1}}\Big)\Big(\frac{\partial}{\partial z_n}\Big)^{m-1}+\cdots\,,$$

where c is a constant and the omitted part consists of terms of order less than $m-1$ with respect to $(\partial/\partial z_n)$. In this case the complex bicharacteristic strip $(z(t), \xi(t))$ through $(0, N)$ is given by the equations

$$z_1(t)=\cdots=z_{n-2}(t)=z_n(t)=0\,,\qquad z_{n-1}(t)=c(i/2)^{m-1}t\,,$$

$$\xi_1(t)=\cdots=\xi_{n-1}(t)=0\,,\qquad \xi_n(t)=i/2\,.$$

Then we may assume that the direction such that the assumptions (A1) and (A2) hold is the $\mathrm{Im}\, z_{n-1}$-axis because the rotation in the z_{n-1}-plane, if needed, is permitted. At this stage we change notations and write s instead of $\mathrm{Im}\, z_{n-1}$ and denote $x=(x_1,\cdots,x_n)$, $y''=(y_1,\cdots,y_{n-2})$, where $z_j=x_j+iy_j$, $j=1,\cdots,n$. Thus the point $(z_1,\cdots,z_n)$ is denoted by (x, y'', s, y_n). Since $\mathrm{grad}_z\phi(0)=(0,\cdots,0,i/2)$, we have

$$\frac{\partial\phi}{\partial x_j}(0)=0,\qquad \frac{\partial\phi}{\partial y_j}(0)=0\,. \tag{4}$$

Now, we may write

$$\phi(x,y'',s)=Q_0(s)+\sum_{j=1}^{n}Q_j(s)x_j+\sum_{j=1}^{n-2}Q_{n+j}(s)y_j$$
$$+Q(x,y'',s)+o(|x|^2+|y''|^2+|s|^k)\,,$$

where $Q_0(s)$ is a polynomial of degree $\leqq k$ in s, $Q_j(s)$ are polynomials of degree $\leqq k-1$ in s, and $Q(x,y'',s)$ is a polynomial of degree $\leqq k$ in (x,y'',s) without terms of degree $\leqq 1$ with respect to (x,y''), and $|x|^2=x_1^2+\cdots+x_n^2$, $|y''|^2=y_1^2+\cdots+y_{n-2}^2$. Then by (4) we have

$$Q_j(0)=0\,,\qquad j=1,\cdots,2n-2\,.$$

Assumption (A1) implies that $Q_0(s)=\mathrm{const.}\, s^k$, where the constant is negative when k is even. By a real contraction on z_{n-1}-axis, we may assume that $Q_0(s)=-s^k$. We remark that after this contraction only the constants may be altered in the formula (3) and the $\mathrm{Im}\, z_{n-1}$-axis (except orientation) is invariant so that the condition (A2) also holds under this new coordinates. If we apply then (A2), we see that

$$|Q_j(s)|\leqq M_1'|s|^{[(k-1)/2]+1}\,,\qquad |s|\ \text{small}\,,$$

here we set $[\alpha]=$ integral part of α. For $|x|$, $|y''|$ and $|s|$ small, we have

$$|Q(x,y'',s)|\leqq M_2(|x|^2+|y''|^2)\,.$$

Let $\varepsilon_1>0$ be arbitrary. We have for $M_1\geqq(n-1)M_1'$,

$$\sum_{j=1}^{n}|Q_j(s)x_j|+\sum_{j=1}^{n-2}|Q_{n+j}(s)y_j|\leqq M_1\varepsilon_1|s|^k+M_1\varepsilon_1^{-1}(|x|^2+|y''|^2)\,.$$

Then we see that for $|x|$, $|y''|$ and $|s|$ sufficiently small,

$$|\phi(x,y'',s)+s^k|\leqq M_1\varepsilon_1|s|^k+M_1\varepsilon_1^{-1}(|x|^2+|y''|^2) \tag{5}$$
$$+M_2(|x|^2+|y''|^2)+\varepsilon_1(|x|^2+|y''|^2+|s|^k)\,.$$

We construct now the function $F(z)$ which satisfies all the conditions in

Lemma 1. We set

$$F(x, y'', s, y_n)=f(x, y'', s)-y_n\,,$$

with

$$f(x, y'', s)=-s^k+\varepsilon\Big(\frac{|x|^2+|y''|^2}{a^2}+\frac{(s-s_0)^2}{s_0^2(1+\eta)}\Big).$$

Here ε, a, s_0, η are positive numbers with the following relations:

$$\varepsilon=\frac{k}{2}(1+\eta)s_0^k\,,\qquad a^2=s_0^{k+1/2}\,,\qquad 0<\eta<1\,. \tag{6}$$

We have then by (3),

$$P_m(z, \mathrm{grad}_z F(z))=c\Big(\frac{1}{2}\Big)^m\Big(\frac{\partial f}{\partial x_n}+i\Big)^{m-1}\Big(\frac{\partial f}{\partial x_{n-1}}-i\frac{\partial f}{\partial s}\Big)+\cdots$$

$$=c\Big(\frac{1}{2}\Big)^m i^{m-1}\Big(\frac{\partial f}{\partial x_{n-1}}-i\frac{\partial f}{\partial s}\Big)+\cdots,$$

where the omitted part is a polynomial of $f_x=\Big(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\Big)$, $f_{y''}=\Big(\frac{\partial f}{\partial y_1},\cdots,\frac{\partial f}{\partial y_{n-2}}\Big)$ and $f_s=\frac{\partial f}{\partial s}$ without any term of degree $\leqq 1$. Here we may suppose that $2^{-m}|c|=1$. Then, we have for $|f_x|$, $|f_{y''}|$ and $|f_s|$ sufficiently small,

$$|P_m(z, \mathrm{grad}_z F(z))|\geqq\Big|\frac{\partial f}{\partial s}\Big|-C(|f_x|^2+|f_{y''}|^2+|f_s|^2)\,, \tag{7}$$

where C is a positive constant depending only on P_m. Now we have

$$\frac{\partial f}{\partial x_j}=2\varepsilon x_j/a^2\,,\qquad \frac{\partial f}{\partial y_j}=2\varepsilon y_j/a^2 \tag{8}$$

and by (6),

$$\frac{\partial f}{\partial s}=-ks_0^{k-1}((s/s_0)^{k-1}-(s/s_0)+1)\,.$$

Then for $\frac{(s-s_0)^2}{s_0^2(1+\eta)}<1$, there are two positive constant $c(k)$ and $C(k)$ depending only on k, such that

$$c(k)s_0^{k-1}\leqq\Big|\frac{\partial f}{\partial s}\Big|\leqq C(k)s_0^{k-1}\,.$$

Thus we have from (7) and (8),

$$|P_m(z, \mathrm{grad}_z F(z))|\geqq c(k)s_0^{k-1}-C(4\varepsilon^2/a^2+C(k)^2 s_0^{2(k-1)})$$

$$\geqq\frac{1}{2}c(k)s_0^{k-1}\,,$$

for $\frac{|x|^2+|y''|^2}{a^2}<1$, $\frac{(s-s_0)^2}{t_0^2(1+\eta)}<1$ and s_0 sufficiently small. Hence $P_m(z, \mathrm{grad}_z F(z))$ does not vanish there. Here we take, as a neighborhood of 0, the set U defined by the inequalities:

$$(9)\qquad \begin{cases} \dfrac{|x|^2+|y''|^2}{a^2}+\dfrac{(s-s_0)^2}{s_0^2(1+\eta)}<1\,, \\ |y_n|<M_0\,, \end{cases}$$

where the constants are chosen so that the conditions in Lemma 1 are satisfied.

Observe that in the set of points (x, y'', s) defined in (9), $\psi(x, y'', s)$ is bounded by a constant $B>0$. We remark that B can be taken arbitrarily small if s_0 is sufficiently small because $\psi(0, 0, 0)=0$. Now we take C_0 and M_0 as

$$(10)\qquad -M_0<C_0<-B-\varepsilon-(3s_0)^k\,.$$

Then, for $z\in U$,

$$\phi(z)\geqq 0 \quad \text{implies} \quad y_n\leqq B\,.$$

On the other hand, since $f(x, y'', s)$ is bounded by $\varepsilon+(3s_0)^k$ in U,

$$F(z)\leqq C_0 \quad \text{implies} \quad y_n>B\,.$$

Thus the condition (iv) in Lemma 1 is satisfied. For the condition (v), it suffices to remark that the point (x, y'', s, y_n) such that $x=0$, $y''=0$, $s=s_0$ and $-C_0\leqq y_n<M_0$ belongs to U. As for (iii), it is sufficient to show that the set $S=\{z\in\bar{U}\,|\,F(z)=C_1\}\cap\partial U$ is contained in the open set $\{z\in U_1\,|\,\phi(z)<0\}$, where U_1 is a suitable open neighborhood of the closure $\bar{U}$ of U. If we take $0<C_1<B$, we have on S

$$\frac{|x|^2+|y''|^2}{a^2}+\frac{(s-s_0)^2}{s_0^2(1+\eta)}=1\,,$$

therefore

$$\begin{aligned}\phi(z)&=\phi(z)-(F(z)-C_1)\\ &=\psi(x, y'', s)+s^k-\varepsilon+C_1\,.\end{aligned}$$

In view of (5),

$$\phi(z)\leqq -\varepsilon+C_1+(M_1+1)\varepsilon_1(3s_0)^k+(M_1\varepsilon_1^{-1}+M_2+\varepsilon_1)a^2\,.$$

If we choose

$$\varepsilon_1=\frac{1}{2}\,\frac{\eta}{1+\eta}\,\frac{1}{(M_1+1)3^k}$$

and s_0 small, we have by (6)

$$(M_1+1)\varepsilon_1(3s_0)^k+(M_1\varepsilon_1^{-1}+M_2+\varepsilon_1)a^2\leqq\frac{1}{2}\,\frac{\eta}{1+\eta}\,\varepsilon\,.$$

This implies that

$$\phi(z)\leqq -\varepsilon+C_1+\frac{1}{2}\,\frac{\eta}{1+\eta}\,\varepsilon\,.$$

Thus if we choose C_1 as

(11) $$C_1 < \left(1-\frac{1}{2}\frac{\eta}{1+\eta}\right)\varepsilon,$$

the condition (iii) is satisfied. Since $F(0)=\varepsilon/(1+\eta)$, (ii) is true if

(12) $$\varepsilon/(1+\eta) < C_1.$$

Lastly if we show that

(13) $$\{z\in\bar{U}\mid F(z)=C\}\cap\{z\in\bar{U}\mid y_n=M_0\}=\emptyset,$$

then (vi) is fulfilled. $F(z)=C$ and $y_n=M_0$ implies that $f(x, y'', s)=M_0+C$, and in $\bar{U}$, $|f|\leqq\varepsilon+(3s_0)^k$ so that if we take

(14) $$M_0+C_0 > \varepsilon+(3s_0)^k,$$

then for every C $(C_0<C<C_1)$ (13) is valid. Consequently if the two constants C_0 and C_1 are taken so as to satisfy (10), (11), (12), (14) and $0<C_1<B$, that is

$$-M_0 < -M_0+\varepsilon+(3s_0)^k < C_0 < -B-\varepsilon-(3s_0)^k < 0,$$

$$0<\frac{\varepsilon}{1+\eta} < C_1 < \left(1-\frac{1}{2}\frac{\eta}{1+\eta}\right)\varepsilon < B,$$

(which are possible if we first fix M_0 and s_0 such that the set U defined by (9) is contained in the set V in Theorem 1, and secondly we change s_0 for a smaller number, if needed, and choose B satisfying $\varepsilon<B$ and $M_0>B+2(\varepsilon+(3s_0)^k)$), then all the conditions in Lemma 1 are satisfied. This completes the proof of Theorem 1 for the special case.

It remains to reduce the general case to the one that we have just studied (Zachmanoglou [**15**], P. 525). We first make a linear change of variables so that $\mathrm{grad}_z\phi(0)=N=(0, \cdots, 0, i/2)$. Let $f(z)$ be a function holomorphic in a neighborhood of 0 and satisfying the conditions

$$P_m(z, \mathrm{grad}_z f(z))=0, \quad f(0)=0, \quad \mathrm{grad}_z f(0)=N.$$

Existence of such a function $f(z)$ follows from Proposition 3 in § 2, since $P_m(0, N)=0$ and $P_m^{(j)}(0, N)\neq 0$ for some j. Then we define the holomorphic transformation of coordinates from z-variables to w-variables as follows:

$$w_j=z_j, \qquad j=1, \cdots, n-1,$$
$$w_n=-2if(z).$$

Since the functional matrix of this transformation is an identity matrix at 0, this is a nonsingular change of variables in a neighborhood of 0. We suppose that $P_m(z, D_z)$ is mapped to $P'_m(w, D_w)$ under this transformation. Since the level surfaces $\{f(z)=\text{constant}\}$ are simply characteristic with respect to $P_m(z, D_z)$, the hyperplanes $\{w_n=\text{constant}\}$ are simply characteristic with respect to $P'_m(w, D_w)$. Moreover we may assume, renaming the variables if necessary, that $P'^{(n-1)}_m(0, N) \neq 0$. At this step, $P'_m(w, D_w)$ can be written as follows:

$$P'_m(w, D_w) = \left(a_1(w)\frac{\partial}{\partial w_1} + \cdots + a_{n-1}(w)\frac{\partial}{\partial w_{n-1}}\right)\left(\frac{\partial}{\partial w_n}\right)^{m-1} + \cdots,$$

where $a_{n-1}(0) \neq 0$ and the omitted part consists of terms of order less than $m-1$ with respect to $(\partial/\partial w_n)$.

We next find the bicharacteristic curve $\{w(t)\}$ with parameter $t = v_{n-1}$ passing through $((v_1, \cdots, v_{n-2}, 0, v_n), N)$ at $t=0$, so that $w_j(v)$ may be written as the following forms:

$$w_j = v_j + g_j(v), \qquad j = 1, \cdots, n-2,$$
$$w_{n-1} = g_{n-1}(v),$$
$$w_n = v_n,$$

where g_j are holomorphic near 0 and $g_j(v'', 0, v_n) = 0$ with $v'' = (v_1, \cdots, v_{n-2})$. Moreover it follows from Hamilton's equation that the Jacobian of w with respect to v at the origin is equal to $P'^{(n-1)}_m(0, N) \neq 0$. Hence there exists a nonsingular holomorphic transformation from w-coordinates to v-coordinates, which maps $P'_m(w, D_w)$ to $P''_m(v, D_v)$. Since the hyperplanes $\{w_n = \text{constant}\}$ are transformed to the hyperplanes $\{v_n = \text{constant}\}$, these hyperplanes are also simply characteristic with respect to $P''_m(v, D_v)$. Moreover Hamilton's equations are invariant, so that we have

$$P''^{(j)}_m(v, N) = \begin{cases} 0 & \text{for } j \neq n-1, \\ 1 & \text{for } j = n-1, \end{cases}$$

when v is in some neighborhood of 0. Therefore we write $P''_m(v, D_v)$ as the following form

$$P''_m(v, D_v) = (-2i)^{m-1}\left(\frac{\partial}{\partial v_{n-1}}\right)\left(\frac{\partial}{\partial v_n}\right)^{m-1} + \cdots,$$

which shows that every differential operator $P(z, D)$ is reduced to the form (3) under the holomorphic change of coordinates.

Lastly we remark that the boundary function $\phi(z)$ may be supposed to have the form (2) (Trèves [**12**], p. 369). In fact, if $\mathrm{grad}_z\phi(0) = N = (0, \cdots, 0, i/2)$, the equation

$$\phi(z', z_n) = 0$$

can be solved with respect to $\mathrm{Im}\, z_n = y_n$. In other words, there exists a C^k function $\psi(z'', x_n)$ in a neighborhood of 0 such that the sets

$$\{z \mid \phi(z) < 0\}, \qquad \{z \mid \psi(z'', x_n) - y_n < 0\}$$

are identical. Then

$$\psi(z'', x_n)-y_n=g(z)\phi(z),$$

where g is a C^k function in $\{z \mid \phi(z)\neq 0\}$, which is positive and C^{k-1} near the origin. Furthermore if D^k is any differentiation of order k, the function

$$(D^k g(z))\phi(z)$$

defined when $\phi(z)\neq 0$, can be extended in a neighborhood of 0 as a continuous function, vanishing for $\phi(z)=0$. Thus if ϕ satisfies (A1) and (A2), so does also $\psi-y_n$.

This completes the proof of Theorem 1.

When $k=2$, condition (A1) with $j=0, 1$ and condition (A2) are always fulfilled. In fact condition (A1) with $j=0$ and condition (A2) are trivial and for (A1) with $j=1$ we have

$$\begin{aligned}\frac{d}{d\tau}\phi(z(\tau t_0)) &= \sum_{j=1}^{n}\Big(\frac{\partial\phi}{\partial z_j}\frac{dz_j}{dt}\frac{dt}{d\tau}+\frac{\partial\phi}{\partial \bar z_j}\frac{d\bar z_j}{d\bar t}\frac{d\bar t}{d\tau}\Big)\\ &= \sum_{j=1}^{n}\Big(\frac{\partial\phi}{\partial z_j}P_m^{(j)}(z, \xi)t_0+\frac{\partial\phi}{\partial \bar z_j}\overline{P_m^{(j)}(z, \xi)}\bar t_0\Big),\end{aligned}$$

thus, if we set $N=(N_1, \cdots, N_n)$, we have for $\tau=0$

$$\begin{aligned}\frac{d}{d\tau}\phi(z(\tau t_0))\Big|_{\tau=0} &= \sum_{j=1}^{n}(N_j t_0 P_m^{(j)}(0, N)+\bar N_j \bar t_0\overline{P_m^{(j)}(0, N)})\\ &= m t_0 P_m(0, N)+m\bar t_0\overline{P_m(0, N)}\\ &=0.\end{aligned}$$

Therefore we have the next corollary.

COROLLARY 1. *Let $P(z, D)$ be a differential operator with holomorphic coefficients in an open set U in $\boldsymbol{C}^n$ and let $\phi(z)$ be a real-valued C^2 function in U whose gradient never vanish. Let $z_0\in U$ be a simply characteristic point of the hypersurface $\{z\in U \mid \phi(z)=\phi(z_0)\}$. We make the following assumption:*

(C) *Let $z(t)$ be the complex bicharacteristic curve through $(z_0, \operatorname{grad}\phi(z_0))$. Then there is a constant $t_0\neq 0$ such that for a real parameter τ,*

$$\frac{d^2}{d\tau^2}\phi(z(\tau t_0))\Big|_{\tau=0}<0.$$

Then there is an open set $U'\ni z_0$ such that every holomorphic function $u(z)$ in $\{z\in U \mid \phi(z)<\phi(z_0)\}$ which satisfies $P(z, D)u(z)=0$, becomes holomorphic in U'.

Condition (C) in the above corollary is, more explicitly stating, as follows:

(15)
$$\sum_{j,k}\Big\{\frac{\partial^2\phi}{\partial z_j\partial z_k}(z_0)t_0P_m^{(j)}(z_0, N)t_0P_m^{(k)}(z_0, N)$$
$$+2\frac{\partial^2\phi}{\partial z_j\partial \bar{z}_k}(z_0)t_0P_m^{(j)}(z_0, N)\bar{t}_0\overline{P_m^{(k)}(z_0, N)}$$
$$+\frac{\partial^2\phi}{\partial \bar{z}_j\partial \bar{z}_k}(z_0)\bar{t}_0\overline{P_m^{(j)}(z_0, N)}\bar{t}_0\overline{P_m^{(k)}(z_0, N)}\Big\}$$
$$+\sum_j \{P_m^{(j)}(z_0, N)P_{m,j}(z_0N)t_0^2+\overline{P_m^{(j)}(z_0, N)}\overline{P_{m,j}(z_0, N)}\bar{t}_0^2\}$$
$$<0,$$

where $N=\mathrm{grad}_z\phi(z_0)$. This follows from the bicharacteristic equations and Euler's identity for homogeneous polynomials.

We say that a domain Ω in C^n is a *domain of holomorphy* with respect to $P(z, D)$, whose coefficients are holomorphic in a neighborhood of $\bar{\Omega}$, if for every point in $\partial\Omega$ there exists a solution $u(z)$ of $P(z, D)u(z)=0$, which is holomorphic in Ω but cannot be holomorphically continued over that point. For example Bengel [**1**] showed that a convex domain is a domain of holomorphy with respect to $P(D)$ whose coefficients are constants, if there is a characteristic supporting hyperplane of the convex domain at every boundary point. Now we suppose that Ω has a C^2 boundary. Zerner [**16**] proved that if Ω is a domain of holomorphy with respect to $P(z, D)$, then every boundary point is characteristic. We give here more precise result.

COROLLARY 2. *Let $\Omega=\{z \mid \phi(z)<0\}$ be a domain of holomorphy with respect to $P(z, D)$, where $\phi(z)$ is a real-valued C^2 function. Then at every boundary point the tangent plane of the surface $\partial\Omega$ is characteristic with respect to $P(z, D)$, and if it is simply characteristic then the left part of the inequality* (15) *is non-negative.*

§ 4. Holomorphic solutions with singularities.

Let $P(z, D)=\sum_{|\alpha|\leq m} a_\alpha(z)\Big(\frac{\partial}{\partial z}\Big)^\alpha$ be a differential operator with holomorphic coefficients in a neighborhood U of 0 in C^n and $P_m(z, D)=\sum_{|\alpha|=m} a_\alpha(z)\Big(\frac{\partial}{\partial z}\Big)^\alpha$ be its principal part. Let $\phi(z)$ be a function holomorphic in U and satisfying $P_m(z, \mathrm{grad}\,\phi)=0$, $\phi(0)=0$ and $\mathrm{grad}\,\phi(z)\neq 0$. We assume that

(16) $$(P_m^{(1)}(z, \mathrm{grad}\,\phi(z)), \cdots, P_m^{(n)}(z, \mathrm{grad}\,\phi(z)))\neq 0$$

in U. Then we construct the solution $u(z)$ of $P(z, D)u(z)=0$ which has singularities on the analytic set $\{z\in U \mid \phi(z)=0\}$. The method of the construction is based on the decomposition of a solution in terms of the function ϕ, and this was effectively used in Y. Hamada [**6**] to study the Cauchy problem with singular initial data.

Under the above conditions on $P(z, D)$ and $\phi(z)$, we have

THEOREM 2. *There exists a solution $u(z)$ of $P(z, D)u=0$ in a neighborhood of* 0 *with the following form:*

$$u(z)=\frac{F(z)}{\phi(z)}+G(z)\log\phi(z)+H(z),$$

where $F(z)$, $G(z)$ and $H(z)$ are holomorphic at 0 *and $u(z)$ is not holomorphic at* 0.

The proof of this theorem consists of two parts. In the first step, 4.1, we construct the formal solution. In the next step, 4.2, we discuss its convergence.

We remark that this theorem is also proved in T. Kawai [**9**]. But our proof of the convergence of the formal solution is self-contained and may be more elementary than that of T. Kawai, so we reproduce it here.

Now, in view of (16), we make a change of variables, if needed, so that $P_m^{(1)}(z, \operatorname{grad}\phi(z))$ does not vanish in a neighborhood of 0. Therefore we may suppose that

$$P_m^{(1)}(z, \operatorname{grad}\phi(z))=1 \tag{17}$$

in a neighborhood of 0. Under this situation, we construct the solution.

4.1. Construction of formal solution.

Let $f_j(s)$ $(j=0, \pm 1, \pm 2, \cdots)$ be functions defined by

$$\begin{cases} f_{-l}(s)=(-1)^l l!\, s^{-l-1}, & l=0, 1, 2, \cdots, \\ f_1(s)=\log s, \\ f_k(s)=\dfrac{s^{k-1}}{(k-1)!}\log s-\dfrac{A_k}{(k-1)!}s^{k-1}, & k=2, 3, \cdots, \end{cases} \tag{18}$$

where $A_k=1+\frac{1}{2}+\cdots+\frac{1}{k-1}$. Thus we have

$$\frac{d}{ds}f_j(s)=f_{j-1}(s). \tag{19}$$

We then assume that the solution $u(z)$ has the form

$$u(z)=\sum_{k=0}^{\infty}f_k(\phi(z))u_k(z), \tag{20}$$

where $u_k(z)$ are functions to be determined. Now we have

$$P(z, D)[f(\phi)u]=f^{(m)}(\phi)P_m(z, \text{grad}\ \phi)u$$
$$+f^{(m-1)}(\phi)\Big\{\sum_{j=1}^{n} P_m^{(j)}(z, \text{grad}\ \phi)\frac{\partial u}{\partial z_j}+c(z)u(z)\Big\}$$
$$+f^{(m-2)}(\phi)L_2[u]+\cdots+f(\phi)L_m[u]\,,$$

where $c(z)$ is holomorphic and L_p ($p=2, \cdots, m$) are linear differential operators of order p with holomorphic coefficients. We remark that these depend only on $P(z, D)$. Therefore using (19) we have formally

$$P(z, D)u(z)=\sum_{k=0}^{\infty}[f_{k-m}(\phi)P_m(z, \text{grad}\ \phi)u_k(z)$$
$$+f_{k-m+1}(\phi)\Big\{\sum_{j=1}^{n} P_m^{(j)}(z, \text{grad}\ \phi)\frac{\partial u_k}{\partial z_j}+c(z)u_k(z)\Big\}$$
$$+f_{k-m+2}(\phi)L_2[u_k]+\cdots+f_k(\phi)L_m[u_k]]$$
$$=0\,.$$

Setting each of the coefficients of $f_{k-m+1}(\phi)$ equal to zero, we have

$$\sum_{j=1}^{n} P_m^{(j)}(z, \text{grad}\ \phi)\frac{\partial u_k}{\partial z_j}+c(z)u_k(z)+L_2[u_{k-1}]+\cdots+L_m[u_{k-m+1}]=0\,,$$

where we set $u_k(z)=0$ if $k<0$.

If we define the operator L as

$$L[v]=\sum_{j=1}^{n} P_m^{(j)}(z, \text{grad}\ \phi)\frac{\partial v}{\partial z_j}+c(z)v(z)\,,$$

we then have the next recursion formulas,

$$\text{(21)}\qquad \begin{cases} L[u_0]=0 \\ L[u_k]=-\sum_{j=2}^{m} L_j[u_{k+1-j}]\,, \qquad k=1, 2, \cdots . \end{cases}$$

Here we change the notations for the convenience and write t instead of z_1 and again $(z_1, \cdots, z_n)$ instead of $(z_2, \cdots, z_n)$. Then, by (17), we can rewrite the equation (21) as follows:

$$\text{(22)}\qquad \begin{cases} L[u_0]=\dfrac{\partial u_0}{\partial t}+\sum_{j=1}^{n} a_j(t, z)\dfrac{\partial u_0}{\partial z_j}+c(t, z)u_0(t, z)=0 \\ L[u_k]=-\sum_{j=2}^{m} L_j[u_{k+1-j}]\,, \qquad k=1, 2, \cdots . \end{cases}$$

We now impose the initial conditions on $u_k(t, z)$ at $t=0$ as

$$\text{(23)}\qquad u_0(0, z)=1\,, \qquad u_k(0, z)=0\,, \qquad k=1, 2, \cdots .$$

Since the hyperplane $\{t=0\}$ is non-characteristic with respect to the operator

L, we can find, by the Cauchy-Kovalevsky theorem, recursively the solutions $u_k(t, z)$ of the initial value problem (22) and (23). Thus we obtain the formal solution.

4.2. Convergence of formal solution.

We prove the convergence of the formal solution given above by the method of majorant functions. The technique used here is due to C. Wagschal [**13**].

For two holomorphic functions at 0, $u(t, z)$, $U(t, z)$, we say that $U(t, z)$ is a majorant of $u(t, z)$ if, for every multi-index α, $|D^\alpha_{t,z}u(0)| \leqq D^\alpha_{t,z}U(0)$, and we denote this by $u \ll U$. Then we have readily that $u \ll U$ implies $D^\alpha u \ll D^\alpha U$ and $u \ll U$, $v \ll V$ implies $uv \ll UV$ and $u+v \ll U+V$.

We set $\varDelta = \{(t, z) \mid |t| \leqq R, |z_j| \leqq R\}$. Then we have the next lemma.

LEMMA 2. *For every function u holomorphic in a neighborhood of $\varDelta$, there exists a constant M such that*

$$u(t, z) \ll M/(R-(t+\sum_{j=1}^{n} z_j)) .$$

The proof is easy from Cauchy's inequalities.

Now we introduce the functions $U_k(t, z)$ as follows:

$$U_k(t, z) = \frac{d^k}{d\xi^k}\frac{1}{r-\xi} = \frac{k!}{(r-\xi)^{k+1}}, \tag{24}$$

where $0 < r < R$, $\xi = \alpha t + \sum_{j=1}^{n} z_j$ and $\alpha \geqq 1$.

Then, we have

LEMMA 3 (C. Wagschal [**13**]).

$$U_k \ll rU_{k+1}, \tag{25}$$

$$\frac{1}{R-\xi}U_k \ll \frac{1}{R-r}U_k, \tag{26}$$

$$u \ll U_k \quad \textit{implies} \quad D^\beta u \ll \alpha^{\beta_0} U_{k+|\beta|}, \tag{27}$$

where $\beta = (\beta_0, \beta_1, \cdots, \beta_n)$ is any multi-index.

The proof is omitted.

We now construct the majorants of the solutions $u_k(t, z)$ of the initial value problem (22) and (23). At first, we choose $R > 0$ and $M > 0$ such that all the coefficients of L, L_j $(j = 2, \cdots, m)$ and $u_0(t, z)$ are holomorphic in a neighborhood of $\varDelta$ and, using Lemma 2, $M/(R-(t+\sum_{j=1}^{n} z_j))$ is a majorant of these functions. Then we show the next proposition.

PROPOSITION 5. *There exists a constant $c > 0$ such that*

(28) $$u_k(t, z) \ll c^{k+1} U_k(t, z), \qquad k=0, 1, 2, \cdots.$$

PROOF. We prove this by induction on k. When $k=0$, (28) is true because $u_0(t, z) \ll M/(R-(t+\sum_{j=1}^{n} z_j)) \ll cU_0(t, z)$ if $c \geqq Mr/R$. We then suppose that (28) is valid for $k=0, 1, \cdots, l-1$. We remark that the Taylor coefficients of $u_k(t, z)$ are uniquely determined by the equations (22) and (23), so that to prove (28) with $k=l$, it is sufficient to show that

(29) $$\frac{\partial}{\partial t} c^{l+1} U_l(t, z) \gg \frac{M}{R-\xi} \Big\{ \sum_{j=1}^{n} \frac{\partial}{\partial z_j} c^{l+1} U_l(t, z) + c^{l+1} U_l(t, z) \Big\} + \sum_{j=2}^{m} \tilde{L}_j [c^{l+2-j} U_{l+1-j}(t, z)],$$

where $\tilde{L}_j$ is an operator obtained by exchanging the coefficients of L_j by those majorants. We have, using (25) and (26),

$$\frac{M}{R-\xi} \frac{\partial}{\partial z_j} U_l(t, z) = \frac{M}{R-\xi} U_{l+1}(t, z) \ll \frac{M}{R-r} U_{l+1}(t, z),$$

$$\frac{M}{R-\xi} U_l(t, z) \ll \frac{M}{R-r} r U_{l+1}(t, z).$$

Thus we have

(30) $$c^{l+1} \frac{M}{R-\xi} \Big\{ \sum_{j=1}^{n} \frac{\partial}{\partial z_j} U_l(t, z) + U_l(t, z) \Big\} \ll c^{l+1} \Big(\frac{Mn}{R-r} + \frac{Mr}{R-r} \Big) U_{l+1}(t, z).$$

Further for a multi-index $\beta = (\beta_0\ \beta_1, \cdots, \beta_n)$,

$$\frac{M}{R-\xi} D^{\beta} U_k(t, z) = \frac{M}{R-\xi} \alpha^{\beta_0} U_{k+|\beta|}(t, z) \ll \frac{M}{R-r} \alpha^{\beta_0} r^{p-|\beta|} U_{k+p}(t, z),$$

and since the order of $\tilde{L}_j$ is j, we have

$$\tilde{L}_j [U_{l+1-j}] \ll \frac{M}{R-r} \alpha^j \Big(\sum_{|\beta| \leqq j} r^{j-|\beta|} \Big) U_{l+1}(t, z),$$

so that

(31) $$\sum_{j=2}^{m} \tilde{L}_j [c^{l+2-j} U_{l+1-j}] \ll \frac{M}{R-r} \Big(\sum_{j=2}^{m} \sum_{|\beta| \leqq j} \alpha^j c^{l+2-j} r^{j-|\beta|} \Big) U_{l+1}(t, z) \ll \frac{M}{R-r} c^l \Big(\sum_{j=2}^{m} \sum_{|\beta| \leqq j} \alpha^j r^{j-|\beta|} \Big) U_{l+1}(t, z).$$

Here we assumed that $c \geqq 1$. Thus from (30) and (31),

$$\left\{c^{l+1}\frac{M}{R-r}(n+r)+\frac{M}{R-r}c^{l}c(\alpha, r, m)\right\}U_{l+1}(t, z)$$

is a majorant of the right hand side of (29), where $c(\alpha, r, m)=\sum_{j=2}^{m}\sum_{|\beta|\leq j}\alpha^{j}r^{j-|\beta|}$. On the other hand,

$$\frac{\partial}{\partial t}c^{l+1}U_{l}(t, z)=c^{l+1}\alpha U_{l+1}(t, z).$$

Therefore if we choose constants c and α as

$$\alpha\geqq\max(1, 2M(n+r)/(R-r)),$$

and

$$c\geqq\max\left(1, \frac{Mr}{R}, \frac{2}{\alpha}\frac{M}{R-r}c(\alpha, r, m)\right),$$

then we have (29). This completes the proof.

We remark that from this proposition every function $u_{k}(t, z)$ is holomorphic in $\{(t, z)\mid\alpha|t|+\sum_{j=1}^{n}|z_{j}|<r\}$. Now we study the convergence of the formal solution (20). At first we have, by Proposition 5 and (24),

$$|u_{k}(t, z)|\leqq c^{k+1}\frac{k!}{(r-\hat{\xi})^{k+1}},$$

where $\hat{\xi}=\alpha|t|+\sum_{j=1}^{n}|z_{j}|$. Then if $|\phi(t, z)c(r-\hat{\xi})^{-1}|<1$,

$$G(t, z)=\sum_{k=0}^{\infty}(k!)^{-1}\phi^{k}u_{k+1},$$

and

$$H(t, z)=-\sum_{k=1}^{\infty}(k!)^{-1}A_{k+1}\phi^{k}u_{k+1}$$

are holomorphic. Thus we have with $F(t, z)=u_{0}(t, z)$,

$$u(t, z)=F(t, z)/\phi(t, z)+G(t, z)\log\phi(t, z)+H(t, z),$$

where F, G and H are holomorphic at 0. We remark that by (23) $F(0, z)=u_{0}(0, z)=1$ and $\phi(0, 0)=0$, so that $u(t, z)$ is not holomorphic at 0. This completes the proof of Theorem 2.

COROLLARY 3. *Let $\phi(z)$ be a holomorphic function in a neighborhood of* 0 *in $\boldsymbol{C}^{n}$ satisfying $P_{m}(z, \operatorname{grad}\phi)=0$, $\phi(0)=0$, $\operatorname{grad}\phi(z)\neq 0$ and $(P_{m}^{(1)}(z, \operatorname{grad}\phi), \cdots, P_{m}^{(n)}(z, \operatorname{grad}\phi))\neq 0$. Then there exists a solution u of $P(z, D)u(z)=0$ which is holomorphic in $\{z\mid\operatorname{Re}\phi(z)<0\}$ and cannot be prolonged over the origin.*

§ 5. Necessary condition for holomorphic continuation.

Let $P(z, D)$ be a differential operator with holomorphic coefficients in a neighborhood of 0 in $\boldsymbol{C}^{n}$ and $\phi(z)$ be a real-valued C^{2} function such that

$\mathrm{grad}_z\phi(z)\neq 0$ and $\phi(0)=0$. Then we seek the conditions on $P(z, D)$ and $\phi(z)$, for which the solution $u(z)$ of $P(z, D)u(z)=0$ which is holomorphic in $\{z\mid\phi(z)<0\}$ can be holomorphically prolonged over 0 or not. For example if $\phi(z)$ does not satisfy the Levi condition at 0, then every holomorphic function in $\{z\mid\phi(z)<0\}$ can be prolonged over the origin (see e.g. M. Hervé [7], p. 44). When the surface $\{z\mid\phi(z)=0\}$ is simply characteristic, we have proved in Corollary 1 in § 3 that if the second derivative of $\phi(z)$ along some direction in the complex bicharacteristic curve is negative at 0, then every solution can be prolonged. Now we study the converse of this corollary.

Let Ω be a domain in C^n with C^2 boundary and $0\in\partial\Omega$. We assume that Ω is strictly pseudo-convex at 0. Then by Proposition 4 in § 2, we find a strictly pluri-subharmonic C^2 function $\phi(z)$ in a neighborhood U of 0 such that

(i) $\Omega\cap U=\{z\in U\mid\phi(z)<0\}$,

(ii) $\mathrm{grad}_z\phi(z)\neq 0$ in U.

We suppose that the surface $\{z\in U\mid\phi(z)=0\}$ is simply characteristic at 0 with respect to a differential operator $P(z, D)$. Under this situation, we have the following theorem.

THEOREM 3. *If assumptions* (B1) *and* (B2) *below are fulfilled, then we can find a solution $u(z)$ of $P(z, D)u(z)=0$ which is holomorphic in $\{z\in V\mid\phi(z)<0\}$ and cannot be holomorphic near the origin, where V is some neighborhood of* 0.

Assumptions: Let $(z(t), \xi(t))$ be the complex bicharacteristic strip of $P(z, D)$ through $(0, \mathrm{grad}_z\,\phi(0))$. *Then we assume that for every complex number $t_0\neq 0$ and a real parameter τ,*

(B1) $$\frac{d^2}{d\tau^2}\phi(z(\tau t_0))\Big|_{\tau=0}>0,$$

(B2) $$\sum_{j,k}\left\{\lambda_k\frac{\partial^2\phi}{\partial z_j\partial z_k}(0)+\bar{\lambda}_k\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(0)\right\}P_m^{(j)}(0, N)+\sum_k\lambda_k P_{m,k}(0, N)=0$$

for all $\lambda=(\lambda_1, \cdots, \lambda_n)$ in C^n, satisfying the equation

$$\sum_j\lambda_j P_m^{(j)}(0, N)=0$$

where $N=\mathrm{grad}_z\phi(0)$.

Since the left hand side of the equality (B2) is the first directional derivative of $P_m(z, \mathrm{grad}\,\phi(z))$ at $z=0$ and $P_m(z, \mathrm{grad}\,\phi(z))$ is invariantly defined, we can change variables if the Jacobian matrix at 0 is a complex orthogonal matrix.

PROOF. We first consider the next special case: the principal part of $P(z, D)$ has the following form

$$(32)\qquad P_m(z, D)=\Big(a_1(z)\frac{\partial}{\partial z_1}+\cdots+a_{n-2}(z)\frac{\partial}{\partial z_{n-2}}+a_n(z)\frac{\partial}{\partial z_n}\Big)\Big(\frac{\partial}{\partial z_{n-1}}\Big)^{m-1}+\cdots,$$

where the omitted part consists of terms of order less than $m-1$ with respect to $(\partial/\partial z_{n-1})$ and $a_j(z)$ are holomorphic in U and satisfy

$$(33)\qquad \begin{cases} a_j(0, \cdots, 0, z_n)=0, & j=1, \cdots, n-2, \\ a_n(0, \cdots, 0, z_n)=1. \end{cases}$$

Further we assume that $\mathrm{grad}_z\phi(0)=N=(0, \cdots, 0, 1, 0)$. In this case the complex bicharacteristic strip $(z(t), \xi(t))$ through $(0, N)$ at $t=0$ is given by the equations

$$z_1(t)=\cdots=z_{n-1}(t)=0, \qquad z_n(t)=t,$$

$$\xi_1(t)=\cdots=\xi_{n-2}(t)=\xi_n(t)=0, \qquad \xi_{n-1}(t)=1.$$

Moreover we have from (32) and (33),

$$(34)\qquad \begin{cases} P_m^{(j)}(0, N)=0, \quad j=1, \cdots, n-1, \quad P_m^{(n)}(0, N)=1 \\ P_{m,j}(0, N)=0, \quad j=1, \cdots, n \end{cases}$$

therefore assumption (B1) means from (15) that there is a constant $\alpha>0$ such that

$$(35)\qquad \frac{\partial^2\phi}{\partial z_n^2}(0)z_n^2+2\frac{\partial^2\phi}{\partial z_n\partial\bar{z}_n}(0)z_n\bar{z}_n+\frac{\partial^2\phi}{\partial\bar{z}_n^2}(0)\bar{z}_n^2\geqq\alpha|z_n|^2.$$

For the condition (B2), we have by (34)

$$(36)\qquad \frac{\partial^2\phi}{\partial z_j\partial z_n}(0)=0, \qquad \frac{\partial^2\phi}{\partial\bar{z}_j\partial z_n}(0)=0, \qquad j=1, \cdots, n-1.$$

Since $\phi(z)$ is strictly pluri-subharmonic in $(z_1, \cdots, z_n)$ variables, it is also strictly pluri-subharmonic in $(z_1, \cdots, z_{n-1})$ variables in the $\{z_n=0\}$ plane. Thus we have

$$(37)\qquad \sum_{j,k=1}^{n-1}\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(0)z_j\bar{z}_k\geqq\gamma|z'|^2,$$

for some constant $\gamma>0$, where we denote $(z_1, \cdots, z_{n-1})$ by z'. Now we have

$$(38)\qquad \phi(z', 0)=z_{n-1}+\bar{z}_{n-1}+\frac{1}{2}\sum_{j,k=1}^{n-1}\Big\{\frac{\partial^2\phi}{\partial z_j\partial z_k}(0)z_jz_k+2\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(0)z_j\bar{z}_k+\frac{\partial^2\phi}{\partial\bar{z}_j\partial\bar{z}_k}(0)\bar{z}_j\bar{z}_k\Big\}+o(|z'|^2).$$

Then we set

$$(39)\qquad f(z')=z_{n-1}+\frac{1}{2}\sum_{j,k=1}^{n-1}\frac{\partial^2\phi}{\partial z_j\partial z_k}(0)z_jz_k.$$

We remark that from (37), (38) and (39), $f(z')=0$ implies that $\phi(z', 0)>0$ in a neighborhood of 0 in the $\{z_n=0\}$ plane except $z'=0$. We then apply the initial value problem, Proposition 3 in §2 with $\phi(z)=f(z')$. Therefore we have a holomorphic function $F(z)$ such that

$$P_m(z, \operatorname{grad} F(z))=0, \tag{40}$$

$$F(z', 0)=f(z'), \tag{41}$$

$$\operatorname{grad}_z F(0)=N=(0, \cdots, 0, 1, 0). \tag{42}$$

Then we show the next lemma in order to estimate the above function $F(z)$.

LEMMA 4. *For a holomorphic function $F(z)$ which satisfies* (40) *and* (42), *we have*

$$\frac{\partial^2 F}{\partial z_j \partial z_n}(0)=0, \qquad j=1, \cdots, n. \tag{43}$$

PROOF. From (40) we have

$$\begin{aligned}\frac{\partial}{\partial z_j} & P_m(z, \operatorname{grad} F(z)) \\ &= \sum_{k=1}^{n} P_m^{(k)}(z, \operatorname{grad} F(z)) \frac{\partial^2 F}{\partial z_k \partial z_j}(z)+P_{m,j}(z, \operatorname{grad} F(z)) \\ &=0.\end{aligned} \tag{44}$$

If we set $z=0$ in (44), we obtain (43) by using (34), which proves this lemma.

We now continue the proof of Theorem 3. From (41), (42) and (43), $F(z)$ may be written as

$$F(z)=f(z')+z_n g(z), \tag{45}$$

where

$$g(z)=O(|z|^2).$$

Thus if $\operatorname{Re} F(z) \geqq 0$, then for some constant C and $|z|$ small, we have

$$\operatorname{Re} f(z') \geqq -C|z_n||z|^2. \tag{46}$$

On the other hand, in a neighborhood of 0, we have

$$\begin{aligned}\phi(z) &= z_{n-1}+\bar{z}_{n-1}+\frac{1}{2} \sum_{j,k=1}^{n} \Big\{ \frac{\partial^2 \phi}{\partial z_j \partial z_k}(0) z_j z_k \\ &\qquad +2 \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k+\frac{\partial^2 \phi}{\partial \bar{z}_j \partial \bar{z}_k}(0) \bar{z}_j \bar{z}_k \Big\}+o(|z|^2) \\ &= f(z')+\overline{f(z')}+\sum_{j,k=1}^{n-1} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k \\ &\quad +\sum_{j=1}^{n-1} \Big\{ \frac{\partial^2 \phi}{\partial z_j \partial z_n}(0) z_j z_n+\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_n}(0) z_j \bar{z}_n+\frac{\partial^2 \phi}{\partial \bar{z}_j \partial z_n}(0) \bar{z}_j z_n\end{aligned}$$

$$+\frac{\partial^2\phi}{\partial\bar{z}_j\partial\bar{z}_n}(0)\bar{z}_j\bar{z}_n\Big\}+\frac{1}{2}\Big\{\frac{\partial^2\phi}{\partial z_n^2}(0)z_n^2$$

$$+2\frac{\partial^2\phi}{\partial z_n\partial\bar{z}_n}(0)z_n\bar{z}_n+\frac{\partial^2\phi}{\partial\bar{z}_n^2}(0)\bar{z}_n^2\Big\}+o(|z|^2).$$

Then by (35), (36) and (37), we have

(47) $$\phi(z)\geqq f(z')+\overline{f(z')}+\gamma|z'|^2+\frac{1}{2}\alpha|z_n|^2+o(|z|^2).$$

Thus if $\operatorname{Re}F(z)\geqq 0$, we have by (46) and (47),

$$\phi(z)\geqq -2C|z_n||z|^2+\varepsilon(|z'|^2+|z_n|^2),$$

for some constant $\varepsilon>0$ and $|z_n|$, $|z'|$ sufficiently small. If $|z_n|<\varepsilon/4C$, we have then

$$\phi(z)\geqq(\varepsilon/2)|z|^2\geqq 0.$$

Therefore there exists a neighborhood V of 0 such that

(48) $$\{z\in V\mid \operatorname{Re}F(z)\geqq 0\}\subset\{z\in V\mid\phi(z)\geqq 0\}.$$

Now we construct the solution $u(z)$ of $P(z, D)u(z)=0$ which is holomorphic in $\{z\in V\mid\phi(z)<0\}$ and cannot be holomorphic near the origin. By Theorem 2 in § 4 we obtain the function $u(z)$ of the form

$$u_1(z)/F(z)+u_2(z)\log F(z)+u_3(z),$$

where $u_1(z)$, $u_2(z)$ and $u_3(z)$ are holomorphic in V (V is sufficiently small) and $u(z)$ is not holomorphic at the origin. Then by (48), $\phi(z)<0$ implies that $\operatorname{Re}F(z)<0$. Therefore we can choose some branch of $\log F(z)$ in $\{z\in V\mid\phi(z)<0\}$. Thus $u(z)$ given by Theorem 2 is holomorphic in $\{z\in V\mid\phi(z)<0\}$ but cannot be holomorphic at 0. This completes the proof of Theorem 3 for the special case.

It remains to reduce the general case to the one we have just studied. We first make a holomorphic linear orthogonal change of variables so that $\operatorname{grad}_z\phi(0)=N=(0,\cdots,0,1,0)$ and $P_m^{(j)}(0,N)=0$, $j=1,\cdots,n-1$, $P_m^{(n)}(0,N)=1$ because N is orthogonal to $(P_m^{(1)}(0,N),\cdots,P_m^{(n)}(0,N))$ by Euler's identity. Then we find a holomorphic function $f(z)$ which satisfies

$$P_m(z,\operatorname{grad}_z f(z))=0,$$

$$f(z',0)=z_{n-1},\ \operatorname{grad}_z f(0)=N.$$

Existence of such a function follows from Proposition 3 in § 2. Now we define the holomorphic transformation of coordinates from z-variables to w-variables as follows:

$$\begin{cases} w_j=z_j, & j=1,\cdots,n-2,n,\\ w_{n-1}=f(z). \end{cases}$$

Since the Jacobian matrix of this transformation is an identity matrix at 0, this transformation is available in this proof (see the remark before this proof). We suppose that $P_m(z, D_z)$ is mapped to $P'_m(w, D_w)$ under this transformation, then $P'_m(w, D_w)$ can be written as follows

(49) $$P'_m(w, D_w) = \left(a_1(w)\frac{\partial}{\partial w_1}+\cdots+a_{n-2}(w)\frac{\partial}{\partial w_{n-2}}+a_n(w)\frac{\partial}{\partial w_n}\right)\left(\frac{\partial}{\partial w_{n-1}}\right)^{m-1}+\cdots,$$

where the omitted part consists of terms of order less than $m-1$ with respect to $(\partial/\partial w_{n-1})$ (see the proof of Theorem 1). Since $P'^{(j)}_m(w, \eta)=\sum_{k=1}^{n} P^{(k)}_m(z, \xi)\frac{\partial w_j}{\partial z_k}$, we have at $w=0$

(50) $$P'^{(j)}_m(0, N)=0, \qquad j=1, 2, \cdots, n-1, \qquad P'^{(n)}_m(0, N)=1.$$

We next find the bicharacteristic strip $(w(t), \eta(t))$ of $P'_m(w, \eta)$ through $(0, N)$ at $t=0$. Then by (49) and (50)

(51) $$\begin{cases} \dfrac{dw_j}{dt}(0)=P'^{(j)}_m(0, N)=0, & j=1, 2, \cdots, n-1, \\ \dfrac{dw_n}{dt}(0)=P'^{(n)}_m(0, N)=1, & w_{n-1}(t)=0. \end{cases}$$

We denote t by v_n and define the transformation

$$\begin{cases} w_j=w_j(v_n)+v_j, \qquad j=1, \cdots, n-2, \\ w_{n-1}=v_{n-1}, \\ w_n=w_n(v_n), \end{cases}$$

then, the Jacobian matrix of this transformation becomes an identity by (51) and is also permissible in our proof. Let $P''_m(v, \lambda)$ be the transform of $P'_m(w, \eta)$, then $P''_m(v, D_v)$ can also be written in the following form:

(52) $$P''_m(v, D_v) = \left(b_1(v)\frac{\partial}{\partial v_1}+\cdots+b_{n-2}(v)\frac{\partial}{\partial v_{n-2}}+b_n(v)\frac{\partial}{\partial v_n}\right)\left(\frac{\partial}{\partial v_{n-1}}\right)^{m-1}+\cdots.$$

Under this system of coordinates, the bicharacteristic strip $(v(t), \lambda(t))$ through $(0, N)$ is given by the next equations

$$v_j(t)=0, \qquad j=1, \cdots, n-1, \qquad v_n(t)=t,$$

$$\lambda_j(t)=0, \qquad j=1, \cdots, n-2, n, \qquad \lambda_{n-1}(t)=1.$$

Then, by Hamilton's equation we have in (52) that

$$b_1(0, \cdots, 0, v_n) = \cdots = b_{n-2}(0, \cdots, 0, v_n) = 0\,,$$
$$b_n(0, \cdots, 0, v_n) = 1\,.$$

Therefore $P''_m(v, D_v)$ obtained above satisfies the conditions (32) and (33), which completes the proof of Theorem 3.

Until now we study only the local properties of holomorphic solutions. Here we have some global problem: For a given domain Ω in $\boldsymbol{C}^n$, we seek a condition under which Ω becomes a domain of holomorphy with respect to $P(z, D)$, or not. We give a necessary condition in Corollary 2 in § 3. We now find a sufficient condition.

Let $P(z, D)$ be a differential operator with holomorphic coefficients in some open set U containing the closure of a given domain Ω. Then we suppose that its principal symbol $P_m(z, \xi)$ does not vanish identically in U. Thus by the Cauchy-Kovalevsky theorem the equation $P(z, D)u(z) = f(z)$ is locally solvable, that is, the following sequence is exact

(53) $$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O} \xrightarrow{P(z,D)} \mathcal{O} \longrightarrow 0\,,$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on U and $\mathcal{S}$ is a solution sheaf of $P(z, D)$ of $\mathcal{O}$ (i.e. $\mathcal{S}$ = kernel of $P(z, D)$ in $\mathcal{O}$). We now consider the next condition (P),

(P) there exists a fundamental system of neighborhoods $\{\Omega_n\}$ of $\bar{\Omega}$ such that

(54) (i) each Ω_n is a domain of holomorphy,

(55) (ii) the equation $P(z, D)u(z) = f(z)$ has a solution $u(z)$ holomorphic in Ω_n for any $f(z)$ holomorphic in Ω_n.

Then we have the next theorem.

Theorem 4. *Let Ω be a domain and $P(z, D)$ be a differential operator which is locally solvable in $U \supset \bar{\Omega}$. We suppose that Ω has a property* (P). *Then if for some point $z_0 \in \partial\Omega$ and a neighborhood V of z_0 there is a function $v(z)$ which is holomorphic on $[\bar{\Omega} - \{z_0\}] \cap V$ and satisfies $P(z, D)v(z) = 0$ but not holomorphic at z_0, we can find a function $u(z)$ which is holomorphic in the whole of Ω and satisfies $P(z, D)u(z) = 0$ but cannot be holomorphic at z_0.*

Proof. Let W be an open set in V containing $[\bar{\Omega} - \{z_0\}] \cap V$ and $v(z)$ is holomorphic in W. Then for some Ω_n, $K = W^c \cap \bar{\Omega}_n \cap V$ is a compact set in V. We set δ = distance $(K, \partial V) > 0$ and set

$$V_1 = \{z \in \Omega_n \mid \text{distance}\,(z, K) > \delta/2\}\,,$$
$$V_2 = V \cap \Omega_n\,.$$

Then $\Omega_n = V_1 \cup V_2$ and $v(z)$ is holomorphic in $V_1 \cap V_2$ because $V_1 \cap V_2 \subset W$.

On the other hand, we have by (53), (54) and (55)

$$H^1(\Omega_n, \mathcal{S})=0. \tag{56}$$

For a locally finite covering $\mathcal{V}$ of Ω_n, we denote by $H^1(\mathcal{V}, \mathcal{S})$ the first Čech cohomology group with respect to $\mathcal{V}$. Then $\Pi: H^1(\mathcal{V}, \mathcal{S})\rightarrow H^1(\Omega_n, \mathcal{S})$ is injective (see J. Morrow and K. Kodaira [11], Proposition 2.2, p. 34). Therefore we have by (56)

$$H^1(\mathcal{V}, \mathcal{S})=0.$$

We now take $\mathcal{V}=\{V_1, V_2\}$. Then $v(z)\in\Gamma(V_1\cap V_2, \mathcal{S})$ is a 1-cocycle, therefore there are 0-cochains $f_j\in\Gamma(V_j, \mathcal{S})$ such that $v(z)=f_1(z)-f_2(z)$. Now we define the function $u(z)$ as follows

$$u(z)=\begin{cases} f_1(z) & \text{for } z\in\Omega\cap V_1, \\ f_2(z)+v(z) & \text{for } z\in\Omega\cap V_2. \end{cases}$$

Then $u(z)$ is holomorphic in Ω and satisfies $P(z, D)u(z)=0$. Moreover $f_2(z)$ is holomorphic near z_0 and $v(z)$ is not holomorphic at z_0. Hence $u(z)$ is singular at z_0. This completes the proof.

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