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Martin boundary over an isolated singularity of rotation free density

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Consider a *density* $P(z)dxdy$ on a Riemann surface R, i.e. a 2-form $P(z)dxdy$ whose coefficients $P(z)$ are nonnegative locally Hölder continuous functions of local parameters $z=x+iy$ on R. Let δ be an isolated parabolic ideal boundary component of R. This means that there exists the base $\{\Omega^*\}$ of neighborhoods of the point δ in the Kerékjártó-Stoilow compactification of R such that each $\Omega=\Omega^*\cap R$ is an *end* of R, i.e. a subregion of R with compact analytic relative boundary $\partial\Omega$ and a single ideal boundary component δ. The parabolicity of δ is characterized by the parabolicity of the double $\hat{\Omega}$ of $\Omega=\Omega^*\cap R$ about $\partial\Omega$ for every $\Omega^*\in\{\Omega^*\}$. To describe a potential theoretic behavior of P at δ we introduce the *P-elliptic dimension*, $\dim_P\delta$, of δ as follows: Let $\mathcal{P}_P(\Omega)$ be the half module of nonnegative solutions u of the elliptic equation

$$\Delta u(z)=P(z)u(z) \qquad (\text{i.e. } d^*du(z)=u(z)P(z)dxdy)$$

on Ω with continuously vanishing boundary values on $\partial\Omega$ for $\Omega=\Omega^*\cap R$ with $\Omega^*\in\{\Omega^*\}$. Since $\mathcal{P}_P(\Omega)$ are isomorphic to each other as half modules for all Ω (Ozawa [**15, 16**]), the common half module structure $\mathcal{P}_P(\delta)$ is determined only by δ and P. Then we define $\dim_P\delta$ to be the dimension of $\mathcal{P}_P(\delta)$, i.e. the minimal cardinal number of sets of generators of $\mathcal{P}_P(\delta)$. The simplest δ is the δ_0 which is represented as the origin $z=0$ of the punctured disk $0<|z|<1$, i.e. there exists $\Omega^*\in\{\Omega^*\}$ such that Ω^* is represented as $|z|<1$ and $\Omega=\Omega^*\cap R$ as $0<|z|<1$. The simplest density is $P\equiv 0$, i.e. $P(z)\equiv 0$ for every z. The P-elliptic dimension of δ with $P\equiv 0$ is in particular referred to as the *harmonic dimension* of δ.

Two opposite extreme cases of the study of the dual dependance of $\dim_P\delta$ on (δ, P) are $\delta\to\dim_P\delta$ with the simplest P, i.e. $P\equiv 0$, and $P\to\dim_P\delta$ with the simplest $\delta=\delta_0$. It is known that there exists δ such that the harmonic dimension $\dim_0\delta$ of δ is either an arbitrary finite cardinal number n (Heins [**4**]), the countably infinite cardinal number $\mathfrak{a}$ (Kuramochi [**8**]), or the cardinal number of continuum $\mathfrak{c}$ (Constantinescu-Cornea [**2**]). These are examples of the study of $\delta\to\dim_P\delta$ with $P\equiv 0$. The starting point of the study of $P \to \dim_P \delta$ with $\delta = \delta_0$ is the *Picard principle*: $\dim_P \delta_0 = 1$ if $P \equiv 0$. Except e. g. a detailed pioneering work of Brelot [1] finding various conditions on P to assure $\dim_P \delta_0 = 1$, our knowledge on $P \to \dim_P \delta_0$ is very slim. We do not even know how widely the range of $P \to \dim_P \delta_0$ can cover the cardinals. The purpose of this paper is to contribute to this latter subject.

We restrict ourselves to tractable densities P which we call *rotation free*, i. e. $P(z) = P(|z|)$ in the closed punctured neighborhood $0 < |z| \leqq 1$ of δ_0. We form the *Martin compactification* U_P^* of U: $0 < |z| < 1$ with respect to the equation $\Delta u = Pu$ such that every boundary point is minimal. We will prove that there corresponds a quantity $\alpha(P) \in [0, 1)$, which we call the *singularity index* of P, to P such that

$$U_P^* \approx (\alpha(P) \leqq |z| \leqq 1)$$

in the sense of homeomorphism, i. e. the homeomorphism $re^{i\theta} \to [(1-\alpha(P))r + \alpha(P)]e^{i\theta}$ of U onto $\alpha(P) < |z| < 1$ is homeomorphically extendable between U_P^* and $\alpha(P) \leqq |z| \leqq 1$. Let $K_P(z, \zeta)$ be the Martin kernel of the equation $\Delta u = Pu$ for $(z, \zeta) \in U \times (\alpha(P) \leqq |z| \leqq 1)$. There exists a bijective correspondence $u \leftrightarrow \mu$ between $\mathscr{P}_P(U)$ and the class of regular Borel measures on the circle $(-\infty, \infty)/\text{mod}\, 2\pi$ with radius $\alpha(P) \geqq 0$ such that

$$u(z) = \int_0^{2\pi} K_P(z, \alpha(P)e^{i\theta}) d\mu(\theta).$$

Therefore $\dim_P \delta_0$ is the cardinal number of $|z| = \alpha(P)$ which is 1 if $\alpha(P) = 0$ and the cardinal number of continuum $\mathfrak{c}$ if $\alpha(P) > 0$. For the particular rotation free densities $P_\lambda(z) = |z|^{-\lambda}$ in U with $\lambda \in [-\infty, \infty)$ we will show that $\alpha(P_\lambda) = 0$ for $\lambda \in [-\infty, 2]$ and $\alpha(P_\lambda) > 0$ for $\lambda \in (2, \infty)$. Therefore the main conclusion of this paper is that the range under $P \to \dim_P \delta_0$ of the class of rotation free densities P is the two element set $\{1, \mathfrak{c}\}$. In particular the Picard principle is valid for rotation free densities P if and only if their singularity indices $\alpha(P) = 0$. For comparison we will append a proof of the *Riemann theorem*: $\lim_{z \to 0} u(z)$ exists for every bounded solution u of $\Delta u = Pu$ on $0 < |z| \leqq 1$ with a rotation free density $P(z)$ on $0 < |z| \leqq 1$ and in fact a bit more general density P which we call almost rotation free.

§ 1. Singularity indices.

1.1. Consider a nonnegative locally Hölder continuous function (*density*) $P(z)$ on the closed punctured disk $0 < |z| \leqq 1$ which is *rotation free* in the sense that $P(z) = P(|z|)$ for every z in $0 < |z| \leqq 1$. The function $P(z)$ may or may not be defined at $z = 0$ and therefore $P(z)$ is supposed to have an isolated singularity at $z = 0$, removable or genuine. To describe the degree of singular behavior of $P(z)$ at $z=0$ we will associate a quantity with P, which will be referred to as the singularity index of P. For this purpose we consider the following system of ordinary differential equations:

$$\frac{d^2}{dt^2}\varphi(t)+\frac{1}{t}\frac{d}{dt}\varphi(t)-\Big(P(t)+\frac{n^2}{t^2}\Big)\varphi(t)=0 \qquad (n=0,1,\cdots) \tag{1}$$

on $(0,1]$. Fix an arbitrary $\rho\in(0,1]$ and an $n=0,1,\cdots$. We will first prove that there exists a *unique* bounded solution $\varphi_n(t;\rho)$ of (1) on $(0,\rho]$ such that $\varphi_n(\rho;\rho)=1$ and $0\leqq\varphi_n(t;\rho)\leqq 1$ on $(0,\rho]$. For this aim observe that there exists a bijective correspondence between solutions $\varphi(t)$ of (1) on (a,b) $(0\leqq a<b\leqq 1)$ and rotation free solutions $u(z)$ (i. e. $u(z)=u(|z|)$) of the elliptic equation

$$L_n u(z)\equiv\Delta u(z)-\Big(P(z)+\frac{|z|^2}{n^2}\Big)u(z)=0$$

on $a<|z|<b$ such that $\varphi(|z|)=u(z)$. Let u_k be the solution of $L_n u=0$ on $1/k<|z|<\rho$ with the boundary values 1 on $|z|=\rho$ and 0 on $|z|=1/k$. Since $\{u_k\}$ forms an increasing sequence for integers $k>1/\rho$, by the maximum principle*), $\{u_k\}$ converges to a solution u of $L_n u=0$ on $0<|z|<\rho$ with the boundary values 1 on $|z|=\rho$ such that $0<u(z)<1$ on $0<|z|<\rho$. Let w be another bounded solution of $L_n u=0$ on $0<|z|<\rho$ with the boundary values 1 on $|z|=\rho$. Since $L_n(\varepsilon\log(\rho/|z|))\leqq 0$ for every $\varepsilon>0$, the maximum principle yields that both $u-w$ and $w-u$ are dominated by $\varepsilon\log(\rho/|z|)$ on $0<|z|<\rho$, i. e. $|u-w|\leqq\varepsilon\log(\rho/|\cdot|)$ for every $\varepsilon>0$ on $0<|z|<\rho$. Therefore $u\equiv w$. Let θ be any fixed real number. Clearly $u(e^{i\theta}z)$ is also a solution of $L_n u=0$ on $0<|z|<\rho$. By the above observation, $u(e^{i\theta}z)=u(z)$ on $0<|z|<\rho$ for every θ. We have thus shown that there exists a unique bounded solution u of $L_n u=0$ on $0<|z|<\rho$ with boundary values 1 on $|z|=\rho$ and that $0<u<1$ on $0<|z|<\rho$ and u is rotation free. Therefore $u(t)$ is the required solution $\varphi_n(t;\rho)$ of (1). We will simply denote by $\varphi_n(t)$ the function $\varphi_n(t;1)$. Then we have the following obvious identity for $0<t<\rho$:

$$\varphi_n(t;\rho)=\varphi_n(t)/\varphi_n(\rho)\,. \tag{2}$$

1.2. We derive some properties of each $\varphi_n(t;\rho)$ and the sequence $\{\varphi_n(t;\rho)\}$. Since $L_n\varphi_n(|z|;\rho)=0$, $L_n\varphi_{n+1}(|z|;\rho)>0$, and $L_n(\varepsilon\log(\rho/|z|))\leqq 0$ for every $\varepsilon>0$ on $0<|z|<\rho$, the maximum principle used as in 1.1 yields $\varphi_n(|z|;\rho)\geqq\varphi_{n+1}(|z|;\rho)$ on $0<|z|<\rho$. Thus we conclude that

*) Let $Au(z)=\Delta u(z)+a(z)u_x(z)+b(z)u_y(z)+c(z)u(z)$ $(c(z)\leqq 0)$ with locally Hölder continuous coefficients on a plane region Ω. The *maximum principle* means that if $Af(z)\leqq 0$ for an $f\in C^2(\Omega)$ with $\liminf_{z\in\Omega,\, z\to\zeta} f(z)\geqq 0$ for every $\zeta\in\partial\Omega$, then $f\geqq 0$ on Ω. This and the solvability of Dirichlet problem will be repeatedly used in this paper. For these refer to e. g. Miranda [**10**].

$$\varphi_0(t\,;\rho)\geqq\varphi_1(t\,;\rho)\geqq\cdots\geqq\varphi_n(t\,;\rho)\geqq\cdots>0 \tag{3}$$

on $(0, \rho]$. Next observe that

$$L_n(|z|^m/\rho^m)=-((n^2-m^2)|z|^{m-2}+P(z)|z|^m)/\rho^m\leqq 0$$

for $m=0,1,\cdots,n$ and as above $L_n(\varepsilon\log(\rho/|z|))\leqq 0$ for every $\varepsilon>0$, the maximum principle implies that $\varphi_n(|z|\,;\rho)\leqq|z|^m/\rho^m+\varepsilon\log(|z|/\rho)$ for every $\varepsilon>0$ on $0<|z|<\rho$. Therefore we have

$$0<\varphi_n(t\,;\rho)\leqq t^m/\rho^m \qquad (m=0,1,\cdots,n) \tag{4}$$

on $(0,\rho]$ for every $n=0,1,\cdots$. Combining (3) and (4) we have $\varphi_n(t)/t^m\leqq\varphi_n(\rho)/\rho^m$ for every t and ρ with $0<t\leqq\rho\leqq 1$. This means that $\varphi_n(t)/t^m$ is an increasing function of t on $(0, 1]$ and hence the same is true of the function $\varphi_n(t\,;\rho)/t^m=\varphi_n(t)/t^m\varphi_n(\rho)$ on $(0,\rho]$. Therefore

$$\frac{d}{dt}\Big(\frac{\varphi_n(t\,;\rho)}{t^m}\Big)\geqq 0 \qquad (m=0,1,\cdots,n) \tag{5}$$

on $(0,\rho]$. In particular on putting $m=0$ we deduce the existence of $\lim_{t\to 0}\varphi_n(t\,;\rho)$ and in fact by (4) we have

$$\lim_{t\to 0}\varphi_0(t\,;\rho)\geqq 0\,, \qquad \lim_{t\to 0}\varphi_n(t\,;\rho)=0 \qquad (n=1,2,\cdots)\,. \tag{6}$$

If $\lim_{t\to 0}\varphi_0(t)>0$, then $\lim_{t\to 0}\varphi_n(t)/\varphi_0(t)$ exists and is zero. Even for the case of $\lim_{t\to 0}\varphi_0(t)=0$ the existence of $\lim_{t\to 0}\varphi_n(t)/\varphi_0(t)$ is assured:

$$\lim_{t\to 0}\varphi_n(t)/\varphi_m(t) \tag{7}$$

exists for every $m=0,1,\cdots$ and every $n>m$. In fact, by (3), $\varphi_m(t\,;\rho)\geqq\varphi_n(t\,;\rho)$ on $(0,\rho]$. Therefore $\varphi_n(t)/\varphi_m(t)\leqq\varphi_n(\rho)/\varphi_m(\rho)$ for every t and ρ with $0<t\leqq\rho\leqq 1$. This means that $\varphi_n(t)/\varphi_m(t)$ is an increasing positive function on $(0, 1]$. Thus the existence of (7) can be deduced.

We wish to compare the magnitude of (7) with $m=0$ for different n. To do this we first compare the magnitude of $\lim_{t\to 0}\varphi_n(t)/\varphi_{n-1}(t)$ for $n=1,2,\cdots$. For this purpose we consider auxiliary functions $\psi_n(t\,;\rho)=\varphi_n(t\,;\rho)/\varphi_{n-1}(t\,;\rho)$ $(n=1,2,\cdots)$ on $(0,\rho]$. We also simply denote by $\psi_n(t)$ the function $\psi_n(t\,;1)$. As a counter part of (2) we obviously have the identity

$$\psi_n(t\,;\rho)=\psi_n(t)/\psi_n(\rho) \tag{8}$$

on $(0,\rho]$. Observe that (3) implies $0<\psi_n(t\,;\rho)\leqq 1$ on $(0,\rho]$ and obviously $\psi_n(\rho\,;\rho)=1$. Thus by (8) we have $0<\psi_n(t)\leqq\psi_n(\rho)$ for every t and ρ with $0<t\leqq\rho\leqq 1$. This means that $\psi_n(t)$ is an increasing function on $(0, 1]$ and the same is true of $\psi_n(t\,;\rho)$ on $(0,\rho]$. Consequently we deduce

(9) $$\frac{d}{dt}\psi_n(t\,;\,\rho)\geqq 0 \qquad (n=1, 2, \cdots)$$

on $(0, \rho]$. Less obvious is the following relation:

(10) $$\frac{d}{dt}\log\left(\frac{\psi_n(t\,;\,\rho)}{t}\right)\leqq 0 \qquad (n=1, 2, \cdots)$$

on $(0, \rho]$. Put $\gamma(t)=\rho^{-1}t\varphi_{n-1}(t\,;\,\rho)$ on $(0, \rho]$. Note that $\gamma(\rho)=1$. By an easy computation we see that

$$L_n\gamma(|z|)=2\varphi_{n-1}(|z|\,;\,\rho)\Big[\frac{d}{dt}\log(\varphi_n(t\,;\,\rho)/t^{n-1})\Big]_{t=|z|}$$

for $0<|z|<\rho$. By (5) we conclude that $L_n\gamma(|z|)\geqq 0$. Since $L_n\varphi_n(|z|\,;\,\rho)=0$ and $L_n(\varepsilon\log(\rho/|z|))\leqq 0$ for every $\varepsilon>0$, the maximum principle yields that $\varphi_n(|z|\,;\,\rho)-\gamma(|z|)+\varepsilon\log(\rho/|z|)\geqq 0$ on $0<|z|<\rho$ for every $\varepsilon>0$. Therefore $\varphi_n(t\,;\,\rho)\geqq\gamma(t)$ on $(0, \rho]$, i.e. $\psi_n(t\,;\,\rho)\geqq t/\rho$ on $(0, \rho]$. By (8) we see that $\psi_n(t)/t\geqq\psi_n(\rho)/\rho$ for every t and ρ with $0<t\leqq\rho\leqq 1$. This means that $\psi_n(t)/t$ and hence $\psi_n(t\,;\,\rho)/t$ is a decreasing function of t on $(0, \rho]$ and the same is true of $\log(\psi_n(t\,;\,\rho)/t)$ and consequently (10) follows.

1.3. To compare $\psi_n(t\,;\,\rho)$ for different $n=1, 2, \cdots$ it is convenient to treat $\psi_n(t\,;\,\rho)$ as a solution of an appropriate differential equation. Observe that $\psi_n(t\,;\,\rho)$ is a solution of

(11) $$\frac{d^2}{dt^2}\psi(t)+\left(\frac{1}{t}+2\frac{d}{dt}\log\varphi_{n-1}(t)\right)\frac{d}{dt}\psi(t)-\frac{2n-1}{t^2}\psi(t)=0$$

on $(0, \rho]$. As in 1.1 there exists a bijective correspondence between solutions $\psi(t)$ of (11) on $(0, \rho]$ and rotation free solutions $v(z)$ of the elliptic equation

$$M_nv(z)\equiv\Delta v(z)+2\nabla\log\varphi_{n-1}(|z|)\cdot\nabla v(z)-\frac{2n-1}{|z|^2}v(z)=0$$

on $0<|z|<\rho$, where ∇v is the gradient vector field $(\partial v/\partial x, \partial v/\partial y)$ of v. Since $M_n\psi_n(|z|\,;\,\rho)=0$ on $0<|z|<\rho$, an easy computation shows that

$$M_{n+1}\psi_n(|z|\,;\,\rho)=-2\psi_n(|z|\,;\,\rho)\Big[\frac{1}{t^2}-\left(\frac{d}{dt}\log\psi_n(t\,;\,\rho)\right)^2\Big]_{t=|z|}$$

on $0<|z|<\rho$. It follows from (10) that $d(\log\psi_n(t\,;\,\rho))/dt-1/t\leqq 0$ and consequently $M_{n+1}\psi_n(|z|\,;\,\rho)\leqq 0$ on $0<|z|<\rho$. By (9), $\nabla\log\psi_{n-1}(|z|)=[d(\log\psi_{n-1}(t))/dt]_{t=|z|}\geqq 0$. Observe that $\Delta\varepsilon\log(\rho/|z|)=0$ and $\nabla\varepsilon\log(\rho/|z|)=-\varepsilon[d(\log t)/dt)_{t=|z|}=-\varepsilon/|z|\leqq 0$. Therefore $M_{n+1}(\varepsilon\log(\rho/|z|))\leqq 0$ for every $\varepsilon>0$ and

$$M_{n+1}(\psi_n(|z|\,;\,\rho)+\varepsilon\log(\rho/|z|)-\psi_{n+1}(|z|\,;\,\rho))\leqq 0$$

on $0<|z|<\rho$ for every $\varepsilon>0$. By the maximum principle, $\psi_n(t\,;\,\rho)+\varepsilon\log(\rho/t)-\psi_{n+1}(t\,;\,\rho)\geqq 0$ on $(0, \rho]$, and on letting $\varepsilon\to 0$ we conclude that $\psi_n(t\,;\,\rho)\geqq\psi_{n+1}(t\,;\,\rho)$, i.e.

$$\psi_1(t;\rho) \geqq \psi_2(t;\rho) \geqq \cdots \geqq \psi_n(t;\rho) \geqq \cdots \geqq 0 \tag{12}$$

on $(0, \rho]$. To deduce a result of the reversed character to (12) we compute

$$M_{n+1}(\psi_n(|z|;\rho)^3)$$
$$=3\psi_n(|z|;\rho)^2\Big[2\frac{d}{dt}\log\psi_n(t;\rho)\cdot\frac{d}{dt}\psi_n(t;\rho)+\frac{4(n-1)}{3t^2}\psi_n(t;\rho)\Big]_{t=|z|},$$

which is nonnegative by (9). Therefore

$$M_{n+1}(\psi_{n+1}(|z|;\rho)-\psi_n(|z|;\rho)^3+\varepsilon\log(\rho/|z|))\leqq 0$$

for every $\varepsilon>0$ and the maximum principle yields

$$\psi_n(t;\rho)^3 \leqq \psi_{n+1}(t;\rho) \qquad (n=1,2,\cdots) \tag{13}$$

on $(0, \rho]$. We are ready to proceed to the definition of singularity indices and to study their mutual relations.

1.4. Let $P(z)$ be a rotation free density on $0<|z|<1$ and $\varphi_n(t)$ be the unique bounded solution of (1) on $(0, 1]$ with $\varphi_n(1)=1$ for each $n=0,1,\cdots$. In view of (7) we can define the quantity

$$\alpha_n=\alpha_n(P)=\lim_{t\to 0}\varphi_n(t)/\varphi_0(t) \tag{14}$$

for each $n=1,2,\cdots$, which will be referred to as the n^{th} *singularity index* of P at $z=0$. In particular we denote by $\alpha=\alpha(P)$ the 1^{st} singularity index $\alpha_1(P)$ and call it simply the *singularity index* of P at $z=0$. It will be seen that $\alpha(P)$ determines generator of positive solutions of $\Delta u=Pu$ at $z=0$. We prove the following

THEOREM 1. *The singularity indices of a rotation free density P satisfy the following fundamental inequalities for all positive integers n:*

$$0\leqq\alpha(P)<1, \qquad (\alpha(P))^{(3^n-1)/2}\leqq\alpha_n(P)\leqq(\alpha(P))^n. \tag{15}$$

It is clear that $\alpha(P)\geqq 0$. By virtue of (9) and $\psi_1(1)=1$. $\alpha(P)=\lim_{t\to 0}\varphi_1(t)/\varphi_0(t)=\lim_{t\to 0}\psi_1(t)\leqq 1$. If $\alpha(P)=1$, then $\psi_1(t)\equiv 1$ on $(0, 1]$ and the constant 1 would be a solution of (11) with $n=1$ on $(0, 1]$, which is clearly impossible and we have $\alpha(P)<1$. We next maintain that $\varphi_n(t)/\varphi_0(t)\leqq(\varphi_1(t)/\varphi_0(t))^n$ for every $n=1,2,\cdots$. This is certainly the case for $n=1$. Suppose it is true for an n. Then by (12) and the assumption of the induction we deduce $\varphi_{n+1}(t)/\varphi_0(t)=\psi_{n+1}(t)(\varphi_n(t)/\varphi_0(t))\leqq\psi_1(t)(\varphi_1(t)/\varphi_0(t))^n=(\varphi_1(t)/\varphi_0(t))^{n+1}$. Thus we conclude that $\alpha_n(P)\leqq(\alpha_1(P))^n$. As a consequence of (13) we have $(\psi_1(t))^{3^n}\leqq\psi_{n+1}(t)$ for every $n=1,2,\cdots$. Hence $(\varphi_n(t)/\varphi_0(t))\cdot(\varphi_1(t)/\varphi_0(t))^{3^n}\leqq\varphi_{n+1}(t)/\varphi_0(t)$. By taking the limit as $t\to 0$ on the both sides of this inequality we deduce $\alpha_n\cdot\alpha_1^{3^n}\leqq\alpha_{n+1}$. Clearly $\alpha_1^{(3^n-1)/2}\leqq\alpha_n$ is true for $n=1$. Assume that it is true for an n. Then $\alpha_{n+1}\geqq\alpha_n\cdot\alpha_1^{3^n}\geqq\alpha_1^{(3^n-1)/2}\cdot\alpha_1^{3^n}=\alpha_1^{(3^{n+1}-1)/2}$. Thus we conclude that $(\alpha(P))^{(3^n-1)/2}\leqq\alpha_n(P)$ for every $n=1,2,\cdots$. The proof of Theorem 1 is herewith complete.

§2. Fourier coefficients of Green's function.

2.1. Let $G(z, \zeta)=G_P(z, \zeta)$ be the Green's function of the equation $\Delta u=Pu$ on the punctured disk $0<|z|<1$ where P is as before a rotation free density on $0<|z|\leqq 1$. It is characterized by the following properties: $G(\cdot, \zeta)$ is a positive solution of $\Delta u=Pu$ on $\{0<|z|<1\}-\{\zeta\}$; $G(z, \zeta)+\log|z-\zeta|=\mathcal{O}(1)$ as $z\to\zeta$; $\lim_{z\to e^{i\theta}} G(z, \zeta)=0$ for every θ and $\limsup_{z\to 0} G(z, \zeta)<\infty$ (cf. e.g. Myrberg [11, 12], Itô [5]). We will later see in Appendix that $\lim_{z\to 0} G(z, \zeta)$ in fact exists but this fact will not be made use of in the main text. Consider Fourier coefficients of $G(z, \cdot)$:

$$
(16)\qquad \begin{cases} c_n(z, t)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} G(z, te^{i\theta})\cos n\theta d\theta & (n=0, 1, \cdots), \\ s_n(z, t)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} G(z, te^{i\theta})\sin n\theta d\theta & (n=1, 2, \cdots) \end{cases}
$$

for $0<|z|<1$ and $t\in(0, 1]$. We will study properties of these functions. By the maximum principle, $G(z, \zeta)\leqq\log(|1-\bar{\zeta}z|/|z-\zeta|)$ and therefore

$$
|c_n(z, t)|,\ |s_n(z, t)|\leqq c_0(z, t)\leqq 2\min\left(\log\frac{1}{|z|}, \log\frac{1}{t}\right)
$$

since $\int_0^{2\pi}\log(|1-te^{-i\theta}z|/|z-te^{i\theta}|)d\theta=-2\pi\max(\log|z|, \log t)$. This means that $c_n(z, t)$ and $s_n(z, t)$ are bounded on $(0, 1]$ as the functions of t for any fixed z in $0<|z|<1$. Moreover $t\to c_n(z, t)$ and $s_n(z, t)$ are continuous on $(0, 1]$ for any fixed z in $0<|z|<1$. The assertion is certainly clear on $(0, |z|)$ and $(|z|, 1]$. To see their continuity at $t=|z|$, let $g(t)$ be $c_n(z, t)$ or $s_n(z, t)$. Let $z=|z|e^{i\theta_0}$, $0<\eta<\pi/2$, and $0<\sigma<\min(|z|, 1-|z|)$. There exists a constant k such that $G(z, te^{i\theta})<-\log|z-te^{i\theta}|+k$ for every (t, θ) in $(|t-|z||<\sigma)\times(|\theta-\theta_0|<\eta)$. Since $-\log|z-te^{i\theta}|\leqq\log|z-|z|e^{i\theta}|$ for every (t, θ) in $(|t-|z||<\sigma)\times(|\theta-\theta_0|<\eta)$, we see that $|g(t)-g(|z|)|$ for $t\in(|z|-\sigma, |z|+\sigma)$ is dominated by

$$
\frac{1}{\pi}\int_{\theta_0+\eta}^{\theta_0+2\pi-\eta}|G(z, te^{i\theta})-G(z, |z|e^{i\theta})|d\theta
$$
$$
+\frac{2}{\pi}\int_{\theta_0-\eta}^{\theta_0+\eta}\log\frac{1}{|e^{i\theta}-e^{i\theta_0}|}d\theta+2\Big(\frac{2}{\pi}\log\frac{1}{|z|}+k\Big)\eta.
$$

Therefore $\limsup_{t\to|z|}|g(t)-g(|z|)|\leqq\mathcal{O}(\eta)$ and on letting $\eta\to 0$ we conclude the continuity of $g(t)=c_n(z, t)$ $(n=0, 1, \cdots)$ or $s_n(z, t)$ $(n=1, 2, \cdots)$ at $t=|z|$. The bounded continuous functions $c_n(z, t)$ $(n=0, 1, \cdots)$ and $s_n(z, t)$ $(n=1, 2, \cdots)$ on $(0, 1]$ also satisfy the ordinary differential equation (1) on $(0, 1]$ except for $t=|z|$. In fact, since $\Delta=\partial^2/\partial t^2+\partial/t\partial t+\partial^2/t^2\partial\theta^2$ for the polar coordinate $te^{i\theta}$, we deduce

$$\begin{aligned}&\left(\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}\right)c_n(z, t)\\&=\frac{1}{\pi}\int_0^{2\pi}\left(\left(\frac{\partial^2}{\partial t^2}+\frac{1}{t}\frac{\partial}{\partial t}\right)G(z, te^{i\theta})\right)\cdot\cos n\theta\, d\theta\\&=\frac{1}{\pi}\int_0^{2\pi}\left(\Delta G(z, te^{i\theta})-\frac{1}{t^2}\frac{\partial^2}{\partial\theta^2}G(z, te^{i\theta})\right)\cdot\cos n\theta\, d\theta\\&=P(t)\cdot\frac{1}{\pi}\int_0^{2\pi}G(z, te^{i\theta})\cdot\cos n\theta\, d\theta-\frac{1}{\pi t^2}\int_0^{2\pi}\left(\frac{\partial^2}{\partial\theta^2}G(z, te^{i\theta})\right)\cdot\cos n\theta\, d\theta\,.\end{aligned}$$

Twice applications of the integration by parts transform the last term of the above to $(n^2/\pi t^2)\int_0^{2\pi}G(z, te^{i\theta})\cdot\cos n\theta\, d\theta$. Consequently we obtain

$$\frac{d^2}{dt^2}c_n(z, t)+\frac{1}{t}\frac{d}{dt}c_n(z, t)-\left(P(t)+\frac{n^2}{t^2}\right)c_n(z, t)=0 \qquad (n=0, 1, \cdots) \tag{17}$$

for $t\in(0, 1]$ with $t\neq|z|$. By the same computation as above we also have

$$\frac{d^2}{dt^2}s_n(z, t)+\frac{1}{t}\frac{d}{dt}s_n(z, t)-\left(P(t)+\frac{n^2}{t^2}\right)s_n(z, t)=0 \qquad (n=1, 2, \cdots) \tag{18}$$

for $t\in(0, 1]$ with $t\neq|z|$. By the result in 1.1, we deduce

$$\frac{c_n(z, t)}{c_n(z, \rho)}=\varphi_n(t\,;\rho)\,, \qquad \frac{s_n(z, t)}{s_n(z, \rho)}=\varphi_n(t\,;\rho)\,, \tag{19}$$

the former for $n=0, 1, \cdots$ and the latter for $n=1, 2, \cdots$, for every t and ρ with $0<t\leqq\rho\leqq|z|$. These are properties of individual c_n and s_n.

2.2. We next study the property of the class $\mathfrak{G}=\{c_n(z, t)\ (n=0, 1, \cdots), s_n(z, t)\ (n=1, 2, \cdots)\}$ as a family of functions of z in $0<|z|<1$. We maintain that the class $\mathfrak{G}$ is *linearly independent* in the following sense: Let $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be arbitrary absolutely convergent real series. Suppose that

$$\sum_{n=0}^{\infty}a_nc_n(z, t)+\sum_{n=1}^{\infty}b_ns_n(z, t)=0 \tag{20}$$

for any fixed t in $(0, 1]$ and z in a nonempty open subset D of $0<|z|<1$. Then $a_n=0$ $(n=0, 1, \cdots)$ and $b_n=0$ $(n=1, 2, \cdots)$. To prove the assertion set

$$\mu(\theta)=\sum_{n=0}^{\infty}a_n\cos n\theta+\sum_{n=1}^{\infty}b_n\sin n\theta$$

which is continuous on $[0, 2\pi]$ with $\mu(0)=\mu(2\pi)$. We also set

$$g(z)=\int_0^{2\pi}G(z, te^{i\theta})\mu(\theta)d\theta\,,$$

which is identical with the left hand side term of (20). By a similar argument as in 2.1, we see that $g(z)$ is continuous on $0<|z|\leqq 1$. In fact, the continuity of $g(z)$ at $|z|\neq t$ is entirely clear. Let $z_0=te^{i\theta_0}$ and $z=|z|e^{i\sigma}$ with $|z-z_0|< t\sin\eta$, η being a small positive number. Let $K=\sup_{\theta\in[0,2\pi]}\mu(\theta)$ and $G(z,\zeta)\leqq -\log|z-\zeta|+k$ with a positive constant k for every z and ζ in $|z-z_0|< t\sin\eta$. Let $z_1=te^{i\sigma}$. Then $|\sigma-\theta_0|<\eta$ and therefore $|g(z)-g(z_0)|$ is dominated by the sum of

$$K\int_{\theta_0+\eta}^{\theta_0+2\pi-\eta}|G(z,te^{i\theta})-G(z_0,te^{i\theta})|\,d\theta$$

and

$$K\int_{\theta_0-\eta}^{\theta_0+\eta}\Big(\log\frac{1}{|e^{i\theta}-e^{i\theta_0}|}+\log\frac{1}{|e^{i\theta}-e^{i\sigma}|}\Big)d\theta+2\Big(2K\log\frac{1}{t}+k\Big)\eta\,,$$

the latter of which is easily seen to be $\mathcal{O}(\eta)$. Therefore $\limsup_{z\to z_0}|g(z)-g(z_0)|\leqq\mathcal{O}(\eta)$ and the continuity of $g(z)$ at $z=z_0$ follows. Since $|g(z)|$ is bounded by $K\min(-\log t, -\log|z|)$ (cf. 2.1) on $0<|z|<1$ and $g(z)$ is a solution of $\Delta u=Pu$ on $0<|z|<1$ less $|z|=t$, the maximum principle yields $\max_{t\leqq|z|\leqq 1}|g(z)|=\max_{|z|=t}|g(z)|$. Again by applying the maximum principle to the function $v(z)=\max_{|z|=t}|g(z)|+\varepsilon\log(t/|z|)\pm g(z)$ which satisfy $(\Delta-P)v\leqq 0$ on $0<|z|<t$ for every $\varepsilon>0$, we conclude that $\sup_{0<|z|\leqq t}|g(z)|=\max_{|z|=t}|g(z)|$. By (20), the solution g of $\Delta u=Pu$ vanishes on a nonempty open subset of $0<|z|<t$ or of $t<|z|<1$. Then $g(z)\equiv 0$ on $0<|z|<t$ or on $t<|z|<1$ (cf. e.g. Miranda [**10**]). By $\sup_{0<|z|\leqq t}|g(z)|=\max_{t\leqq|z|\leqq 1}|g(z)|=\max_{|z|=t}|g(z)|$, we can now conclude that $g(z)\equiv 0$ on $0<|z|\leqq 1$.

2.3. We complete the proof of linear independence of $\mathfrak{G}$ by showing $\mu(\theta)\equiv 0$ on $[0,2\pi]$. If this were not the case there would exist a $\theta_0\in(0,2\pi)$ with $\mu(\theta_0)\neq 0$. We may assume that $\mu>0$ on $(\theta_0-\eta,\theta_0+\eta)\subset(0,2\pi)$ with an $\eta>0$. Choose an $h\in C^1[0,2\pi]$ such that $h\geqq 0$, $h(\theta_0)=1$, and the support of h is contained in $(\theta_0-\eta,\theta_0+\eta)$. Let u_k be the solution of $\Delta u=Pu$ on $1/k<|z|<t$ with boundary values 0 on $|z|=1/k$ and $h(\arg z)$ on $|z|=t$ for each integer $k>1/t$. By the maximum principle, $\{u_k\}$ forms an increasing sequence and therefore $\{u_k\}$ and $\{{}^*du_k\}$ converge to a solution u of $\Delta u=Pu$ on $0<|z|<t$ with the boundary values $h(\arg z)$ on $|z|=t$ and ${}^*du$, respectively, and the convergences are uniform on $\sigma\leqq|z|\leqq t$ for every $\sigma\in(0,t)$ (cf. e.g. [**10**]). We also consider the solution $\hat{u}$ of $\Delta u=Pu$ on $t<|z|<1$ with boundary values $h(\arg z)$ on $|z|=t$ and 0 on $|z|=1$. Let $G_k(z,\zeta)$ be the Green's function of $\Delta u=Pu$ on $1/k<|z|<1$. Similarly as above $\{G_k(z,\zeta)\}$ and $\{{}^*d_\zeta G_k(z,\zeta)\}$ converge to $G(z,\zeta)$ and ${}^*d_\zeta G(z,\zeta)$, respectively, uniformly on $\sigma\leqq|\zeta|\leqq 1$ less an arbitrary small disk about z for every fixed $\sigma\in(0,t)$.

Fix any z in $0<|z|<t$. The Green formula yields

$$2\pi u_k(z)=\int_{|\zeta|=t}(G_k(z, \zeta)^*d_\zeta u_k(\zeta)-u_k(\zeta)^*d_\zeta G_k(z, \zeta))$$

and on making $k\to\infty$ we deduce

$$-2\pi u(z)=\int_0^{2\pi}\Big(G(z, te^{i\theta})\frac{\partial}{\partial t}u(te^{i\theta})-h(\theta)\frac{\partial}{\partial t}G(z, te^{i\theta})\Big)td\theta .$$

Again by the Green formula applied to $G(z, \zeta)$ and $\hat{u}(\zeta)$ on $t\leqq|\zeta|\leqq 1$ we deduce as above

$$0=\int_0^{2\pi}\Big(G(z, te^{i\theta})\frac{\partial}{\partial t}\hat{u}(te^{i\theta})-h(\theta)\frac{\partial}{\partial t}G(z, te^{i\theta})\Big)td\theta .$$

Subtraction of the last two formulas gives

$$u(z)=\int_0^{2\pi}G(z, te^{i\theta})\nu(\theta)d\theta \tag{21}$$

where $\nu(\theta)=(t/2\pi)\partial(\hat{u}(te^{i\theta})-u(te^{i\theta}))/\partial t$ is a continuous function on $[0, 2\pi]$ with $\nu(0)=\nu(2\pi)$. The identity (21) is derived under the assumption $0<|z|<t$. As in 2.2 the right hand side of (21) is continuous on $0<|z|\leqq 1$ and the same is obviously true of $u(z)$. Therefore (21) is valid for $0<|z|\leqq t$ and in particular for $|z|=t$. However if $z=te^{i\theta}$, then (21) takes on the form

$$h(\theta)=\int_0^{2\pi}G(te^{i\theta}, te^{i\tau})\nu(\tau)d\tau .$$

On integrating both sides of $g(te^{i\tau})=0$, i.e.

$$\int_0^{2\pi}G(te^{i\tau}, te^{i\theta})\mu(\theta)d\theta=0 ,$$

with respect to the measure $\nu(\tau)d\tau$ on $[0, 2\pi]$, we obtain

$$\int_0^{2\pi}\Big(\int_0^{2\pi}G(te^{i\tau}, te^{i\theta})\mu(\theta)d\theta\Big)\nu(\tau)d\tau=0 .$$

By the Fubini theorem and the symmetry of $G(z, \zeta)$, we conclude that

$$0=\int_0^{2\pi}h(\theta)\mu(\theta)d\theta=\int_{\theta_0-\eta}^{\theta_0+\eta}h(\theta)\mu(\theta)d\theta ,$$

which is clearly a contradiction since the continuous function $h(\theta)\mu(\theta)$ is nonnegative and $h(\theta)\mu(\theta)\not\equiv 0$ on $[\theta_0-\eta, \theta_0+\eta]$. Thus we have shown that $\mu(\theta)\equiv 0$, i.e.

$$\sum_{n=0}^{\infty}a_n\cos n\theta+\sum_{n=1}^{\infty}b_n\sin n\theta\equiv 0$$

on $[0, 2\pi]$. By the orthogonality of the system $\{\cos n\theta\ (n=0, 1, \cdots), \sin n\theta\ (n=1, 2, \cdots)\}$, this is only possible when $a_n=0\ (n=0, 1, \cdots)$ and $b_n=0\ (n=1, 2, \cdots)$.

We restate the result in

THEOREM 2. *The system* $\mathfrak{S}=\{c_n(z, t), s_n(z, t)\}$ *of Fourier coefficients of the Green's function* $G_P(z, te^{i\theta})$ *is linearly independent as functions of* z.

§ 3. Martin kernel.

3.1. Let $K(z, \zeta)=K_P(z, \zeta)=K_P(z, \zeta\,;z_0)$ be the *Martin kernel* of the equation $\Delta u=Pu$ on the punctured disk $0<|z|<1$ with P, as before, a rotation free density on $0<|z|\leqq 1$, i.e. $K(z, \zeta)=G(z, \zeta)/G(z_0, \zeta)$. We study the limiting function of $K(z, \zeta)$ as ζ converges to the boundary of $0<|z|<1$, the essential part of which is for the case $\zeta\to 0$. We denote by $\mathcal{P}_P$ the half module of nonnegative solutions of $\Delta u=Pu$ on $0<|z|<1$. A nonzero solution $u\in\mathcal{P}_P$ is said to be *minimal* if $u\geqq v$ with a $v\in\mathcal{P}_P$ implies that v/u is constant on $0<|z|<1$. Let $\mathcal{I}_P$ be the subclass of $\mathcal{P}_P$ consisting of solutions u with boundary values zero on $|z|=1$ and $\mathcal{S}_P$ be the subclass of $\mathcal{P}_P$ consisting of solutions u such that u is bounded in a punctured neighborhood of $z=0$. We have the direct sum decomposition

$$\mathcal{P}_P=\mathcal{I}_P+\mathcal{S}_P\,, \qquad \mathcal{I}_P\cap\mathcal{S}_P=\{0\}\,. \tag{22}$$

In fact, let $u\in\mathcal{P}_P$. Denote by v_n (w_n, resp.) the solution of $\Delta u=Pu$ on $\varepsilon_n<|z|<1-\varepsilon_n$ with boundary values u (0, resp.) on $|z|=\varepsilon_n$ and 0 (u, resp.) on $|z|=1-\varepsilon_n$, where $\{\varepsilon_n\}$ is a decreasing zero sequence in $(0, 1/4)$. By the maximum principle, $0<v_n$, $w_n<u$, and $u=v_n+w_n$. By the Harnack principle for solutions of $\Delta u=Pu$ (cf. e.g. [**10**]), on taking a suitable subsequence if necessary, we may assume that $\{v_n\}$ and $\{w_n\}$ converge to solutions v and w in $\mathcal{P}_P$ uniformly on each compact subset of $0<|z|<1$. Let $M=\max_{|z|=1/2} u(z)$. By the maximum principle, $v_n\leqq -M(\log|z|)/(\log 2)$ on $1/2<|z|<1-\varepsilon_n$ and $w_n\leqq M$ on $\varepsilon_n<|z|<1/2$. Therefore $v\leqq -M(\log|z|)/(\log 2)$ on $1/2<|z|<1$ and $w\leqq M$ on $0<|z|<1/2$, i.e. $v\in\mathcal{I}_P$ and $w\in\mathcal{S}_P$. Let $u\in\mathcal{I}_P\cap\mathcal{S}_P$. By the maximum principle, $u<-\varepsilon\log|z|$ for every $\varepsilon>0$ and thus $u=0$. We first cite the following result of Itô [**6**]:

$$\lim_{r\to 1,\,\sigma\to\theta} K(z, re^{i\sigma})=\left[\frac{\partial}{\partial r}G(z, re^{i\theta})\right]_{r=1}\Big/\left[\frac{\partial}{\partial r}G(z_0, re^{i\theta})\right]_{r=1}\equiv k_1(z, \theta) \tag{23}$$

uniformly for z in any compact subset of $0<|z|<1$; $k_1(z, \theta)\in\mathcal{S}_P$; $k_1(z_0, \theta)=1$; $k_1(z, \theta)>0$ on $0<|z|<1$; $k_1(z, \theta)$ has boundary values zero on $|z|=1$ less $e^{i\theta}$; $k_1(z, \theta)$ is minimal. It is convenient to use the one-dimensional *torus* $T=(-\infty, \infty) \bmod 2\pi$ as the representation of $|z|=1$. By the above, $k_1(z, \theta_1)$ and $k_1(z, \theta_2)$ are nonproportional (i.e. $k_1(\cdot, \theta_1)/k_1(\cdot, \theta_2)$ is not constant) if $\theta_1\neq\theta_2$ in T.

3.2. We next prove the counter part of (23) for $r\to 0$, which is the main part of this paper. Fix any z in $0<|z|<1$ and any $\rho\in(0, |z|]$, and let $r\in(0, \rho)$. Since the function $\sigma\to G(z, re^{i\sigma})$ is of class C^1 on T, we can expand G into its Fourier series by using Fourier coefficients (16):

$$G(z, re^{i\sigma})=\frac{1}{2}c_0(z, r)+\sum_{n=1}^{\infty}(c_n(z, r)\cos n\sigma+s_n(z, r)\sin n\sigma).$$

Substitution of coefficients by (19) gives

$$G(z, re^{i\sigma})=\frac{1}{2}c_0(z, \rho)\varphi_0(r; \rho)+\sum_{n=1}^{\infty}(c_n(z, \rho)\cos n\sigma+s_n(z, \rho)\sin n\sigma)\varphi_n(r; \rho).$$

For short we set $A_n(z, \sigma; \rho)=c_n(z, \rho)\cos n\sigma+s_n(z, \rho)\sin n\sigma$. We first discuss the existence of the limit as $r\to 0$ and $\sigma\to\theta$ of the function

$$G(z, re^{i\sigma})/\varphi_0(r; \rho)=\frac{1}{2}c_0(z, \rho)+\sum_{n=1}^{\infty}A_n(z, \sigma; \rho)\varphi_n(r; \rho)/\varphi_0(r; \rho).$$

Observe that

$$\varphi_n(r; \rho)/\varphi_0(r; \rho)=\psi_1(r; \rho)\cdot\psi_2(r; \rho)\cdots\psi_n(r; \rho).$$

By (9), $\psi_1(r; \rho)$ is increasing on $(0, \rho]$ and $\psi_1(\rho; \rho)=1$. If $\psi_1(\rho_1; \rho)=1$ for a $\rho_1\in(0, \rho)$, then $\psi_1(r; \rho)\equiv 1$ on (ρ_1, ρ) which must be a solution of (11) with $n=1$ on (ρ_1, ρ). This is clearly a contradiction, and consequently

$$\sup_{r\in(0,\tau]}\psi_1(r; \rho)\equiv\lambda_\tau<1$$

for every $\tau\in(0, \rho)$. By (12) and by the estimate right after (16), we now conclude that

$$|A_n(z, \sigma: \rho)\varphi_n(r; \rho)/\varphi_0(r; \rho)|\leqq(-4\log\rho)\lambda_\tau^n$$

for every $r\in(0, \tau]$ with $\tau\in(0, \rho)$. By (14) we have

$$\lim_{r\to 0, \sigma\to\theta}A_n(z, \sigma; \rho)\varphi_n(r; \rho)/\varphi_0(r; \rho)=A_n(z, \theta; \rho)\alpha_n(P)\varphi_0(\rho)/\varphi_n(\rho).$$

Therefore by the Weierstrass double convergence theorem

$$\lim_{r\to 0, \sigma\to\theta}G(z, re^{i\sigma})/\varphi_0(r; \rho)\equiv L(z, \theta; \rho)$$

exists, where $L(z, \theta; \rho)=c_0(z; \rho)/2+\sum_{n=1}^{\infty}A_n(z, \theta; \rho)\alpha_n(P)\varphi_0(\rho)/\varphi_n(\rho)$ for any z in $0<|z|<1$, any $\rho\in($ $, |z|]$, and any $\theta\in T$. We have to show that

$$L(z, \theta; \rho)>0 \tag{24}$$

for any z in $0<|z|<1$, any $\rho\in(0, |z|)$, and any $\theta\in T$. In fact by the Harnack principle, $z\to L(z, \theta; \rho)$ is a nonnegative solution of $\Delta u=Pu$ on $\rho<|z|<1$. If $L(z, \theta; \rho)=0$ for some z, θ, and ρ $(\leqq|z|)$, then $L(z, \theta; \rho)\equiv 0$ for every z in $\rho<|z|<1$, i.e.

$$\sum_{n=0}^{\infty} a_n c_n(z, \rho) + \sum_{n=1}^{\infty} b_n s_n(z, \rho) \equiv 0$$

on $\rho < |z| < 1$ with $a_0 = 1/2$, $a_n = \alpha_n(P)(\varphi_0(\rho)/\varphi_n(\rho)) \cos n\theta$, and $b_n = \alpha_n(P)(\varphi_0(\rho)/\varphi_n(\rho)) \sin n\theta$ $(n = 1, 2, \cdots)$. Since, as above, $|a_n|$, $|b_n| < \lambda_\tau^n$ $(\tau \in (0, \rho)$, $0 < \lambda_\tau < 1)$ for every $n = 1, 2, \cdots$, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent real series. By Theorem 2, we must have $a_n = 0$ $(n = 0, 1, \cdots)$ and $b_n = 0$ $(n = 1, 2, \cdots)$. However this is impossible since $a_0 = 1/2$, and thus we see the validity of (24). In view of

$$K(z, re^{i\sigma}) = (G(z, re^{i\sigma})/\varphi_0(r; \rho))/(G(z_0, re^{i\sigma})/\varphi_0(r; \rho)),$$

we conclude the existence of the limit

$$\lim_{r \to 0, \sigma \to \theta} K(z, re^{i\sigma}) = L(z, \theta; \rho)/L(z_0, \theta; \rho) \equiv k_0(z, \theta) \tag{25}$$

for every z in $0 < |z| < 1$ and $\theta \in T$, where ρ is any number in $(0, |z|) \cap (0, |z_0|)$. Since the left hand side term of (25) is independent of ρ,

$$L(z, \theta; \rho)/L(z_0, \theta; \rho) = L(z, \theta; \bar{\rho})/L(z_0, \theta; \bar{\rho})$$

for any ρ and $\bar{\rho}$ in $(0, |z|) \cap (0, |z_0|)$. Since $k_0(z_0, \theta) = 1$, $k_0(z, \theta) > 0$. As a counter part of $k_1(z, \theta) \in \mathcal{S}_P$, we maintain that $k_0(z, \theta) \in \mathcal{P}_P$ for every $\theta \in T$. Let $0 < \rho < |z_0|$. Since $K(z_0, re^{i\theta}) = 1$ for any $r \in (0, \rho]$, the Harnack inequality shows that $\sup\{K(z, re^{i\theta});\ |z| = |z_0|,\ r \in (0, \rho]\} = l < \infty$. Let w be the solution of $\Delta u = Pu$ on $|z_0| < |z| < 1$ with boundary values l on $|z| = |z_0|$ and 0 on $|z| = 1$. By the maximum principle,

$$0 < K(z, re^{i\sigma}) \leqq w(z)$$

on $|z_0| < |z| < 1$ for every $r \in (0, \rho]$, and therefore $0 < k_0(z, \theta) \leqq w(z)$. In particular $k_0(z, \theta)$ has the boundary values zero on $|z| = 1$, i.e. $k_0(z, \theta) \in \mathcal{P}_P$.

3.3. To study the dependence of $k_0(z, \theta)$ on $\theta \in T$ we first observe the following transition property:

$$k_0(z, \theta + \beta) = k(\theta, \beta) \cdot k_0(e^{-i\beta} z, \theta) \tag{26}$$

for $0 < |z| < 1$, where θ and $\beta \in T$ and $k(\theta, \beta)$ is a positive constant determined only by θ and β not dependent on z. Since $P(z)$ is rotation free, $G(z, re^{i(\theta+\beta)}) = G(e^{-i\beta} z, re^{i\theta})$, and hence $L(z, \theta + \beta; \rho) = L(e^{-i\beta} z, \theta; \rho)$. Therefore we have (26) with $k(\theta, \beta) = L(z_0, \theta + \beta; \rho)/L(e^{-i\beta} z_0, \theta; \rho) > 0$. In particular, (26) shows that $k_0(z, \theta)$ are simultaneously minimal or nonminimal for all $\theta \in T$. Actually we will later see in §4 that the former is the case. To study whether $k_0(z, \theta_1)$ and $k_0(z, \theta_2)$ are identical or nonproportional for $\theta_1 \neq \theta_2$ in T, we need to consider the singularity index $\alpha(P) = \alpha_1(P)$ introduced in 1.4.

Assume that $\alpha(P)=0$. By Theorem 1, $\alpha_n(P)=0$ $(n=1, 2, \cdots)$. Therefore $L(z, \theta; \rho)=c_0(z; \rho)/2$ and $k_0(z, \theta)=c_0(z; \rho)/c_0(z_0; \rho)$, i.e. $k_0(z, \theta)$ does not depend on $\theta \in T$. Hence (25) can be sharpened as follows:

$$\lim_{\zeta\to 0} K(z, \zeta)=c_0(z; \rho)/c_0(z_0; \rho) \equiv k_0(z) \tag{27}$$

exists for every $0<|z|<1$ with $\rho \in (0, |z|)$ if $\alpha(P)=0$. In general $c_0(z; \rho)=c_0(|z|; \rho)$. Thus in the present situation, $k(\theta, \beta)=L(z_0, \theta+\beta; \rho)/L(e^{-i\beta}z_0, \theta, \rho)=c_0(z_0; \rho)/c_0(e^{i\beta}z_0; \rho)=c_0(|z_0|; \rho)/c_0(|e^{-i\beta}z_0|; \rho)=1$. Thus (26) means that $k_0(z)=k_0(e^{-i\beta}z)$ for every $\beta \in T$, i.e.

$$k_0(z)=k_0(|z|)\,. \tag{28}$$

Assume next that $\alpha(P)>0$. By Theorem 1, $\alpha_n(P)>0$ $(n=1, 2, \cdots)$. Suppose $k_0(z, \theta_1)\equiv k_0(z, \theta_2)$ on $0<|z|<1$ for θ_1 and θ_2 in T. Fix a $\rho \in (0, |z_0|)$. By (25) we have $L(z_0, \theta_2; \rho)L(z, \theta_1; \rho)-L(z_0, \theta_1; \rho)L(z, \theta_2; \rho)\equiv 0$ on $\rho<|z|<1$. This relation can be rewritten as

$$\sum_{n=0}^{\infty} a_n c_n(z; \rho)+\sum_{n=1}^{\infty} b_n s_n(z; \rho)\equiv 0$$

for $\rho<|z|<1$, where $a_0=(L(z_0, \theta_2; \rho)-L(z_0, \theta_1; \rho))/2$ and

$$\begin{cases} a_n=(L(z_0, \theta_2; \rho)\cos n\theta_1-L(z_0, \theta_1; \rho)\cos n\theta_2)\alpha_n(P)(\varphi_0(\rho)/\varphi_n(\rho))\,, \\ b_n=(L(z_0, \theta_2; \rho)\sin n\theta_1-L(z_0, \theta_1; \rho)\sin n\theta_2)\alpha_n(P)(\varphi_0(\rho)/\varphi_n(\rho))\,. \end{cases}$$

Here, as in 3.2, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent real series and Theorem 2 implies that $a_n=0$ $(n=0, 1, \cdots)$ and $b_n=0$ $(n=1, 2, \cdots)$. First $a_0=0$ implies $L(z_0, \theta_2; \rho)=L(z_0, \theta_1; \rho)$, and then, in view of $\alpha_n(P)(\varphi_0(\rho)/\varphi_n(\rho))>0$, $a_n=b_n=0$ $(n=1, 2, \cdots)$ imply that $\cos n\theta_1=\cos n\theta_2$ and $\sin n\theta_1=\sin n\theta_2$ for every $n=1, 2, \cdots$. Only a part of this (i.e. for $n=1$) implies that $\theta_1=\theta_2$ in T. If $k_0(z, \theta_i)$ $(i=1, 2)$ are proportional, then, since $k_0(z_0, \theta_i)=1$, $k_0(z, \theta_1)\equiv k_0(z, \theta_2)$. Therefore $k_0(z, \theta_1)$ and $k_0(z, \theta_2)$ are nonproportional on $0<|z|<1$ for $\theta_1 \neq \theta_2$ in T if $\alpha(P)>0$.

§ 4. Martin compactification.

4.1. We denote by U the punctured disk $0<|z|<1$. Using the set $\{z_n\}_1^\infty$ of rational points z_n in U, we consider a metric $d=d_P$ on U given by the following:

$$d(\zeta_1, \zeta_2)=\sum_{n=1}^{\infty} \frac{1}{2^n}\cdot\frac{|K(z_n, \zeta_1)-K(z_n, \zeta_2)|}{1+|K(z_n, \zeta_1)-K(z_n, \zeta_2)|} \tag{29}$$

for ζ_1 and ζ_2 in U, where $K(z, \zeta)=K_P(z, \zeta)=G_P(z, \zeta)/G_P(z_0, \zeta)$ is the Martin kernel on U with respect to $\Delta u=Pu$ with P a rotation free density on

$0<|z|\leqq 1$. We denote by $U^*=U_P^*$ the completion of U with respect to d_P. As a consequence of the Harnack principle for nonnegative solutions of $\Delta u=Pu$, the completion U^* is in fact a compactification of U, and for this reason, $U^*=U_P^*$ is referred to as the *Martin compactification* of U with respect to P or to the equation $\Delta u=Pu$. The function $(z, \zeta)\to K(z, \zeta)$ is continuously extendable to $U\times U^*$ and the extended function $K(z, \zeta^*)$ $((z, \zeta^*)\in U\times U^*)$ is again called the *Martin kernel* with respect to P. The function $z\to K(z, \zeta^*)$ is a positive solution of $\Delta u=Pu$ on $U-\{\zeta^*\}$ for every fixed $\zeta^*\in U^*$. Then the extension of the metric d to U^* is given by

$$d(\zeta_1^*, \zeta_2^*)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{|K(z_n, \zeta_1^*)-K(z_n, \zeta_2^*)|}{1+|K(z_n, \zeta_1^*)-K(z_n, \zeta_2^*)|}$$

for ζ_1^* and ζ_2^* in U^*. A point ζ^* in the boundary U^*-U (the *Martin boundary* of U with respect to P) is said to be *minimal* if $K(z, \zeta^*)$ is a minimal solution in $\mathcal{P}_P$. The set of minimal point will be denoted by $\Gamma=\Gamma_P$. It is easy to see that the compactification U^* as well as the minimality of boundary points are determined only by P independent of the choice of the reference point $z_0\in U$. The fundamental theorem of Martin says that there exists a bijective correspondence $u\leftrightarrow\mu$ between the class $\mathcal{P}_P$ and the class of regular Borel measures μ on U^*-U with $\mu((U^*-U)-\Gamma)=0$ such that

$$u(z)=\int_{\Gamma}K(z, \zeta^*)d\mu(\zeta^*)\,. \tag{30}$$

Martin [**9**] proved (30) for the case of subregion U in Euclidean ε-space with $P\equiv 0$ (see a comprehensive exposition of Constantinescu-Cornea [**3**]). For a proof of (30) for the case of open Riemann surface U with general density P we refer to Nakai [**13**] and, for more general case, to Itô [**6**] and Šur [**18**], among others. We are going to determine the exact shapes of U^* and $K(z, \zeta^*)$ in order to compute $\dim_P 0$.

4.2. We define a projection $\pi: U^*\to(0\leqq|\zeta|\leqq 1)$ as a surjective continuous mapping as follows. First let $\pi(\zeta)=\zeta$ for $\zeta\in U$. Let $\zeta^*\in U^*-U$ and $\{\zeta_n\}$ be a sequence in U such that $d(\zeta_n, \zeta^*)\to 0$ $(n\to\infty)$. We assert that $\{\zeta_n\}$ is convergent in $0\leqq|\zeta|\leqq 1$. Observe that $\{\zeta_n\}$ does not accumulate in U and that $K(z, \zeta_n)\to K(z, \zeta^*)$ $(n\to\infty)$ uniformly on each compact subset of U. If $\{\zeta_n\}$ were not convergent in $0\leqq|\zeta|\leqq 1$, there would exist two subsequences $\{\zeta_n'\}$ and $\{\zeta_n''\}$ such that $\zeta_n'\to\zeta'$ and $\zeta_n''\to\zeta''$ $(n\to\infty)$ in $0\leqq|\zeta|\leqq 1$ with $|\zeta'|, |\zeta''|=0$ or 1 and $\zeta'\neq\zeta''$. Suppose $\zeta'=e^{i\theta'}$. Then $|\zeta''|$ must be 1. For, if $\zeta''=0$, then we can choose a subsequence $\{\zeta_n'''\}$ of $\{\zeta_n''\}$ such that $\sigma_n=\arg\zeta_n'''\to\theta'''$ in T. Then by (23) and (25), $\lim_n K(z, \zeta_n')=k_1(z, \theta')$ and $\lim_n K(z, \zeta_n''')=k_0(z, \theta''')$. On the other hand, since $\lim_n K(z, \zeta_n)=\lim_n K(z, \zeta_n')=\lim_n K(z, \zeta_n''')$, we must have $K(z, \zeta^*)\equiv k_1(z, \theta')\equiv k_0(z, \theta''')$. But this is impossible because $k_1\in\mathcal{S}_P$,

$k_0 \in \mathcal{F}_P$, and $\mathcal{S}_P \cap \mathcal{F}_P = \{0\}$. Therefore $\zeta'' = e^{i\theta''}$. Again by (23), $\lim_n K(z, \zeta_n'') = k_1(z, \theta'')$, and hence $k_1(z, \theta') = k_1(z, \theta'')$. Again by (23) we have $\theta' = \theta''$ in T, which shows that $\zeta' = \zeta''$, a contradiction. Hence we have seen that if $\{\zeta_n\}$ contains a subsequence converging to a point on $|\zeta| = 1$, then $\{\zeta_n\}$ is convergent. If $\{\zeta_n\}$ does not contain any such subsequence, then $\{\zeta_n\}$ itself converges to 0. Thus we have seen that $\{\zeta_n\}$ is convergent to a ζ in $|\zeta| = 0$ or 1. By the same argument as above we see that ζ is determined only by ζ^* not depending on the choice of $\{\zeta_n\}$. We define $\pi(\zeta^*) = \zeta$. Conversely give any ζ in $|\zeta| = 0$ or 1. Let $\{\zeta_n\}$ be a sequence in U converging to ζ in $0 \leqq |\zeta| \leqq 1$. The compactness of U^* assures the existence of a subsequence $\{\zeta_n'\}$ of $\{\zeta_n\}$ such that $\zeta_n' \to \zeta^* \in U^* - U$ $(n \to \infty)$. Then obviously $\pi(\zeta^*) = \zeta$. Thus clearly $\pi : U^* \to (0 \leqq |\zeta| \leqq 1)$ is a surjective continuous mapping. We moreover have that $\pi : \pi^{-1}(0 < |\zeta| \leqq 1) \to (0 < |\zeta| \leqq 1)$ is homeomorphic. To see this we only have to show that $\pi^{-1}(\zeta)$ with $|\zeta| = 1$ contains only a single point. Let ζ_1^* and ζ_2^* be in $\pi^{-1}(\zeta)$ and $\theta = \arg \zeta$ in T. Then $K(z, \zeta_j^*) \equiv k_1(z, \theta)$ $(j = 1, 2)$ and consequently $\zeta_1^* = \zeta_2^*$. Denote by $\Gamma_1 = \pi^{-1}(|\zeta| = 1)$. Again by (23) we see that $\Gamma_1 \subset \Gamma$ and $K(z, \zeta^*) = k_1(z, \arg \pi(\zeta^*))$ for every ζ^* in Γ_1.

Next let $\zeta^* \in \pi^{-1}(0)$ and $\{\zeta_n\}$ be a sequence in U with $d(\zeta_n, \zeta^*) \to 0$ $(n \to \infty)$. Let $\{\zeta_n'\}$ be a subsequence of $\{\zeta_n\}$ such that $\arg \zeta_n' \to \theta$ $(n \to \infty)$. By (25), $K(z, \zeta^*) = k_0(z, \theta)$. Therefore by (26), $K(z, \zeta^*)$ are simultaneously minimal or nonminimal for all $\zeta^* \in \pi^{-1}(0)$. If $K(z, \zeta^*)$ were nonminimal for all $\zeta^* \in \pi^{-1}(0)$, then $\Gamma = \Gamma \cap (U^* - U) = (\Gamma \cap \pi^{-1}(0)) \cup (\Gamma \cap \pi^{-1}(|\zeta| = 1)) = \Gamma_1$. Let u be in $\mathcal{F}_P$ with $u > 0$. For example let $u(z) = k_1(z, \theta)$. By (30) there exists a regular Borel measure μ with $\mu((U^* - U) - \Gamma_1) = 0$ such that

$$u(z) = \int_{\Gamma_1} K(z, \zeta^*) d\mu(\zeta^*) .$$

By $K(z_0, \zeta^*) = 1$ for any $\zeta^* \in \Gamma_1$, the Harnack inequality shows that $K(z, \zeta^*) < k < \infty$ for every z in $|z| = |z_0|$ and every $\zeta^* \in \Gamma_1$. Since $K(z, \zeta^*) = k_1(z, \arg \pi(\zeta^*)) \in \mathcal{S}_P$ for $\zeta^* \in \Gamma_1$, i.e. $K(z, \zeta^*)$ is bounded in a punctured neighborhood of $z = 0$ for every $\zeta^* \in \Gamma_1$, the maximum principle yields $K(z, \zeta^*) \leqq \varepsilon \log(|z_0|/|z|) + k$ on $0 < |z| \leqq |z_0|$ for every $\varepsilon > 0$, and thus $K(z, \zeta^*) \leqq k$ on $0 < |z| \leqq |z_0|$ for every $\zeta^* \in \Gamma_1$. Since $u(z_0) = \mu(\Gamma_1)$,

$$u(z) = \int_{\Gamma_1} K(z, \zeta^*) d\mu(\zeta^*) \leqq \int_{\Gamma_1} k d\mu(\zeta^*) = k u(z_0)$$

on $0 < |z| \leqq |z_0|$, i.e. $u \in \mathcal{S}_P$. Therefore $u \in \mathcal{F}_P \cap \mathcal{S}_P = \{0\}$ contradicts $u > 0$. This means that every $\zeta^* \in \pi^{-1}(0)$ is minimal and thus

$$U^* - U = \Gamma = \pi^{-1}(0) \cup \pi^{-1}(|\zeta| = 1) . \tag{31}$$

We state this in the following

THEOREM 3. *Every boundary point of the Martin compactification* U_P^* *of* *the punctured disk $U: 0<|z|<1$ with respect to any rotation free density P on the closed punctured disk $0<|z|\leqq 1$ is minimal.*

In the proof of (31) we have shown that if μ is a regular Borel measure on U^*-U then $\int_{\Gamma_1} K(\cdot, \zeta^*)d\mu(\zeta^*)\in \mathcal{S}_P$. As above $K(z, \zeta^*)<k<\infty$ on $|z|=|z_0|$ for every $\zeta^*\in\pi^{-1}(0)$. Let w be the solution of $\Delta u=Pu$ on $|z_0|<|z|<1$ with boundary values k on $|z|=|z_0|$ and 0 on $|z|=1$. Since $K(z, \zeta^*)=k_0(z, \theta)$ for some $\theta\in T$, $K(z, \zeta^*)\in \mathcal{F}_P$ and by the maximum principle $K(z, \zeta^*)\leqq w(z)$ for every z in $|z_0|<|z|<1$ and $\zeta^*\in\pi^{-1}(0)$. Let μ be any regular Borel measure on U^*-U. Then $\mu(\pi^{-1}(0))=c<\infty$ since $\pi^{-1}(0)$ is compact. Therefore $\int_{\pi^{-1}(0)} K(z, \zeta^*)d\mu(\zeta^*)\leqq cw(z)$ on $|z_0|<|z|<1$. This shows that $\int_{\pi^{-1}(0)} K(\cdot, \zeta^*)d\mu(\zeta^*)\in\mathcal{F}_P$. These with (30) and (31) imply the following

THEOREM 4. *There exists a bijective correspondence $u\leftrightarrow\mu$ between the class $\mathcal{F}_P$ and the class of regular Borel measures μ on $\pi^{-1}(0)$ such that*

$$u=\int_{\pi^{-1}(0)} K(\cdot, \zeta^*)d\mu(\zeta^*)\,. \tag{32}$$

4.3. By the above theorem the dimension of the half module $\mathcal{F}_P$ is identical with the dimension of the dual space $C(\pi^{-1}(0))^*$ of the space $C(\pi^{-1}(0))$ of continuous functions on the compact metric space $\pi^{-1}(0)$, which in turn is identical with the cardinal number of $\pi^{-1}(0)$. We only have to determine $\pi^{-1}(0)$. For this purpose we need to consider the following two cases separately: $\alpha(P)=0$ and $\alpha(P)>0$. First suppose that $\alpha(P)=0$. Then by (27), $K(z, \zeta^*)=k_0(z)$ for every $\zeta^*\in\pi^{-1}(0)$, i.e. $\pi^{-1}(0)$ consists of a single point ζ_0^*. Therefore $\pi: U^*\to(0\leqq|\zeta|\leqq 1)$ is homeomorphic, i.e. U^* is homeomorphic to a closed disk. Next suppose that $\alpha(P)>0$. For $\zeta^*\in U^*$ let $r(\zeta^*)\in[0, 1]$ and $\theta(\zeta^*)\in T$ be defined as follows. First if $\pi(\zeta^*)\neq 0$, then $r(\zeta^*)=|\pi(\zeta^*)|$ and $\theta(\zeta^*)=\arg\pi(\zeta^*)$. If $\pi(\zeta^*)=0$, then by the first part of 4.2, $K(z, \zeta^*)=k_0(z, \theta)$ for some $\theta\in T$ and by the last part of 3.3 such θ is uniquely determined in T. In this case we set $r(\zeta^*)=0$ and $\theta(\zeta^*)=\theta$. By (23) and (25) we can easily deduce that $r: U^*\to[0, 1]$ and $\theta: U^*\to T$ are continuous. Therefore $(r, \theta): U^*\to[0, 1]\times T$ is continuous, and by 3.1 and 3.3, it is bijective. This means that the homeomorphism $\rho: U\to(0, 1)\times T$ given by $\rho(z)=(|z|, \arg z)$ can be extended to a homeomorphism $\rho^*: U^*\to[0, 1]\times T$. To describe the above two cases simultaneously we consider the homeomorphism $\rho_P: U\to(\alpha(P)<|z|<1)$ given by

$$\rho_P(z)=[(1-\alpha(P))|z|+\alpha(P)]e^{i\arg z}\,.$$

We maintain that ρ_P is extended to a homeomorphism $\rho_P^*: U^*\to(\alpha(P)\leqq|z|\leqq 1)$. If $\alpha(P)=0$, then $\rho_P=\pi$ on U and the assertion follows. If $\alpha(P)>0$, then $\tau: [0, 1]\times T\to(\alpha(P)\leqq|z|\leqq 1)$ given by

$$\tau(t, \theta)=[(1-\alpha(P))t+\alpha(P)]e^{i\theta}$$

is a homeomorphism and $\rho_P=\tau\circ\rho$. Then $\rho_P^*=\tau\circ\rho^*$ is a homeomorphism of U^* onto $\alpha(P)\leqq|z|\leqq 1$. We thus can state our *main result* in this paper as follows:

THEOREM 5. *The Martin compactification U_P^* of the punctured disk $U: 0<|z|<1$ with respect to any rotation free density P on the closed punctured disk $0<|z|\leqq 1$ is homeomorphic to the closed annulus $A_P: \alpha(P)\leqq|z|\leqq 1$ with $\alpha(P)$ the singularity index of P at $z=0$.*

We simply write $K_P(z, \zeta)$ instead of $K_P(z, \rho_P^{*-1}(\zeta))$ for $\zeta\in A_P$. By (32) we have $\mathcal{P}_P=\{\mu\cdot K_P(z, 0); \mu\in R$ (real numbers)$\}$ for $\alpha(P)=0$, i.e. $\dim\mathcal{P}_P=1$ for $\alpha(P)=0$, and that there exists a bijective correspondence $u\leftrightarrow\mu$ between $\mathcal{P}_P$ and the class of regular Borel measures μ on $[0, 2\pi]$ such that

$$u=\int_0^{2\pi}K_P(\cdot, \alpha(P)e^{i\theta})d\mu(\theta) \tag{33}$$

if $\alpha(P)>0$, i.e. $\dim\mathcal{P}_P=\mathfrak{c}$ (the cardinal number of continuum) for $\alpha(P)>0$.

§ 5. Picard principle.

5.1. Let Q be a density (nonnegative locally Hölder continuous function) on a plane region D and w be a boundary point of D such that $\{0<|z-w|<\varepsilon\}\subset D$ for some $\varepsilon>0$. Let $\mathcal{P}_Q(w, \varepsilon)$ be the half module of nonnegative solution u of $\Delta u=Qu$ on $0<|z-w|<\varepsilon$ with boundary values 0 on $|z-w|=\varepsilon$. It is easy to see that $\mathcal{P}_Q(w, \varepsilon)$ are mutually isomorphic for all admissible $\varepsilon>0$ and thus we can consider the common half module structure $\mathcal{P}_Q(w)$. We call the dimension $\dim\mathcal{P}_Q(w)$ the *Q-elliptic dimension* of w, $\dim_Q w$ in notation, where $\dim\mathcal{P}_Q(w)$ is the minimal cardinal number of sets of generators of $\mathcal{P}_Q(w)$. The notion $\dim_Q w$ can be given as described in the introduction to an isolated parabolic ideal boundary component of a Riemann surface but at present we confine ourselves to the case of zero genus. By a translation and a similarity we only have to consider $\dim_Q 0=\dim\mathcal{P}_Q(0, 1)$ for the study of Q-elliptic dimension. We say that the *Picard principle* is valid for Q at 0 if $\dim_Q 0=1$. Let $G(z, \zeta)$ be the Green's function of $\Delta u=Qu$ on $0<|z|<1$. The validity of the Picard principle is equivalent to the existence of the limit

$$\lim_{\zeta\to 0}G(z, \zeta)/G(z_0, \zeta)$$

for every $0<|z|<1$ with a fixed z_0 in $0<|z|<1$. A straightforward sufficient condition for the existence of the above limit is the *boundedness* of Q. It seems to be very difficult to give a comprehensive necessary and sufficient condition on Q for $\dim_Q 0=1$ (cf. Brelot [1]). We only have it for rotation free densities as a direct consequence of 4.3:

THEOREM 6. *The P-elliptic dimension of* 0 *for any rotation free density* P *around* 0 *is either* 1 *or the cardinal number of continuum* $\mathfrak{c}$ *according as the singularity index* $\alpha(P)$ *of* P *at* 0 *is zero or strictly positive. In particular the Picard principle is valid if and only if* $\alpha(P)=0$.

There is left a task to show, in fact, that there exist rotation free densities P with $\alpha(P)=0$ or $\alpha(P)>0$. Then the range of the mapping $P\rightarrow \dim_P 0$ from the class of rotation free densities P is the two element set $\{1, \mathfrak{c}\}$. Instead of finding individual P with $\alpha(P)=0$ or $\alpha(P)>0$, we will compute $\alpha(|z|^{-\lambda})$ for $\lambda\in[-\infty, \infty)$ and will show that $\alpha(|z|^{-\lambda})=0$ for $\lambda\in[-\infty, 2]$ and $\alpha(|z|^{-\lambda})>0$ for $\lambda\in(2, \infty)$. In this connection we insert here an open question. Let P_1 and P_2 be *arbitrary* densities on an open Riemann surface R and $P_j^+(R)$ be the half module of nonnegative solutions of $\Delta u=P_j u$ on R $(j=1,2)$. We know that $\dim P_j^+(R)\geqq 1$ (Myrberg [**12**]). Our first question is: What would be the "natural isomorphism" between $P_1^+(R)$ and $P_2^+(R)$? Assume there exists a constant $c\in[1, \infty)$ such that

$$c^{-1}P_1(z)\leqq P_2(z)\leqq cP_1(z) \tag{34}$$

on R. After the first question is somehow settled we next ask: Is the *order comparison theorem* valid, i.e. are $P_1^+(R)$ and $P_2^+(R)$ "naturally isomorphic" if (34) is postulated? These types of questions are positively settled for all known modules of solutions with various boundedness properties such as sup-norm finiteness, Dirichlet-norm finiteness, energy finiteness, etc. (Royden [**17**], Nakai [**14**]). For the present class $\mathscr{F}_P(0, 1)$ with rotation free P the first question is of no interest or of trivial character, and the second question should be asked whether (34) implies $\dim_{P_1}0=\dim_{P_2}0$. It seems quite likely that this is the case for rotation free densities.

5.2. Let $P_\lambda(z)=|z|^{-\lambda}$ on $0<|z|<1$ with $\lambda\in[-\infty, \infty)$. Here we understand that $P_{-\infty}(z)\equiv 0$. Clearly P_λ can be extended to a density on $0<|z|\leqq 1$. We will show that

$$\alpha(P_\lambda)=0 \quad (\lambda\in[-\infty, 2]), \qquad \alpha(P_\lambda)>0 \ (\lambda\in(2, \infty)). \tag{35}$$

First let $\lambda\in[-\infty, 2]$. We estimate $\varphi_0(t;\rho)$ and $\varphi_1(t;\rho)$ for the density P_λ, where $P_{-\infty}\equiv 0$. Observe that $|z|^c$ $(c>0)$ is a rotation free solution of $\Delta u(z)=Q_c(z)u(z)$ with $Q_c(z)=c^2|z|^{-2}$ on $0<|z|<1$. Let $\lambda=2$. Then $\varphi_0(t)=t$ and $\varphi_1(t)=t^{\sqrt{2}}$ and thus $\varphi_1(t)/\varphi_0(t)=t^{\sqrt{2}-1}$. Therefore $\alpha(P_2)=\lim_{t\to 0}\varphi_1(t)/\varphi_0(t)=0$. Similarly trivial is the case $\lambda=-\infty$. In this case $\varphi_0(t)=1$ and $\varphi_1(t)=t$, and thus $\alpha(P_{-\infty})=\lim_{t\to 0}\varphi_1(t)/\varphi_0(t)=0$. For $-\infty<\lambda<2$ let $\lambda=2-2\mu$ $(\mu>0)$. Fix a $\rho\in(0,1)$. Then $P_\lambda(z)=|z|^{2\mu}\cdot|z|^{-2}\leqq\rho^{2\mu}|z|^{-2}=Q_{\rho^\mu}(z)$ on $0<|z|\leqq\rho$. The function

$$w(z)=\varphi_0(|z|;\rho)-|z|^{\rho^\mu}/\rho^{\rho^\mu}+\varepsilon\log(\rho/|z|)$$

has nonnegative boundary values (including $+\infty$) on the boundary of $0<|z|<\rho$ and satisfies $\Delta w(z)=P_{\lambda}(z)\varphi_0(|z|;\rho)-Q_{\rho\mu}(z)(|z|^{\rho\mu}/\rho^{\rho\mu})\leqq P_{\lambda}(z)(\varphi_0(|z|;\rho)-|z|^{\rho\mu}/\rho^{\rho\mu})$, i.e. $(\Delta-P_{\lambda}(z))w(z)\leqq 0$ for every $\varepsilon>0$. The maximum principle yields $w(z)\geqq 0$ and thus $\varphi_0(|z|;\rho)\geqq|z|^{\rho\mu}/\rho^{\rho\mu}$ on $0<|z|<\rho$. On the other hand, $P_{\lambda}(z)+|z|^{-2}\geqq Q_1(z)$. Since $\Delta\varphi_1(|z|;\rho)=(P_{\lambda}(z)+|z|^{-2})\varphi_1(|z|;\rho)$ and $\Delta(|z|/\rho)=Q_1(z)(|z|/\rho)$ on $0<|z|<\rho$, the maximum principle applied as above yields $\varphi_1(|z|;\rho)\leqq|z|/\rho$ on $0<|z|<\rho$. Therefore $\varphi_1(t;\rho)/\varphi_0(t;\rho)\leqq(t/\rho)/(t^{\rho\mu}/\rho^{\rho\mu})$, i.e.

$$\varphi_1(t)/\varphi_0(t)\leqq(\varphi_1(\rho)/\varphi_0(\rho))\cdot\rho^{\rho\mu-1}\cdot t^{1-\rho\mu}$$

for $t\in(0,\rho]$. Since $1-\rho^{\mu}>0$, we conclude that $\alpha(P_{\lambda})=\lim_{t\to 0}\varphi_1(t)/\varphi_0(t)=0$.

Next we show $\alpha(P_{\lambda})>0$ for $\lambda\in(2,\infty)$. We will do this by an indirect way. Set $\lambda=2+2\mu$ $(\mu>0)$. Observe that functions

$$\begin{cases} c_{\mu}(z)=\exp((\cos\mu\theta)/\mu r^{\mu}) \\ s_{\mu}(z)=\exp((\sin\mu\theta)/\mu r^{\mu}) \end{cases}$$

are solutions of $\Delta u(z)=P_{\lambda}(z)u(z)$ on $0<|z|\leqq 1$, where $z=re^{i\theta}$. Let $\bar{c}_{n\mu}(z)$ ($\bar{s}_{n\mu}(z)$, resp.) be the solution of $\Delta u=P_{\lambda}u$ on $n^{-1}<|z|<1$ with boundary values $c_{\mu}(s_{\mu}$, resp.) on $|z|=1$ and 0 (0, resp.) on $|z|=n^{-1}$ $(n=2,3,\cdots)$. By the maximum principle, $\bar{c}_{n\mu}\leqq c_{\mu}$ ($\bar{s}_{n\mu}\leqq s_{\mu}$, resp.) and $\{\bar{c}_{n\mu}\}$ ($\{\bar{s}_{n\mu}\}$, resp.) is increasing $(n=1,2,\cdots)$. Therefore $\bar{c}_{\mu}=\lim_{n\to\infty}\bar{c}_{n\mu}$ ($\bar{s}_{\mu}=\lim_{n\to\infty}\bar{s}_{n\mu}$) exists and is a *bounded* solution of $\Delta u=P_{\lambda}u$ on $0<|z|<1$ with boundary values c_{μ} (s_{μ}, resp.) on $|z|=1$ and $\bar{c}_{\mu}\leqq c_{\mu}$ ($\bar{s}_{\mu}\leqq s_{\mu}$, resp.). Then $c_{\mu}^{*}=c_{\mu}-\bar{c}_{\mu}\in\mathcal{P}_P(0,1)$ ($s_{\mu}^{*}=s_{\mu}-\bar{s}_{\mu}\in\mathcal{P}_P(0,1)$, resp.). Obviously c_{μ}^{*} and s_{μ}^{*} are strictly positive since c_{μ} and s_{μ} are unbounded. Contrary to the assertion assume that $\alpha(P_{\lambda})=0$. Then Theorem 5 would imply $\dim\mathcal{P}_P(0,1)=1$, i.e. there exists a constant $a\in(0,\infty)$ such that $c_{\mu}^{*}\equiv as_{\mu}^{*}$ on $0<|z|<1$. Consider functions c_{μ}^{*} and s_{μ}^{*} on the radius $l_0:\arg z=0$. Then $c_{\mu}^{*}(r)=\exp(1/\mu r^{\mu})+\mathcal{O}(1)$ and $s_{\mu}^{*}(r)=\mathcal{O}(1)$ as $r\to 0$ on l_0. Therefore $c_{\mu}^{*}=as_{\mu}^{*}$ implies a contradiction: $\exp(1/\mu r^{\mu})=\mathcal{O}(1)$ on l_0 as $r\to 0$. Thus we must have $\alpha(P_{\lambda})>0$ for $\lambda\in(2,\infty)$.

Appendix: The Riemann theorem.

A.1. Closely related to the Picard principle is the Riemann theorem. Let P be a density on $0<|z|\leqq 1$. We say that the *Riemann theorem* is valid for P at $z=0$ if the limit $\lim_{z\to 0}u(z)$ exists for every *bounded* solution of $\Delta u=Pu$ on $0<|z|\leqq 1$. That the Riemann theorem need not necessarily be valid for every density P is examplified by the following: Let $\lambda\in(0,1)$ and $\{a_n\}$ be a sequence in $(0,1)$ such that $\lambda^n<a_n<\lambda^{n-1}$ $(n=1,2,\cdots)$. Let

$$0 < r_n < \min(e^{-2^n n}, (\lambda^{n-1} - a_n)/2, (a_n - \lambda^n)/2)$$

and $B_n(\rho)$ be the open disk about a_n with radius $\rho > 0$. Let w be the harmonic Green's function of the region $D = \{|z| < 3\} - \bigcup_{n=1}^{\infty} \overline{B_n(r_n)}$ with pole at $z = 2$. We continue w to $\{|z| < 3\}$ by setting $w = 0$ on $\bigcup_{n=1}^{\infty} \overline{B_n(r_n)}$. Observe that the harmonic capacity γ_n of $\overline{B_n(r_n)}$ is r_n which is, by the choice of r_n, less than $e^{-2^n n}$ and therefore

$$\sum_{n=1}^{\infty} n(\log \gamma_n^{-1})^{-1} < \sum_{n=1}^{\infty} 2^{-n} < \infty .$$

The Wiener criterion (cf. e.g. Kellog [7]) then assures that $z = 0$ is an irregular boundary point of the region D. By the Bouligand theorem (cf. e.g. Tsuji [19]),

$$\limsup_{z \to 0} w(z) > 0 .$$

Let $\{b_n\}$ be a sequence in $\{0 < |z| < 1\} - \bigcup_{n=1}^{\infty} \overline{B_n(r_n)}$ such that $\lim_n b_n = 0$ and $\lim_n w(b_n) = b > 0$. Let $r_n' \in (r_n, 2r_n)$ be such that $0 < w < b/2$ in $B(r_n') - B(r_n)$ for each $n = 1, 2, \cdots$. We modify w in each $B_n(r_n') - \overline{B_n(r_n/2)}$ $(n = 1, 2, \cdots)$ so that the resulting function ω is c^2 subharmonic in $0 < |z| \leqq 1$ such that $\omega | \left(\{0 < |z| \leqq 1\} - \bigcup_{n=1}^{\infty} \overline{B_n(r_n')} \right) \cup \left(\bigcup_{n=1}^{\infty} \overline{B_n(r_n/2)} \right) = w$. Set

$$P(z) = (\Delta \omega(z))/(\omega(z) + 1) \tag{36}$$

on $0 < |z| \leqq 1$. Then $P(z)$ is a density on $0 < |z| \leqq 1$ and $u(z) = \omega(z) + 1$ is a bounded solution of $\Delta u = Pu$ on $0 < |z| \leqq 1$. Clearly $\lim_n a_n = \lim_n b_n = 0$ but

$$\lim_{n \to \infty} u(a_n) = 1 \neq 1 + b = \lim_{n \to \infty} u(b_n) .$$

Therefore the Riemann theorem is not valid at $z = 0$ for the density P given by (36). A trivial sufficient condition for P for which the Riemann theorem is valid is the existence of the limit

$$\lim_{z \to 0} \int_{0 < |\zeta| < 1} P(\zeta) \log \left| \frac{1 - \bar{\zeta} z}{z - \zeta} \right| d\xi d\eta \qquad (\zeta = \xi + i\eta) \tag{37}$$

(see (40) below) and this is the case for densities P which are bounded in $0 < |z| < 1$ (cf. e.g. Miranda [10]). We have seen in the main text that the Picard principle may not necessarily be valid for rotation free densities. In contrast we will see that *the Riemann theorem is always valid for every rotation free density*, and in fact for a bit more general densities which we call almost rotation free. This is one of the motivation for that we feel the Picard principle is more delicate recognition than the Riemann theorem.

A.2. We say that a density $P(z)$ on $0<|z|<1$ is *almost rotation free* if there exists a constant $c\in[1, \infty)$ such that

$$c^{-1}P(|z|)\leqq P(z)\leqq cP(|z|) \tag{38}$$

for every z in $0<|z|\leqq 1$. A rotation free density is a special member of the class of almost rotation free densities, i.e. $c=1$. We maintain:

THEOREM 7. *The Riemann theorem is valid at $z=0$ for every almost rotation free density P on $0<|z|\leqq 1$, i.e. $\lim_{z\to 0} u(z)$ exists for every bounded solution of $\Delta u=Pu$ on $0<|z|\leqq 1$.*

Take an arbitrary bounded solution u of $\Delta u=Pu$ on $0<|z|\leqq 1$. We have to show the existence of $\lim_{z\to 0} u(z)$. Let u_n^+ (u_n^-, resp.) be the solution of $\Delta u=Pu$ on $1/n\leqq|z|\leqq 1$ with boundary values $\max(u(z), 0)$ ($\max(-u(z), 0)$, resp.) $(n=2, 3, \cdots)$. Since $\{u_n^+\}$ ($\{u_n^-\}$, resp.) is increasing and bounded by $\sup_{0<|z|<1}|u|$, by the maximum principle, $u^+=\lim_n u_n^+$ ($u^-=\lim_n u_n^-$, resp.) exists and is a nonnegative bounded solution of $\Delta u=Pu$ on $0<|z|<1$. Clearly $u=u_n^+-u_n^-$ on $1/n\leqq|z|\leqq 1$ and thus $u=u^+-u^-$ on $0<|z|<1$. Therefore we may assume without loss of generality that $u\geqq 0$ on $0<|z|<1$. Let $Q(z)=c^{-1}P(|z|)$. Then Q is a rotation free density on $0<|z|\leqq 1$ such that

$$Q(z)\leqq P(z)\leqq c^2Q(z) \tag{39}$$

for every z in $0<|z|\leqq 1$. We denote by $v_n(z)$ the solution of $\Delta v=Qv$ on $1/n<|z|<1$ with boundary values u $(n=2, 3, \cdots)$. By the maximum principle, $u\leqq v_n\leqq \sup_{0<|z|\leqq 1} u(z)$ and thus $\{v_n\}$ is increasing. Hence $v=\lim_n v_n$ exists on $0<|z|\leqq 1$ and v is a bounded solution of $\Delta v=Qv$ on $0<|z|<1$ with boundary values u on $|z|=1$ and $u\leqq v$ on $0<|z|<1$. Let $H_n(z, \zeta)$ be the *harmonic* Green's function on $1/n<|z|<1$ $(n=2, 3, \cdots)$ and set

$$h_n(z)=v(z)+\frac{1}{2\pi}\int_{1/n<|z|<1} H_n(z, \zeta)Q(\zeta)v(\zeta)d\xi d\eta$$

for z in $1/n<|z|<1$. Since

$$\Delta_z\Big(\frac{1}{2\pi}\int_{1/n<|z|<1} H_n(z, \zeta)Q(\zeta)v(\zeta)d\xi d\eta\Big)=-Q(z)v(z)$$

in the genuine sense (cf. e.g. Miranda [**10**]), h_n is harmonic on $1/n<|z|<1$ with boundary values v, and therefore the maximum principle yields $0\leqq h_n \leqq \sup_{0<|z|<1} v(z)$ there. Observe that $\lim_{n\to\infty} H_n(z, \zeta)=H(z, \zeta)=\log(|1-\bar{\zeta}z|/|z-\bar{\zeta}|)$ increasingly on $|z|\leqq 1$. By the Lebesgue-Fatou theorem we conclude that $h=\lim_n h_n$ exists on $0<|z|\leqq 1$ and

$$h(z)=v(z)+\frac{1}{2\pi}\int_{0<|\zeta|<1} H(z, \zeta)Q(\zeta)v(\zeta)d\xi d\eta\,. \tag{40}$$

Here $h(z)$ is bounded and harmonic on $0<|z|<1$ and, by the classical Riemann theorem, $h(z)$ is harmonic on $|z|<1$ with boundary values $v=u$. By the same reasoning as above we also have

(41) $$h(z)=u(z)+\frac{1}{2\pi}\int_{0<|\zeta|<1} H(z, \zeta)P(\zeta)u(\zeta)d\xi d\eta\,.$$

Let the sequence $\{\varphi_n(t)\}_{n=0}^{\infty}$ be given as in 1.1 with respect to the density Q. Fix an arbitrary $t\in(0, 1)$ and set

$$\begin{cases} c_n(t)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} v(te^{i\theta})\cos n\theta d\theta & (n=0, 1, \cdots)\,, \\ s_n(t)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} v(te^{i\theta})\sin n\theta d\theta & (n=1, 2, \cdots)\,. \end{cases}$$

By the same argument as in 2.1 we see that $c_n(t)=c_n\varphi_n(t)$ and $s_n(t)=s_n\varphi_n(t)$ (cf. (19)), where $c_n=c_n(1)$ and $s_n=s_n(1)$ are bounded by $K=2\max_{|z|=1} v(z)$ in the absolute values. By (4) with $m=n$, $|c_n(t)|\leqq Kt^n$ and $|s_n(t)|\leqq Kt^n$ $(n=1, 2, \cdots)$. Expand $v(te^{i\theta})$ into its Fourier series:

$$v(te^{i\theta})=\frac{1}{2}c_0(t)+\sum_{n=1}^{\infty}(c_n(t)\cos n\theta+s_n(t)\sin n\theta)\,.$$

By the above estimates of coefficients we have

$$|v(te^{i\theta})-c_0(1)\varphi_0(t)/2|\leqq 2K\sum_{n=1}^{\infty}t^n=\frac{2K}{1-t}\cdot t\,.$$

By (6) and by the above inequality we deduce the existence of

$$\lim_{t\to 0} v(te^{i\theta})=\frac{c_0(1)}{2}\lim_{t\to 0}\varphi_0(t)\,.$$

Therefore the function $v(z)$ is continuous on (or continuously extendable to) $|z|\leqq 1$. By (40), the potential $\int_{0<|\zeta|<1} H(z, \zeta)Q(\zeta)v(\zeta)d\xi d\eta=2\pi(h(z)-v(z))$ is therefore extendable to a continuous function $q(z)$ on $0\leqq|z|\leqq 1$. At this stage we do not yet know that $q(0)=\int_{0<|\zeta|<1} H(0, \zeta)Q(\zeta)v(\zeta)d\xi d\eta$. Put

$$q_n(z)=\int_{1/n<|\zeta|<1} H(z, \zeta)Q(\zeta)v(\zeta)d\xi d\eta$$

for $n=2, 3, \cdots$. Then $\{q_n(z)\}$ is an increasing sequence of continuous functions on $|z|\leqq 1$ such that $q(z)=\lim_n q_n(z)$ for each z in $0<|z|\leqq 1$. Clearly $\lim_{n\to\infty} q_n(0)\leqq q(0)$. For any $\varepsilon>0$ there exists a $\delta\in(0, 1)$ such that $q(z)\geqq q(0)-\varepsilon/2$ on $|z|\leqq\delta$. By the Dini theorem $\lim_n q_n(z)=q(z)$ uniformly on $|z|=\delta$. Thus there exists an n such that $q_n(z)\geqq q(z)-\varepsilon/2$ on $|z|=\delta$, and hence $q_n(z)\geqq q(0)-\varepsilon$ on $|z|=\delta$. The superharmonic function $q_n(z)-(q(0)-\varepsilon)$ on $|z|<\delta$ has nonnegative boundary values on $|z|=\delta$ and thus $q_n(z)\geqq q(0)-\varepsilon$ on $|z|\leqq\delta$, and in particular $q_n(0)\geqq q(0)-\varepsilon$. Therefore $q(0)\geqq \lim_n q_n(0)\geqq q(0)-\varepsilon$. By letting $\varepsilon\to 0$, we can now conclude that

$$q(0)=\lim_{n\to\infty} q_n(0)=\lim_{n\to\infty}\int_{1/n<|\zeta|<1} H(0,\zeta)Q(\zeta)v(\zeta)d\xi d\eta$$
$$=\int_{0<|\zeta|<1} H(0,\zeta)Q(\zeta)v(\zeta)d\xi d\eta\,.$$

From this again by the Dini theorem it follows that the sequence $\{q_n(z)\}$ converges to $q(z)$ uniformly on $|z|\leqq 1$, i.e. let ε be any positive number; then there exists an $N\geqq 2$ such that

$$\int_{0<|z|<1/n} H(z,\zeta)Q(\zeta)v(\zeta)d\xi d\eta=q(z)-q_n(z)<\varepsilon$$

for every $n\geqq N$ and every z in $0\leqq|z|\leqq 1$. By (39) and $u\leqq v$ on $0<|z|\leqq 1$, the above inequality implies that

$$\int_{0<|z|<1/n} H(z,\zeta)P(\zeta)u(\zeta)d\xi d\eta<c^2\varepsilon \tag{42}$$

for every $n\geqq N$ and every z in $0\leqq|z|\leqq 1$. Put

$$p_a(z)=\int_{a<|z|<1} H(z,\zeta)P(\zeta)u(\zeta)d\xi d\eta$$

for $a\in[0,1)$ on $|z|\leqq 1$. Since

$$p_0(z)=p_{1/n}(z)+\int_{0<|z|<1/n} H(z,\zeta)P(\zeta)u(\zeta)d\xi d\eta\,,$$

if we fix an $n\geqq N$, then by (42) we have

$$|p_0(z)-p_0(0)|\leqq|p_{1/n}(z)-p_{1/n}(0)|+2c^2\varepsilon$$

for any z in $0<|z|\leqq 1$. In view of the continuity of $p_a(z)$ on $|z|\leqq 1$ for $a>0$ we have $\limsup_{z\to 0}|p_0(z)-p_0(0)|\leqq 2c^2\varepsilon$ and thus $\lim_{z\to 0}|p_0(z)-p_0(0)|=0$. This means that the potential part $(1/2\pi)p_0(z)$ in (41) is continuous at $z=0$ and so is the harmonic function $h(z)$ in (41), i.e. we have the existence of the limit $\lim_{z\to 0} u(z)$. The proof of Theorem 7 is herewith complete.

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