J. Math. Soc. Japan
Vol. 26, No. 3, 1974

Integral equation associated with some non-linear evolution equation

By Kenji MARUO

(Received March 23, 1973)
(Revised July 26, 1973)

§1. Introduction and Theorem.

In [1] G. Webb established the existence and uniqueness of a global solution of the integral equation

$$U(t)x=T(t)x-\int_{0}^{t}T(t-s)Bu(s)x\,ds\,.$$

associated with the non-linear evolution equation

$$du/dt+Au(t)+Bu(t)=0$$

in some Banach space X. Here A is a closed, densely defined, linear m-accretive operator from X to itself, $T(t)$ is the semigroup generated by $-A$, and B is a continuous, everywhere defined, non-linear accretive operator from X to itself. This result was extended by K. Maruo and N. Yamada [2] to the case where A and B are both dependent on t. In the present paper it is shown that a similar result remains valid if B is a not necessarily everywhere defined operator depending on t provided that $-A$ is the infinitesimal generator of an analytic semigroup.

Throughout this paper X will denote a Banach space with norm $\|\ \|$. We impose the following conditions on the operator A and $B(t)$, $0\leqq t\leqq T<+\infty$:

(I) A is a closed, densely defined, linear m-accretive operator from X to itself. $T(t)$ which is the semigroup generated by $-A$ is an analytic semigroup.

In what follows we assume that the origin belongs to the resolvent set of A without loss of generality.

(II) For each $t\in[0, T]$ $B(t)$ is an accretive, nonlinear operator from X to itself.

(III) There exist numbers α, α' with $\alpha>0$, $\alpha'\geqq 0$, $\alpha+\alpha'<1$ and a positive non-decreasing function $l(x)$ defined on $[0, \infty)$ such that

(i) $D(A^{\alpha})\subset D(B(t))$ for $0\leqq t\leqq T$;

(ii) for any $\varepsilon>0$ and $t\in[0, T]$ there exists a positive number δ depending only on ε and t such that

$$\|B(t)u-B(s)v\| \leqq \varepsilon l(\|A^{\alpha}u\|+\|A^{\alpha}v\|)$$

for any $u, v \in D(A^{\alpha})$, $\|u-v\| < \delta$ and $|t-s| < \delta$;

(iii) there is a positive constant K_L depending on $L>0$ such that

$$\|A^{-\alpha'}B(t)u\| \leqq K_L(1+\|A^{\alpha}u\|)$$

for any $u \in D(A^{\alpha})$ with $\|u\| \leqq L$.

REMARK 1. It follows from the well known inequality

$$\|A^{\gamma}u\| \leqq C\|A^{\alpha}u\|^{\gamma/\alpha}\|u\|^{1-\gamma/\alpha} \qquad \text{if} \quad 0<\gamma<\alpha$$

that when $\|A^{\alpha}u\|$ and $\|A^{\alpha}v\|$ are both bounded $\|A^{\gamma}(u-v)\|$ can be made arbitrarily small by letting $\|u-v\|$ be sufficiently small. Thus the continuity assumption (ii) of (III) is weaker than the case $0 \leqq \alpha < \beta < 1$ of that of Theorem 8 of T. Kato [3].

REMARK 2. It follows from Heine-Borel's theorem that the constant δ in the assumption (ii) of (II) can be taken independently also of t.

We use the usual notations $C([0, T]; X)$, $L^1([0, T], X)$ etc., to denote various spaces of functions with values in X. By $[D(A^{\alpha})]$ we denote the subspace $D(A^{\alpha})$ equipped with the graph norm of A^{α}.

THEOREM. *Suppose that the assumptions stated above are satisfied. Then for any $x \in X$ there exists a function $u(t, x)$ belonging to $C([0, T], X) \cap C((0, T]; [D(A^{\alpha})])$ satisfying*

$$u(t, x) = T(t)x - \int_0^t T(t-s)B(s)u(s, x)ds\,, \qquad 0 \leqq t \leqq T\,. \tag{1.1}$$

Furthermore the solution of (1.1) *having this property is unique.*

The author wishes to thank the referee for helpful advices.

§2. Existence of the local solution.

LEMMA 2.1. *For any $x \in D(A^{\alpha})$ there exists a positive number T_0 and a function $u \in C([0, T_0], [D(A^{\alpha})])$ which satisfies* (1.1) *in* $[0, T_0]$.

PROOF. We follow the method of G. Webb [1]. Let

$$V = \{y \in D(A^{\alpha}) / \|A^{\alpha}(x-y)\| < \delta_0\}\,.$$

If δ_0 and T_1 are sufficiently small positive numbers, then in view of the continuity of $B(t)$ there exists a constant M such that $\sup_{y \in V} \sup_{0 \leqq t \leqq T_1} \|B(t)y\| \leqq M$. We put $v = T(t)x + \omega$. Then we can choose $T_2 > 0$ so small that v belongs to V for any $0 \leqq t \leqq T_2$ if $\omega \in D(A^{\alpha})$ and $\|A^{\alpha}\omega\| \leqq C_{\alpha}T_2^{1-\alpha}M(1-\alpha)^{-1}$, where C_{α} is a constant such that $\|A^{\alpha}T(t)\| \leqq C_{\alpha}t^{-\alpha}$. Set $T_0 = \min(T_2, T_1)$. Let n be a positive integer. Let $t_0^n = 0$ and $u_n(t_0^n) = x$. Inductively for each positive integer i we shall define t_i^n. Assume that t_j^n have been defined for any $j=0,1,2,\cdots,k-1$. For $t_{k-1}^n \leqq t \leqq T_0$ we put

$$u_n(t)=T(t)x-\sum_{i=1}^{k-1}\int_{t_{i-1}^n}^{t_i^n} T(t-s)B(t_{i-1}^n)u_n(t_{i-1}^n)ds$$
$$-\int_{t_{k-1}^n}^{t} T(t-s)B(t_{k-1}^n)u_n(t_{k-1}^n)ds\,. \tag{2.1}$$

Then it is easy to see that $u_n(t)\in V$. Now we take t_k^n such that

$$t_k^n=\min\{\min(t:\ \|B(t)u_n(t)-B(t_{k-1}^n)u_n(t_{k-1}^n)\|\geqq 1/n),\ T_0\}\,.$$

Evidently $A^\alpha u_n(t)$ is continuous. Next we shall show that there exists some positive integer N such that $t_N^n=T_0$. Assume that $t_i^n<T_0$ for all i. Let $T_3=\lim_{i\to\infty} t_i^n$. Since we find $\|(T(t)-I)y\|\leqq C_\alpha t^{1-\alpha}\|A^\alpha y\|$ for any $y\in D(A^\alpha)$, $\|B(t_{i-1}^n)u_n(t_{i-1}^n)\|\leqq M$ and (2.1) we see that $A^\alpha u_n(t)$ is Hoelder continuous of order h where $h=\max(\alpha,1-\alpha)$. Hence $\lim_{t\to T_3} u_n(t)=z_0$ and $\lim_{t\to T_3} A^\alpha u_n(t)=\mu$ exist and $A^\alpha z_0=\mu$ and $z_0\in V$. Then we take a integer k such that

$$\|B(t_k^n)u_n(t_k^n)-B(T_3)z_0\|\leqq 1/2n\,.$$

We find that this is a contradiction to the definition of t_{k+1}^n. Hence $t_N^n=T$ for some N.

Using the method of the proof of the proposition (3.1) of G. Webb [**1**], we can show that the sequence $\{u_n(t)\}$ is uniformly convergent. Next we will show that $\{A^\alpha u_n(t)\}$ is a Cauchy sequence in $C([0,T_0],X)$. We put $\{t_i^n\}\cup\{t_i^m\}=\{t_j^{n,m}\}$ and

$$u_n^*(t)=T(t)x-\sum_{j=1}^{k-1}\int_{t_{j-1}^{n,m}}^{t_j^{n,m}} T(t-s)B(t_{j-1}^{n,m})u_n(t_{j-1}^{n,m})ds$$
$$-\int_{t_{k-1}^{n,m}}^{t} T(t-s)B(t_{k-1}^{n,m})u_n(t_{k-1}^{n,m})ds \tag{2.2}$$

for $t_{k-1}^{n,m}\leqq t<t_k^{n,m}$. If $t_{i-1}^n=t_{j-1}^{n,m}<t_j^{n,m}<\cdots<t_{j+l}^{n,m}=t_i^n$ then for $0\leqq p<l+1$

$$\|B(t_{i-1}^n)u_n(t_{i-1}^n)-B(t_{j+p-1}^{n,m})u_n(t_{j+p-1}^{n,m})\|\leqq 1/n$$

from the definition of $u_n(t)$ and t_i^n. Hence, with the aid of the condition we get

$$\left\|\int_{t_{i-1}^n}^{t_i^n} A^\alpha T(t-s)B(t_{i-1}^n)u_n(t_{i-1}^n)ds-\sum_{p=0}^{l}\int_{t_{j+p-1}^{n,m}}^{t_{j+p}^{n,m}} A^\alpha T(t-s)B(t_{j+p-1}^{n,m})u_n(t_{j+p-1}^{n,m})ds\right\|$$
$$\leqq 1/n\int_{t_{i-1}^n}^{t_i^n} C_\alpha(t-s)^{-\alpha}ds\,. \tag{2.3}$$

Similarly replacing u_n by u_m in (2.2) we define u_m^*. It follows from (2.3) that

$$\|A^\alpha\{u_n^*(t)-u_n(t)\}\|\leqq C_\alpha(1-\alpha)^{-1}T^{1-\alpha}/n \tag{2.4}$$

and analogously we get

$$\|A^{\alpha}\{u_m^*(t)-u_m(t)\}\| \leqq C_\alpha(1-\alpha)^{-1}T^{1-\alpha}/m\,. \tag{2.5}$$

On the other hand we know the inequality

$$\|A^{\alpha}\{u_n^*(t)-u_m^*(t)\}\| \leqq \sum_{j=1}^{k-1}\int_{t_{j-1}^{n,m}}^{t_j^{n,m}} C_\alpha(t-s)^{-\alpha}\|B(t_{j-1}^{n,m})u_n(t_{j-1}^{n,m})-B(t_{j-1}^{n,m})u_m(t_{j-1}^{n,m})\|ds$$

$$+\int_{t_{k-1}^{n,m}}^{t} C_\alpha(t-s)^{-\alpha}\|B(t_{k-1}^{n,m})u_n(t_{k-1}^{n,m})-B(t_{k-1}^{n,m})u_m(t_{k-1}^{n,m})\|ds$$

$$\text{for}\quad t_{k-1}^{n,m} \leqq t \leqq t_k^{n,m}\,.$$

Since $\{u_n\}$ is a Cauchy sequence in $C([0, T_0]: X)$ and $\{A^\alpha u_n(t)\}$ is uniformly bounded it follows noting Remark 2 that

$$\lim_{n,m\to\infty} \|A^{\alpha}\{u_n^*(t)-u_m^*(t)\}\| = 0 \tag{2.6}$$

uniformly in $0 \leqq t \leqq T_0$. In view of (2.4), (2.5) and (2.6) we get

$$\lim_{n,m\to\infty} \|A^{\alpha}\{u_n(t)-u_m(t)\}\| = 0$$

uniformly in $0 \leqq t \leqq T_0$. It follows from the manner of defining t_i^n, δ_i^n, $u_n(t_{i-1}^n)$ that

$$\|B(t)u_n(t)-B(t_{i-1}^n)u_n(t_{i-1}^n)\| \leqq 1/n \qquad \text{for}\quad t_{i-1}^n \leqq t \leqq t_i^n\,.$$

Hence if we note Remark 2 we can easily show that $u(t)=\lim_{n\to\infty} u_n(t)$ is the desired solution.

LEMMA 2.2. *Let $x, y \in D(A^\alpha)$. If $u(t, x)$ and $v(t, x)$ are solutions of* (1.1) *such that $A^\alpha u(t, x)$ and $A^\alpha v(t, y)$ are continuous on $[0, T_1]$ and $[0, T_2]$ respectively, then*

$$\|u(t, x)-v(t, y)\| \leqq \|x-y\|$$

for $0 \leqq t \leqq T = \min(T_1, T_2)$. Consequently the solution (1.1) *is unique and*

$$u(t+t^1, x) = u(t, u(t^1, x))$$

for $t>0$, $t^1>0$, $0<t^1+t \leqq T_1$.

PROOF. Let $\{t_i^n\}_{i=0}^n$ be a partition of $[0, T]$ such that the mesh of $\{t_i^n\}$ goes to zero with n. We put, for $t_{k-1}^n < t < t_k^n$,

$$u_n(t, x) = T(t)x - \sum_{i=1}^{k-1}\int_{t_{i-1}^n}^{t_i^n} T(t-s)B(t_{i-1}^n)u(t_{i-1}^n, x)ds$$

$$-\int_{t_{k-1}^n}^{t} T(t-s)B(t_{k-1}^n)u_n(t_{k-1}^n)ds\,,$$

and

$$v_n(t, y) = T(t)y - \sum_{i=1}^{k-1}\int_{t_{i-1}^n}^{t_i^n} T(t-s)B(t_{i-1}^n)v(t_{i-1}^n, y)ds$$

$$-\int_{t_{k-1}^n}^{t} T(t-s)B(t_{k-1}^n)v(t_{k-1}^n, y)ds\,.$$

Then $u_n(t, x)$ and $A^{\alpha}u_n(t, x)$ converge to $u(t, x)$ and $A^{\alpha}u(t, x)$, respectively, as $n\rightarrow\infty$ and similarly for $v_n(t, y)$ and $A^{\alpha}v_n(t, y)$.

Using the method of the proof of Proposition (3.6) of G. Webb [**1**], we complete the proof of the lemma.

§ 3. Proof of the Theorem.

LEMMA 3.1. *For any* $x\in D(A^{\alpha})$ *there exists a global solution of* (1.1) *such that* $A^{\alpha}u(t, x)$ *is continuous on* $[0, T]$.

PROOF. Let $u(t, x)$ be a solution of (1.1) on $[0, T_0)$. Using the method of the proof of proposition (3) of K. Maruo and N. Yamada [**2**], we find

$$\|u(t, x)\|\leqq\|x\|+\int_0^{T_0}\|B(s)0\|ds=M_1<+\infty\,.$$

On the other hand from our assumption (iii) of (III) it follows that

$$\|A^{\alpha}u(t, x)\|\leqq\|A^{\alpha}x\|+\int_0^t\|A^{\alpha}T(t-s)A^{\alpha'}A^{-\alpha'}B(s)u(s, x)\|ds$$

$$\leqq C\Big\{1+\int_0^t(t-s)^{-(\alpha+\alpha')}\|A^{\alpha}u(s, x)\|ds\Big\}$$

where C is a constant depending only on M_1, $\|A^{\alpha}x\|$ and T_0. Hence for some constant M_2 we have

$$\|A^{\alpha}u(t, x)\|<M_2 \tag{3.1}$$

for any $0\leqq t<T_0$. Combining (3.1) and (iii) of (III) we get

$$\sup_{0\leqq t<T_0}\|A^{-\alpha'}B(t)u(t, x)\|<+\infty\,.$$

Using the method of the proof of Proposition (3) of [**2**], we find that $\lim_{t\rightarrow T_0}A^{\alpha}u(t, x)$ and $\lim_{t\rightarrow T_0}u(t, x)$ exist. Thus the proof of Lemma 3.1 is complete.

We fix any point $x\in X$. We denote by $\{x_n\}_{n=0}^{\infty}\subset D(A^{\alpha})$ a sequence converging to x. Let $u_n(t, x_n)$, $0\leqq t\leqq T$, be the solution of (1.1) with x replaced by x_n whose existence was established in Lemma 3.1.

In view of our assumption (iii) of (III) we find

$$t^{\alpha}\|A^{\alpha}u_n(t, x_n)\|\leqq K\Big\{1+\int_0^t t^{\alpha}(t-s)^{-(\alpha+\alpha')}s^{-\alpha}\cdot s^{\alpha}\|A^{\alpha}u_n(s, x_n)\|\,ds\Big\}\,. \tag{3.2}$$

It follows from (3.2) that there is a constant K_T dependent only on $\|x\|$ and T such that

$$\|A^{\alpha}u_n(t, x_n)\|\leqq K_T t^{-\alpha}\,. \tag{3.3}$$

On the other hand in view of Lemma 2

$$\|u_n(t, x_n)-u_m(t, x_m)\| \leqq \|x_n-x_m\| . \tag{3.4}$$

Combining (3.3), (3.4) and noting Remark 2 we get

$$\lim_{n, m\to\infty} \|B(t)u_n(t, x_n)-B(t)u_m(t, x_m)\|=0$$

uniformly in the wider sence on $0<t\leqq T$. Thus we find that $A^{\alpha}u_n(t, x_n)$ is uniformly convergent in any compact set of $(0, T]$ as $n\to\infty$ to complete the proof of the Theorem.

§4. Application.

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. We put $X=L_2(\Omega)$. We consider the initial boundary value problem

$$\begin{cases} du/dt+(-\varDelta)^m u+a(x, t)|u|^{2l}u=0 \\ u(0)=x\in L_2(\Omega) \\ u=(\partial/\partial\nu)u=\cdots=(\partial/\partial\nu)^{m-1}u=0 \quad \text{on} \quad \partial\Omega , \end{cases}$$

where l is some positive integer, $a(x, t)$ is a positive continuous function in $\Omega\times[0, T]$ and $\partial/\partial\nu$ denotes the outer normal derivative. We assume

$$nl/2m<1 . \tag{4.1}$$

It is known that the operator A defined by

$$\begin{aligned} &D(A)=H_{2m}(\Omega)\cap\mathring{H}_m(\Omega) , \\ &Au=(-\varDelta)^m u \qquad \text{for} \quad u\in D(A) \end{aligned} \tag{4.2}$$

satisfies the assumption (I).

If we put

$$B(t)u=a(x, t)|u|^{2l}u ,$$

$$D(B(t))=\{u\in L_2(\Omega)/B(t)u\in L_2(\Omega)\} .$$

We know that from (4.1), (4.2) and Sobolev Lemma, the operator $B(t)$ satisfies the assumption (II) and (III) with $nl/2m<\alpha<1$, $\alpha'=0$ and $l(x)=x+1$.

Bibliography

[1] G. Webb, Continuous non-linear perturbations of accretive operators in Banach spaces, J. Fun. Anal., 10 (1972), 191-203.
[2] K. Maruo and N. Yamada, A Remark on integral equation in a Banach space,

Proc. Japan Acad., **49** (1973), 13-16.
[3] T. Kato, Nonlinear evolution equations in Banach spaces, Proc. Sym. Appl. Math., **17** (1965), 50-67.

Kenji MARUO
Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka
Japan