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Cohomology of Lie algebras over a manifold, I

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We shall here try to give a generalization of the de Rham theory towards higher order jet spaces. Our investigation is motivated by the recent works by M. V. Losik [**12**] and I. M. Gelfand and D. B. Fuks [**7**]. Their results, being entirely full of originality, might well be regarded as a part of a big theory to be formed on the theory of cohomology of vector fields which they have been rapidly developing since 1968. We shall, however, take somewhat different ways from theirs. Since the de Rham theory is of fundamental importance, we have more or less a prospect for obtaining a higher viewpoint in the manifold theory in our attempt to develop the de Rham theory. Actually, one of the most important contributions in this direction was made by Atiyah, Bott and Singer in the theory on elliptic complexes. Taking account of this successful theory, we are led to try a realization of a possibility of extending the de Rham theory from a point of view commanding both that of the cohomology of vector fields and that of the elliptic complexes. In a series of the papers, we shall give a constructive method of elliptic complexes in the jet spaces and investigate these complexes in various aspects.

We are going to describe our setting. Let M be a smooth manifold with a countable basis and E a smooth vector bundle over M. We denote by $\Gamma(E)$ the smooth cross-section space of E. We say that $\Gamma(E)$ is a Lie algebra over M if there is a Lie algebra structure on $\Gamma(E)$ endowed with the bracket rule $[\xi, \eta]$, satisfying a continuity condition and $\mathrm{supp}\,[\xi, \eta] \subset \mathrm{supp}\,\xi \cap \mathrm{supp}\,\eta$. Let W be another vector bundle over M. A Lie algebra representation φ of $\Gamma(E)$ to $\mathrm{Hom}\,(\Gamma(W), \Gamma(W))$ is called a differential representation if φ satisfies a continuity condition and $\mathrm{supp}\,\varphi(\xi)\eta \subset \mathrm{supp}\,\xi \cap \mathrm{supp}\,\eta$. By virtue of the cohomology theory of Lie algebras, we can then canonically construct a complex $\{C^p, d\}$ associated to the Lie algebra $\Gamma(E)$ and the representation φ. Here there is room for choice of the cochain spaces C^p. Indeed, it is natural to take as C^p the space of continuous alternating p-linear maps from $\Gamma(E) \times \cdots \times \Gamma(E)$ (p times) to $\Gamma(W)$. As far as we know, however, it seems to be as yet unsettled how to build up a reasonable cohomology theory corresponding to these cochain spaces.

Actually, we restrict our attention only to "support preserving" p-cochains. That is, we denote by $C^p[E, W]$ the space of continuous p-cochains on $\Gamma(E)$ with value in $\Gamma(W)$, satisfying

$$\operatorname{supp} L(\xi_1, \cdots, \xi_p) \subset \operatorname{supp} \xi_1 \cap \cdots \cap \operatorname{supp} \xi_p .$$

(Gelfand and Fuks call this the diagonal p-cochain space.) Then we have a complex

$$\cdots \xrightarrow{d} C^p[E, W] \xrightarrow{d} C^{p+1}[E, W] \xrightarrow{d} \cdots$$

with

$$dL(\xi_1, \cdots, \xi_{p+1}) = \sum_{s=1}^{p+1} (-1)^{s-1} \varphi(\xi_s) L(\xi_1, \cdots, \check{\xi}_s, \cdots, \xi_{p+1})$$
$$+ \sum_{s<t} (-1)^{s+t} L([\xi_s, \xi_t], \xi_1, \cdots, \check{\xi}_s, \cdots, \check{\xi}_t, \cdots, \xi_{p+1}) .$$

Main purpose of our investigation is to study the structure of the complex $\{C^p[E, W], d\}$ from the cohomological point of view. Also we are concerned with a similar problem for a subalgebra of $\Gamma(E)$.

A special feature of the complex $\{C^p[E, W], d\}$ is caused by the fact that the cochain space $C^p[E, W]$ admits a filtration

$$C_0^p[E, W] \subset C_1^p[E, W] \subset \cdots \subset C_k^p[E, W] \subset \cdots \longrightarrow C^p[E, W]$$

at least as far as M is compact. Here use is made of the identifications

$$C^p[E, W] = \Gamma(\varinjlim \operatorname{Hom}(\wedge^p J^k(E), W))$$

$$C_k^p[E, W] = \Gamma(\operatorname{Hom}(\wedge^p J^k(E), W)) , \qquad k = 0, 1, 2, \cdots ,$$

$J^k(E)$ being the k-th jet bundle of E. We say that $\{C^p[E, W], d\}$ has the stable jet range $k \geqq k_0$ if, for $k \geqq k_0$, the subcomplex $\{C_k^p[E, W], d\}$ is well-defined and the injection $C_k^p[E, W] \hookrightarrow C^p[E, W]$ induces an isomorphism on the cohomology level.

We are in a position to state Losik's result. Let $\tau(M)$ be the tangent bundle of M. Losik considered the Lie algebra $\mathfrak{A}(M) = \Gamma(\tau(M))$ of vector fields on M and studied the complex $\{C^p[\tau(M), \varepsilon^1], d\}$ (ε^1 is the trivial 1-dimensional bundle) associated to the natural representation of $\mathfrak{A}(M)$ on $\Gamma(\varepsilon^1)$ ($= C^\infty(M)$) as differential operators. Then the subcomplex $\{C_0^p[\tau(M), \varepsilon^1], d\}$ is well-defined and is nothing but the de Rham complex. Hence $\{C^p[\tau(M), \varepsilon^1], d\}$ is regarded as an extension of the de Rham complex in the direction of the higher order jets of $\tau(M)$. Really, for each k ($k = 0, 1, 2, \cdots$) $\{C_k^p[\tau(M), \varepsilon^1], d\}$ gives rise to an elliptic complex over M. Losik proved that $\{C^p[\tau(M), \varepsilon^1], d\}$ has the stable jet range $k \geqq 1$, and determined its cohomology group explicitly.

The present paper consists of four sections. Sections 1 and 2 deal with the general theory outlined above, some of which will be used in the forthcoming papers. In Section 3 we introduce a notion of local Lie algebras, and apply this to the Lie algebra of formal vector fields $\mathfrak{a}_n$ which is obtained as a localization of $\mathfrak{A}(M)$. Let L_0 be a subalgebra of $\mathfrak{a}_n$ consisting of formal vector fields without constant terms. Then we can prove that the cohomology group of L_0 associated to a finite-dimensional representation becomes finite-dimensional if the representation is decomposable (as to the precise definition, see Section 3). Section 4 concerns the cohomology of vector fields. Let $G(h)$ be the Lie group consisting of the h-jets of local diffeomorphisms of $\boldsymbol{R}^n$ around 0. Assume that a finite-dimensional representation ρ of $G(h)$ on V be given. Since the Lie algebra of $G(h)$ is L_0/L_h, the lifting of $d\rho$ to L_0 gives rise to a finite-dimensional representation of L_0 on V. Hence we have the complex $\{C^p(L_0, V), d\}$ on the Lie algebra L_0 associated to $d\rho$. On the other hand, let $P(h)$ be the principal $G(h)$-bundle associated to $J^{h-1}(\tau(M))$, and put

$$W=P(h)\times_\rho V.$$

Then we can show that the representation $d\rho$ canonically induces a differential representation $\rho^\#$ of $\mathfrak{A}(M)$ on $\Gamma(W)$, and hence we obtain a complex $\{C^p[\tau(M), W], d\}$ on $\mathfrak{A}(M)$ associated to $\rho^\#$. Our main result is to clarify the relation between these two complexes. Specifically, there is a spectral sequence $\{E_r^{p,q}, d_r\}$ which converges to the cohomology group of $\{C^p[\tau(M), W], d\}$ and possesses E_2-terms with the form

$$E_2^{p,q}=H^p(M, \underline{H}^q(L_0, V));$$

here, $\underline{H}^q(L_0, V)$ denotes a locally constant sheaf over M, any stalk of which is isomorphic to the q-dimensional cohomology group of $\{C^p(L_0, V), d\}$. Accordingly, the behaviour of local feature gives us an information to the global cohomology.

Short announcements of this investigation have been published in [**14**] and [**15**].

§1. Inductive vector bundles and differential cochains.

Let M be an n-dimensional smooth manifold with a countable basis. By a vector bundle we mean a finite-dimensional, smooth vector bundle over M. We usually will not refer to the underlying field of vector bundles, since either choice of $\boldsymbol{R}$ or $\boldsymbol{C}$ is generally irrelevant to our discussions. Given a vector bundle E, we denote by $\tilde{\Gamma}(E)$ the space of continuous cross-sections of E, and by $\Gamma(E)$ the space of smooth cross-sections of E. For any $\sigma\in\Gamma(E)$,

$\operatorname{supp}\sigma$ is, by definition, the closed set of M given by $\operatorname{supp}\sigma=\overline{\{x \mid \sigma(x)\neq 0\}}$. We endow $\Gamma(E)$ with the uniform convergence topology of all derivatives on each compact set of M. Then $\Gamma(E)$ becomes a locally convex topological vector space; specifically, $\Gamma(E)$ becomes a Fréchet nuclear space and thus a Montel space. For any compact set $K\subset M$, $\Gamma(E|K)$ means the space of cross-sections on K, each of which consists of the restriction of some $\sigma\in\Gamma(E)$ to K. The n-dimensional trivial bundle is denoted by ε^n. $\mathcal{E}=\Gamma(\varepsilon^1)$ means the algebra of smooth functions on M.

DEFINITION 1.1. Suppose that a sequence of vector bundles $\{E_k\}$ $(k=0,1,2,\cdots)$ is given and satisfies the conditions

i) each E_k is a subbundle of E_{k+1} $(k=0,1,2,\cdots)$,

ii) $\lim\dim E_k=\infty$.

Then the inductive limit space $E=\varinjlim E_k$ is called an *inductive vector bundle* over M.

Let $\pi_k: E_k\to M$ be the projection. Then $\pi=\varinjlim\pi_k$ gives a continuous projection from E onto M, whence E is regarded as a fibre space over M. $\tilde{\Gamma}(E)$ denotes the space of continuous cross-sections of E. Let K be a compact set of M. Then for any $\sigma\in\tilde{\Gamma}(E)$ $\sigma(K)$ becomes a compact set of E. Hence from a known property of inductive limit topology it follows that $\sigma(K)$ is contained in some E_k, or $\sigma|K\in\tilde{\Gamma}(E_k|K)$.

DEFINITION 1.2. $\sigma\in\tilde{\Gamma}(E)$ is called *smooth* if for any compact set $K\subset M$

$$\sigma|K\in\bigcup_{k=0}^{\infty}\Gamma(E_k|K).$$

The space of smooth cross-sections of E, denoted by $\Gamma(E)$, has an $\mathcal{E}$-module structure in a natural way. Noting that $\Gamma(E_k|K)$ is a Fréchet nuclear space in the C^∞ topology (cf. [**10**] II, p. 88), we shall introduce the inductive limit topology to $\Gamma(E|K)$ relative to the filtration by $\Gamma(E_k|K)$ $(k=0,1,2,\cdots)$:

$$\Gamma(E|K)=\varinjlim\Gamma(E_k|K).$$

$\Gamma(E|K)$ is thus a nuclear space of type $(\mathcal{LF})$. Using this topology of $\Gamma(E|K)$, we shall now endow $\Gamma(E)$ with the weakest topology such that the restriction map $r_k:\Gamma(E)\to\Gamma(E|K)$ becomes continuous for each compact set $K\subset M$.

PROPOSITION 1.1.

i) *$\Gamma(E)$ is a nuclear topological vector space.*

ii) *Let B be a bounded set of $\mathcal{E}$. Then the multiplication*

$$\begin{array}{cccc}\mu: & B\times\Gamma(E) & \longrightarrow & \Gamma(E)\\ & \Cup & & \Cup\\ & (f,\sigma) & \longmapsto & f\cdot\sigma\end{array}$$

gives rise to a continuous map.

PROOF. i) Immediate from [**10**; II, p. 48, Corollaire 2].

ii) It is sufficient to prove the continuity

$$r_K \circ \mu : B \times \Gamma(E) \longrightarrow \Gamma(E|K)$$

for each K. Note that, on any local coordinates neighborhood U with the compact closure, each partial derivative of $f \in B$ has an upper bound uniformly for all $f \in B$. Hence it follows easily that the map $r_K \circ \mu$ induces a continuous map from $B|K \times \Gamma(E_k|K)$ to $\Gamma(E_k|K)$ ($\subset \Gamma(E|K)$) for $k = 0, 1, 2, \cdots$. Since $B|K$ is a precompact set of $\mathcal{E}|K$, we have $B|K \times \Gamma(E|K) = \varinjlim (B|K \times \Gamma(E_k|K))$. Thus

$$r_K \circ \mu | K : B|K \times \Gamma(E|K) \longrightarrow \Gamma(E|K)$$

is continuous. Set $\tilde{r}_K : B \times \Gamma(E) \to B|K \times \Gamma(E|K)$ for the restriction map. Then $\tilde{r}_K$ is clearly continuous, and we have $r_K \circ \mu = (r_K \circ \mu | K) \circ \tilde{r}_K$. Hence $r_K \circ \mu$ is continuous. This completes the proof.

Two inductive vector bundles $E = \varinjlim E_k$ and $F = \varinjlim F_k$ are called *isomorphic* if there is an isomorphism $\Phi : \Gamma(E) \cong \Gamma(F)$ as topological vector spaces and $\mathcal{E}$-modules. It is clear that E and F are isomorphic if and only if, for any compact set $K \subset M$, there is an isomorphism $\Phi_K : \Gamma(E|K) \to \Gamma(F|K)$ as topological vector spaces and $\mathcal{E}$-modules such that we have the compatibility condition $\Phi_{K'}|K = \Phi_K$ for $K \subset K'$.

PROPOSITION 1.2. *Let $E = \varinjlim E_k$ and $F = \varinjlim F_k$ be inductive vector bundles with the projections π and $\tilde{\pi}$ respectively. Then E is isomorphic to F if and only if there is a bijection map Ψ from E to F with the following properties:*

(i) *$\tilde{\pi} \circ \Psi = \pi$; Ψ is linear on each fibre.*

(ii) *For any compact set $K \subset M$, there are increasing sequences*

$$l(0) < l(1) < \cdots < l(k) < \cdots$$

$$k(0) < k(1) < \cdots < k(l) < \cdots$$

such that

$$\Psi(E_k|K) \subset F_{l(k)}|K \tag{1.1}$$

$$\Psi^{-1}(F_l|K) \subset E_{k(l)}|K; \tag{1.2}$$

moreover, Ψ and Ψ^{-1} are smooth on each $E_k|K$ and $F_l|K$.

PROOF. Assume that such a map Ψ exists. Then (1.1) implies that Ψ_* induces a continuous map from $\Gamma(E_k|K)$ to $\Gamma(F|K)$ ($k = 0, 1, 2, \cdots$), whence Ψ_* gives a continuous map from $\Gamma(E|K)$ to $\Gamma(F|K)$. The similar reasoning applies to Ψ^{-1}, which shows that Ψ_* gives an isomorphism from $\Gamma(E|K)$ to $\Gamma(F|K)$ as topological vector spaces. By (i) Ψ_* is also an isomorphism as

$\mathcal{E}$-modules. Let $\{f_\alpha\}$ be a partition of unity subordinate to a covering of M consisting of compact neighborhoods. For any $\sigma \in \Gamma(E)$, put $\Psi_*(\sigma) = \sum \Psi_*(f_\alpha \sigma)$. Then a routine argument shows that Ψ_* is a well-defined isomorphism from $\Gamma(E)$ to $\Gamma(F)$ as topological vector spaces and $\mathcal{E}$-modules. Thus we get $E \cong F$.

Conversely, assume that $E \cong F$ and so we have an isomorphism $\Phi : \Gamma(E) \to \Gamma(F)$. Let K be a compact set of M. Then Φ naturally induces the isomorphism $\Phi_K : \Gamma(E|K) \to \Gamma(F|K)$. Since each $\Gamma(E_k|K)$ is a finitely generated $\mathcal{E}$-module, we can take a set of generators $\sigma^1, \cdots, \sigma^s$ of $\Gamma(E_k|K)$. Then $\Phi_K(\sigma^1), \cdots, \Phi_K(\sigma^s)$ are contained in $\Gamma(F_{l(k)}|K)$ for some $l(k)$. From this follows immediately that Φ_K induces a smooth injective map Ψ_K from $E_k|K$ to $F_{l(k)}|K$. These Ψ_K, piecing together, give rise to a desired bijection $\Psi : E \to F$, as is easily verified. This completes the proof.

Note that a map Ψ satisfying (i) and (ii) stated in Proposition 1.2 necessarily gives a homeomorphism from E to F. We are often concerned with the isomorphism class of inductive vector bundles. In particular, an inductive vector bundle E is said to have a *filtration*

$$F_0 \subset F_1 \subset \cdots \subset F_k \subset \cdots$$

if each F_k is a subbundle of E and the inductive vector bundle $\varinjlim F_k$ becomes isomorphic to E. (Also, for convenience' sake, we always assume that any vector bundle W has a unique filtration $\varinjlim W_k = W$, where $W_k = W$ for $k = 0, 1, 2, \cdots$.)

Suppose that a sequence of inductive vector bundles $E^{(0)}, E^{(1)}, E^{(2)}, \cdots$ be given, where $E^{(s)} = \varinjlim_k E_k^{(s)}$ and $E_k^{(s)} \subset E_k^{(s')}$ for $s < s'$. For any increasing sequence of integers

$$\kappa:\ k_1 < k_2 < \cdots < k_i < \cdots$$

$$\sigma:\ s_1 < s_2 < \cdots < s_i < \cdots,$$

put $E(\kappa, \sigma) = \varinjlim_i E_{k_i}^{(s_i)}$. Then from Proposition 1.2 it follows directly that for other increasing sequences κ' and σ' we have $E(\kappa, \sigma) = E(\kappa', \sigma')$. Hence the isomorphism class of $E(\kappa, \sigma)$ is well-defined, which we shall denote by $\varinjlim E^{(s)}$. Similarly, if a multi-sequence $\{E_{k_1 \cdots k_p}\}$ $(k_1, \cdots, k_p = 0, 1, 2, \cdots)$ of vector bundles over M is given and satisfies

$$E_{k_1 \cdots k_p} \subset E_{k'_1 \cdots k'_p} \quad \text{for} \quad k_1 \leqq k'_1, \cdots, k_p \leqq k'_p,$$

then the isomorphism class of inductive vector bundle

$$\varinjlim_{k_1, \cdots, k_p \to \infty} E_{k_1 \cdots k_p}$$

is well-defined.

Let $E=\varinjlim E_k$ and $F=\varinjlim F_k$ be inductive vector bundles. The inductive vector bundle $E\oplus F=\varinjlim (E_k\oplus F_k)$ is called the Whitney sum of E and F. We have clearly $\Gamma(E\oplus F)=\Gamma(E)\oplus\Gamma(F)$, and if $E\cong E_1$ and $F\cong F_1$ then $E\oplus F\cong E_1\oplus F_1$. The isomorphism classes of inductive vector bundles make us define a K-group under the operation of Whitney sum. We denote this K-group by $K_{\text{ind}}(M)$. Set $\varepsilon^\infty=\varinjlim \varepsilon^k$.

PROPOSITION 1.3. $K_{\text{ind}}(M)=0$.

PROOF. Let $E=\varinjlim E_k$ and $F=\varinjlim F_k$ be any two inductive vector bundles. It is sufficient to prove that we can take an inductive vector bundle $G=\varinjlim G_k$ so as to satisfy

$$E\oplus G\cong\varepsilon^\infty \quad \text{and} \quad F\oplus G\cong\varepsilon^\infty .$$

Assume that G_l is already defined for $l<2k$. Take vector bundles E'_k and F'_k such that $E_k\oplus E'_k$ and $F_k\oplus F'_k$ are both trivial bundles. Put

$$G_{2k}=G_{2k-1}\oplus F_{k-1}\oplus E'_k ,$$

$$G_{2k+1}=G_{2k}\oplus E_k\oplus F'_k .$$

Thus we can inductively define vector bundles $G_0, G_1, \cdots$ with the properties that $E_k\oplus G_{2k}$ and $F_k\oplus G_{2k+1}$ are trivial for $k=0, 1, \cdots$. Let $G=\varinjlim G_k$. Then we have

$$E\oplus G\cong E\oplus(\varinjlim G_{2k})\cong\varinjlim (E_k\oplus G_{2k})\cong\varepsilon^\infty$$

$$F\oplus G\cong F\oplus(\varinjlim G_{2k+1})\cong\varinjlim (F_k\oplus G_{2k+1})\cong\varepsilon^\infty ,$$

which completes the proof.

Among many examples of inductive vector bundles we are mainly concerned with the following type. Let E be a vector bundle over M. For any non-negative integer k, denote by $J^k(E)$ the k-th jet bundle of E. The natural projection map

$$\lambda_k : J^{k+1}(E) \longrightarrow J^k(E) \qquad (k=0, 1, 2, \cdots)$$

canonically induces the injective map

$$(\lambda_k)^* : \operatorname{Hom}(J^k(E), W) \longrightarrow \operatorname{Hom}(J^{k+1}(E), W)$$

for any vector bundle W. Hence we can get an inductive vector bundle

$$\operatorname{Hom}(J(E), W)=\varinjlim \operatorname{Hom}(J^k(E), W) .$$

More generally, denoting by $\wedge^p J^k(E)$ the p-th exterior product of $J^k(E)$ we have an inductive vector bundle

$$\operatorname{Hom}(\wedge^p J(E), W)=\varinjlim \operatorname{Hom}(\wedge^p J^k(E), W) .$$

It should be noted that the same method of construction applies to the case where W is an inductive vector bundle.

In the next Proposition 1.4, we shall give a crucial interpretation to the cross-section space of $\mathrm{Hom}(\bigwedge^p J(E), W)$. To describe this, first consider the jet extension map $j_k: \Gamma(E) \to \Gamma(J^k(E))$. Let W be a vector bundle over M. Then this induces a bilinear map

$$\begin{array}{rccc} j^k: & \Gamma(\mathrm{Hom}(J^k(E), W)) \times \Gamma(E) & \longrightarrow & \Gamma(W) \\ & \Cup & & \Cup \\ & (\varphi(x), \xi(x)) & \longmapsto & \varphi(j_k(\xi)(x))(x)\,. \end{array}$$

The following diagram is clearly commutative:

$$\begin{array}{ccc} \Gamma(\mathrm{Hom}(J^k(E), W)) \times \Gamma(E) & \xrightarrow{j^k} & \Gamma(W) \\ \Big\downarrow (\lambda_k)^* \times \mathrm{id} & & \Big\downarrow \mathrm{id} \\ \Gamma(\mathrm{Hom}(J^{k+1}(E), W)) \times \Gamma(E) & \xrightarrow{j^{k+1}} & \Gamma(W)\,. \end{array}$$

Hence, passing to the inductive limit, we obtain a bilinear map

$$j^\infty: \varinjlim \Gamma(\mathrm{Hom}(J^k(E), W)) \times \Gamma(E) \longrightarrow \Gamma(W)\,.$$

On the other hand, for any compact set $K \subset M$, we have

$$\varinjlim \Gamma(\mathrm{Hom}(J^k(E), W) | K) = \Gamma(\mathrm{Hom}(J(E), W) | K)\,.$$

Thus, for any compact set $K \subset M$, we have

$$j^\infty | K: \Gamma(\mathrm{Hom}(J(E), W) | K) \times \Gamma(E | K) \longrightarrow \Gamma(W | K)\,.$$

Take a locally finite covering $\{U_\alpha\}$ of M, U_α being a compact neighborhood, and let $\{f_\alpha\}$ be partition of unity subordinate to $\{U_\alpha\}$. Set

$$j = \sum f_\alpha (j^\infty | U_\alpha) \circ r_{U_\alpha}$$

where r_{U_α} stands for the restriction map to U_α. Then j gives rise to a well-defined bilinear map

$$j: \Gamma(\mathrm{Hom}(J(E), W)) \times \Gamma(E) \longrightarrow \Gamma(W)\,.$$

It is easily verified that, for any fixed $\varphi \in \Gamma(\mathrm{Hom}(J(E), W))$, $j(\varphi, \xi)$ is continuous in ξ. We denote by $\mathrm{Hom}(\Gamma(E), \Gamma(W))$ the space of continuous linear maps from $\Gamma(E)$ to $\Gamma(W)$. Then, it follows that j induces a linear map

$$\tilde{j}: \Gamma(\mathrm{Hom}(J(E), W)) \longrightarrow \mathrm{Hom}(\Gamma(E), \Gamma(W))$$

by putting $\tilde{j}(\varphi)(\xi) = j(\varphi, \xi)$.

More generally, we can define in a similar way a linear map

$$^{(p)}\tilde{j}: \Gamma(\mathrm{Hom}\,(\wedge^p J(E),\, W)) \longrightarrow \mathrm{Hom}^p(\Gamma(E),\, \Gamma(W))$$

for $p=1, 2, 3, \cdots$, where $^{(1)}\tilde{j}=\tilde{j}$ and $\mathrm{Hom}^p(\Gamma(E), \Gamma(W))$ denotes the space of the continuous p-linear alternating maps from $\Gamma(E)\times \cdots \times\Gamma(E)$ (p times) to $\Gamma(W)$.

Note that the map $^{(p)}\tilde{j}$ may be defined in a more general case where we admit to take an inductive vector bundle as W.

DEFINITION 1.3. A continuous p-multilinear alternating map L from $\Gamma(E)\times \cdots \times\Gamma(E)$ (p times) to $\Gamma(W)$ is called a *differential p-cochain* if it satisfies

$$\mathrm{supp}\, L(\xi_1, \cdots, \xi_p) \subset \mathrm{supp}\, \xi_1 \cap \cdots \cap \mathrm{supp}\, \xi_p$$

where $\xi_1, \cdots, \xi_p \in \Gamma(E)$.

The totality of differential p-cochains is denoted by $C^p[E, W]$ ($p=1, 2, \cdots$); we set $C^0[E, W]=\Gamma(W)$. $C^p[E, W]$ is a subspace of $\mathrm{Hom}^p(\Gamma(E), \Gamma(W))$. We shall cite a theorem due to Peetre in the following form ([**13**]; cf. [**4**]):

PEETRE'S THEOREM. $C^1[E, W]$ is canonically isomorphic to $\Gamma(\mathrm{Hom}\,(J(E), W))$; this isomorphism is realized by the map $\tilde{j}: \Gamma(\mathrm{Hom}\,(J(E), W)) \to C^1[E, W]$.

Hence the following proposition may be regarded as a generalization of Peetre's theorem.

PROPOSITION 1.4. *For each $p=1, 2, \cdots$ $^{(p)}\tilde{j}$ gives an isomorphism*

$$\Gamma(\mathrm{Hom}\,(\wedge^p J(E),\, W)) \cong C^p[E, W].$$

PROOF. It is clear that the image of $^{(p)}\tilde{j}$ is contained in $C^p[E, W]$. Since at each point $x \in M$ we have

$$\bigcup_{\xi \in \Gamma(E)} j_k(\xi) \cap J^k(E)_x = J^k(E)_x,$$

it follows that the map $^{(p)}\tilde{j}$ is injective. Hence it is sufficient to show the surjectivity of $^{(p)}\tilde{j}$. We proceed to prove this by the induction on p. In case $p=1$, Peetre's theorem clearly assures the surjectivity of $^{(p)}\tilde{j}$. Assume that the assertion has been established till p. Let $L \in C^{p+1}[E, W]$ be given. An $\eta \in \Gamma(E)$ being fixed, we set

$$L_\eta(\xi_1, \cdots, \xi_p) = L(\eta, \xi_1, \cdots, \xi_p)$$

and regard L_η as a differential p-cochain with parameter η. Apply the induction hypothesis to L_η. Then we find that there exists a unique $\Phi_\eta \in \Gamma(\mathrm{Hom}\,(\wedge^p J(E), W))$ satisfying

$$^{(p)}\tilde{j}(\Phi_\eta) = L_\eta.$$

Take a compact set K of M and an open neighborhood U of K with the compact closure; K and U are arbitrary but fixed for a while. Then there is an integer k such that

$$\Phi_\eta | \bar{U} \in \Gamma(\operatorname{Hom}(\wedge^p J^k(E), W) | \bar{U}). \tag{1.3}$$

We denote by $\kappa(\Phi_\eta)$ the smallest integer of k's which satisfy (1.3). Put

$$E(k) = \{\eta \,|\, \kappa(\Phi_\eta) \leqq k\}.$$

Then $\Gamma(E) = \bigcup_{k=0}^{\infty} E(k)$ and each $E(k)$ becomes a subspace of $\Gamma(E)$ since

$$\kappa(\Phi_{\eta+\eta'}) \leqq \operatorname{Max}(\kappa(\Phi_\eta), \kappa(\Phi_{\eta'})).$$

Moreover, each $E(k)$ is closed in $\Gamma(E)$. In fact, this can be checked from the observation that $\eta \in E(k)$ is equivalent to the following statement: For any point $x_0 \in \bar{U}$, take $\xi_1, \cdots, \xi_p$ such that for at least one ξ_i we have $j_k(\xi_i)(x_0) = 0$; then for any choice of such $\xi_1, \cdots, \xi_p$, L always satisfies

$$L(\eta, \xi_1, \cdots, \xi_p)(x_0) = 0.$$

Since $\Gamma(E)$ is a Fréchet space, from the above we can conclude that there exists a k_0 with $\Gamma(E) = E(k_0)$.

Accordingly, for any $\eta \in \Gamma(E)$, we have

$$L(\eta, \xi_1, \cdots, \xi_p) | \bar{U} = \Phi_\eta(j_{k_0}(\xi_1), \cdots, j_{k_0}(\xi_p)) | \bar{U}$$

where $\xi_1, \cdots, \xi_p \in \Gamma(E)$. Thus the correspondence $\eta | \bar{U} \rightarrow \Phi_\eta | \bar{U}$ gives rise to a linear map

$$\tilde{\Phi} : \Gamma(E | \bar{U}) \longrightarrow \Gamma(\operatorname{Hom}(\wedge^p J^{k_0}(E), W) | \bar{U}).$$

From the continuity of L it follows immediately that $\tilde{\Phi}$ becomes continuous. Hence we have

$$\tilde{\Phi} \in \operatorname{Hom}((\Gamma(E | \bar{U}), \Gamma(\operatorname{Hom}(\wedge^p J^{k_0}(E), W) | \bar{U})).$$

Note that, if $x \notin \operatorname{supp} \eta$, we have

$$\tilde{\Phi}(\eta)(j_{k_0}(\xi_1), \cdots, j_{k_0}(\xi_p))(x) = L(\eta, \xi_1, \cdots, \xi_p)(x) = 0$$

by the support-preserving property of L. It follows that $\operatorname{supp} \tilde{\Phi}(\eta) \subset \operatorname{supp} \eta$. We are now in a position to apply Peetre's theorem to $\tilde{\Phi}$ on U, and then restrict our considerations only to the behaviour of $\tilde{\Phi}$ on K. Then we find that there are non-negative integer k_1 and

$$\Psi' \in \Gamma(\operatorname{Hom}(J^{k_1}(E), \operatorname{Hom}(\wedge^p J^{k_0}(E), W)) | K)$$

such that

$$\tilde{\Phi}(\eta) = \Psi'(j_{k_1}(\eta)), \qquad \eta \in \Gamma(E | K).$$

Using the identification

$$\Gamma(\operatorname{Hom}(J^{k_1}(E), \operatorname{Hom}(\wedge^p J^{k_0}(E), W)) | K)$$
$$\cong \Gamma(\operatorname{Hom}(J^{k_1}(E) \otimes \wedge^p J^{k_0}(E), W) | K)$$

and noting that L is alternating, we can finally obtain an element

$$\Psi \in \Gamma(\mathrm{Hom}(\wedge^{p+1} J^k(E), W) | K), \qquad k = \mathrm{Max}(k_0, k_1)$$

such that for any $\eta, \xi_1, \cdots, \xi_p \in \Gamma(E|K)$ we have

$$L(\eta, \xi_1, \cdots, \xi_p) = \tilde{\Phi}(\eta)(\xi_1, \cdots, \xi_p)$$
$$= \Psi(j_k(\eta), j_k(\xi_1), \cdots, j_k(\xi_p)).$$

Now in order to complete the induction, we have only to point out that such a Ψ obtained for each compact set K can be pieced together, which gives rise naturally to some $\Psi_0 \in \Gamma(\mathrm{Hom}(\wedge^{p+1} J(E), W))$. It is then clear that ${}^{(p+1)}\tilde{j}(\Psi_0) = L$, which completes the proof.

It may be better to add a remark that Proposition above can be generalized to the case where W is an inductive vector bundle. Note first that the definition of $C^p[E, W]$ is naturally extended to this case. We have

COROLLARY. *Let E be a vector bundle and $W = \varinjlim W_l$ an inductive vector bundle. Then we have a canonical isomorphism*

$$C^p[E, W] \cong \Gamma(\mathrm{Hom}(\wedge^p J(E), W)).$$

PROOF. Take an $L \in C^p[E, W]$ and let K be a compact set of M. Then we have

$$L: \Gamma(E|K) \times \cdots \times \Gamma(E|K) \longrightarrow \Gamma(W|K) = \varinjlim \Gamma(W_l | K).$$

Since the left-hand side is a Fréchet space, for some l_0 we have

$$L: \Gamma(E|K) \times \cdots \times \Gamma(E|K) \longrightarrow \Gamma(W_{l_0}|K) \ (\subset \Gamma(W|K)).$$

Thus L is realized as an element of $\Gamma(\mathrm{Hom}(\wedge^p J^{k_0}(E), W_{l_0})|K)$ for some k_0. From this we can easily deduce the assertion.

Similar remark will be applied to the results of this section, which will assure the validity of the results even to the case where W is an inductive vector bundle. However, to avoid some complications, we shall be only concerned with the case where W is a vector bundle (E is always assumed to be a vector bundle).

We set

$$C^p(E, W) = \mathrm{Hom}(\wedge^p J(E), W)$$

for $p = 1, 2, \cdots$; $C^p(E, W)$ are inductive vector bundles over M. Also set $C^0(E, W) = W$. Then Proposition 1.4 shows that there are canonical identifications

$$C^p[E, W] = \Gamma(C^p(E, W)), \qquad p = 0, 1, 2, \cdots.$$

Now we shall give a local description of $L \in C^p[E, W]$. Take an open set U such that $\bar{U}$ is contained in a local coordinates neighborhood $\left\{(x^1, \cdots, x^n) \,\middle|\, \sum_{\mu=1}^{n} (x^\mu)^2 < 1\right\}$. Assume that local triviality of E and W be given on $\bar{U}$ in terms of the local basis $e_1, \cdots, e_s$ of $E|\bar{U}$ and $f_1, \cdots, f_t$ of $W|\bar{U}$. Hence any $\sigma \in \Gamma(E|\bar{U})$ and $\tau \in \Gamma(W|\bar{U})$ are written in the form

$$\sigma(x) = \sum_{\lambda=1}^{s} \sigma^\lambda(x) e_\lambda\,, \qquad \tau(x) = \sum_{\beta=1}^{t} \tau^\beta(x) f_\beta\,.$$

We put

$$\omega_A^\lambda(\sigma) = \frac{\partial^{|A|}\sigma^\lambda}{\partial x^A} \quad \left(= \frac{\partial^{|A|}\sigma^\lambda}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}\right)$$

where $A = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i \geqq 0$, denote multi-indices and $\lambda = 1, \cdots, s$. Then $L \in C^p(E|\bar{U}, W|\bar{U})$ can be expressed in the following form:

$$L = \sum_{\beta,\lambda,A} L^{\beta A_1 \cdots A_p}_{\lambda_1 \cdots \lambda_p}(x) \omega_{A_1}^{\lambda_1} \wedge \cdots \wedge \omega_{A_p}^{\lambda_p} \otimes f_\beta\,, \tag{1.4}$$

where the summation is finite, and $\lambda_1, \cdots, \lambda_p = 1, \cdots, s$, $\beta = 1, \cdots, t$ and A's run over multi-indices; $L^{\beta A_1 \cdots A_p}_{\lambda_1 \cdots \lambda_p}(x)$ are smooth functions on $\bar{U}$. That L admits such a local expression follows immediately from the fact that $\omega_{A_1}^{\lambda_1} \wedge \cdots \wedge \omega_{A_p}^{\lambda_p}$, $|A_i| \leqq k$, form a local basis of $\mathrm{Hom}(\wedge^p J^k(E), \varepsilon^1)$ on $\bar{U}$.

As we have introduced above, $C^p[E, W]$ has the nuclear topology as the cross-section space of $C^p(E, W)$.

For reference we put

DEFINITION 1.4. The nuclear topology of $C^p[E, W]$ is called the *jet topology* of $C^p[E, W]$.

We shall make explicit what subsets of $C^p[E, W]$ form a basis of neighborhoods of 0 in the jet topology. Choose a locally finite open covering $\{U_i\}$ $(i = 1, 2, \cdots)$ of M such that each U_i is a local coordinates neighborhood and that $E|U_i$ and $W|U_i$ are trivial bundles on each U_i. Let $V_i \Subset U_i$ $(i = 1, 2, \cdots)$ be open sets satisfying $\cup V_i = M$. (Here the notation $V_i \Subset U_i$ means that $\bar{V}_i \subset U_i$ and $\bar{V}_i$ is compact.) From (1.4) any $L \in C^p[E, W]_{\bar{V}_i}$ can be expressed as

$$L = \sum L^{\beta A_1 \cdots A_p}_{\lambda_1 \cdots \lambda_p}(x\,;\, \bar{V}_i) \omega_{A_1}^{\lambda_1} \wedge \cdots \wedge \omega_{A_p}^{\lambda_p} \otimes f_\beta$$

where $L^{\beta A_1 \cdots A_p}_{\lambda_1 \cdots \lambda_p}(x\,;\, \bar{V}_i)$ means a smooth function on $\bar{V}_i$. Take any finite number of V_i, say, $V_{i_1}, \cdots, V_{i_l}$. We assume that for each index $(j\,;\, \beta\,;\, \lambda_1, \cdots, \lambda_p\,;\, A_1, \cdots, A_p)$ a neighborhood $\mathcal{U}^{\beta \lambda_1 \cdots \lambda_p}_{A_1 \cdots A_p}|\bar{V}_{i_j}$ of 0 in $\mathcal{E}|\bar{V}_{i_j}$ is given, where $j = 1, \cdots, l\,;\, \beta = 1, \cdots, t\,;\, \lambda_1, \cdots, \lambda_p = 1, \cdots, s$: A_i ranges over all the multi-indices. Then the following proposition is an immediate consequence of the definition of the jet topology:

PROPOSITION 1.5. *The totality of* $L\in C^p[E, W]$ *satisfying*

$$L^{\beta A_1\cdots A_p}_{\lambda_1\cdots\lambda_p}(x;\ \bar{V}_{i_j})\in \mathcal{V}^{\beta\lambda_1\cdots\lambda_p}_{A_1\cdots A_p}|\bar{V}_{i_j}$$

for each index $(j;\ \beta;\ \lambda_1, \cdots, \lambda_p;\ A_1, \cdots, A_p)$ *form a neighborhood of* 0 *in* $C^p[E, W]$. *Actually, all such neighborhoods occuring in the various choice of* $V_{i_j}, \cdots, V_{i_l}$ *and* $\mathcal{V}$ *form a basis of neighborhoods of* 0 *in* $C^p[E, W]$.

$C^p[E, W]$ $(p\geqq 1)$, however, is equipped with another natural topology too, being called the *distributional topology*. We are now going to describe this. Let $\mathcal{D}(E)$ be the Schwartz space consisting of smooth cross-sections of E with compact supports. Then any bounded set B of $\mathcal{D}(E)$ is characterized by the following properties:

i) the support of $\sigma\in B$ is all contained in a fixed compact set $K\subset M$, and

ii) any differential operator with a finite order from $\mathcal{D}(E)$ to $\mathcal{E}$ maps B into a bounded set of $\mathcal{E}$. The distributional topology of $C^p[E, W]$ is then defined to be the bounded convergence topology when we regard any element of $C^p[E, W]$ as a continuous p-linear map from $\mathcal{D}(E)\times\cdots\times\mathcal{D}(E)$ to $\Gamma(W)$.

Note that the distributional topology of $C^p[E, W]$ is completely determined by the distributional topology of $C^p[E, W]|K$ on each compact set K; the latter topology is clearly given by the bounded convergence topology on $\Gamma(E|K)$. More precisely, once we have introduced the distributional topology to $C^p[E, W]|K$, then the distributional topology of $C^p[E, W]$ is characterized as the weakest one for which the restriction map $C^p[E, W]\to C^p[E, W]|K$ becomes continuous for each compact set K.

The following notations will be frequently used:

$$C^p_k(E, W)=\mathrm{Hom}\,(\wedge^p J^k(E), W)\,,$$

$$C^p_k[E, W]=\Gamma(C^p_k(E, W))\,;\qquad p, k=0, 1, 2, \cdots.$$

PROPOSITION 1.6. i) *On the space* $C^p[E, W]$, *the distributional topology is weaker than the jet topology.*

ii) *On* $C^p_k[E, W]$ $(k=0, 1, 2, \cdots)$ *the jet topology and the distributional one induce the same topology.*

PROOF. i): From the remarks above, it suffices to show that, given a compact set $K\subset M$, the distributional topology is weaker than the jet topology on $C^p[E, W]|K$. This, however, is further reduced to the case where $K=\bar{V}$; here V is an open set whose closure is contained in a local coordinates neighborhood, and $E|\bar{V}$, $W|\bar{V}$ are trivial. Let $B\subset\Gamma(E|\bar{V})$ be a bounded set. We shall use the similar notations to those given in (1.4). Then for each $(\lambda_1, \cdots, \lambda_p;\ A_1, \cdots, A_p)$ there exists a positive integer $m(\lambda_1, \cdots, \lambda_p;\ A_1, \cdots, A_p)$ with

$$|\omega_{A_p}^{\lambda_1}\wedge\cdots\wedge\omega_{A_p}^{\lambda_p}(\sigma_1,\cdots,\sigma_p)|\leqq m(\lambda_1,\cdots,\lambda_p;A_1,\cdots,A_p)$$

whenever $\sigma_1,\cdots,\sigma_1\in B$. Let $\mathcal{W}$ be any convex neighborhood of 0 in $\Gamma(W|\bar{V})$. We may assume that $\mathcal{W}$ is expressed as

$$\mathcal{W}=\mathcal{U}_1\times\cdots\times\mathcal{U}_\beta\times\cdots\times\mathcal{U}_t$$

according to the local trivialization of $\Gamma(W|\bar{V})$ on $\bar{V}$, where $\mathcal{U}_\beta$ means a convex neighborhood of 0 in $\mathcal{E}|\bar{V}$. Then a basis of neighborhoods of 0 in the distributional topology consists of the sets with the form

$$\mathcal{X}(B,\mathcal{W})=\{L|L(\sigma_1,\cdots,\sigma_p)\in\mathcal{W}\text{ for }\sigma_1,\cdots,\sigma_p\in B\}\ .$$

Set

$$\mathcal{U}^{\beta\lambda_1\cdots\lambda_p}_{A_1\cdots A_p}|\bar{V}=\frac{1}{m(\lambda_1,\cdots,\lambda_p;A_1,\cdots,A_p)}\mathcal{U}_\beta\,.$$

Referring to Proposition 1.5, we find that these $\mathcal{U}^{\beta\lambda_1\cdots\lambda_p}_{A_1\cdots A_p}|\bar{V}$ together determine a neighborhood $\mathcal{U}$ of 0 in the jet topology of $C^p[E,W]|\bar{V}$. We have

$$\mathcal{U}\subset\mathcal{X}(B,\mathcal{W})$$

which completes the proof of i).

ii): The jet topology induces a Fréchet topology on $C_k^p[E,W]$, which is identified with that of $\Gamma(\mathrm{Hom}(\wedge^pJ^k(E),W))$. On the other hand, the distributional topology induces a weaker one than the jet topology. Hence it suffices to prove that the distributional topology induces a Fréchet topology on $C_k^p[E,W]$, or what amounts to the same, the distributional topology on $C_k^p[E,W]$ is complete. This, however, can be seen as follows: For any Cauchy sequence $\{L_n\}$ of $C_k^p[E,W]$, it is easy to verify that L_n converges to $L\in C^p[E,W]$ in the distributional topology; this L really belongs to $C_k^p[E,W]$ because $L\in C_k^p[E,W]$ is equivalent to the statement that $L(\xi_1,\cdots,\xi_p)(x)=0$ holds whenever for some ξ_i we have $j_k(\xi_i)(x)=0$. Hence $C_k^p[E,W]$ is a Fréchet space, which completes the proof.

PROPOSITION 1.7. *Let*

$$d:C^p[E,W]\longrightarrow C^{p+1}[E,W]$$

be a linear map satisfying the following conditions:

i) $\mathrm{supp}\,dL\subset\mathrm{supp}\,L,\ L\in C^p[E,W]$.

ii) *d is continuous when $C^p[E,W]$ and $C^{p+1}[E,W]$ both are endowed with the distributional topology. Then d becomes continuous even in the jet topology.*

PROOF. By i) and the definition of the jet topology, we have only to check that, for each compact set $K\subset M$, $d:C^p[E,W]|K\to C^{p+1}[E,W]|K$ is continuous in the jet topology. Since

$$C^p[E,W]|K=\varinjlim C_k^p[E,W]|K,$$

for this purpose it is sufficient to show the continuity of d on each $C_k^p[E, W]|K$ ($k=0, 1, 2, \cdots$) in the jet topology. But in the distributional topology $C_k^p[E, W]|K$ is a Fréchet space by Proposition 1.6. Hence for each k there is an integer l such that

$$d(C_k^p[E, W]|K) \subset C_l^{p+1}[E, W]|K .$$

From ii) and Proposition 1.6 ii) it follows that d is continuous from $C_k^p[E, W]|K$ to $C^{p+1}[E, W]|K$ in the jet topology, which completes the proof.

§ 2. Lie algebras over a manifold.

We shall first summarize some basic facts about cohomology theory of Lie algebras [11]. Let $\mathfrak{g}$ be a topological Lie algebra over $\boldsymbol{K}$ ($\boldsymbol{K}=\boldsymbol{R}$ or $\boldsymbol{C}$). Let V be a topological $\boldsymbol{K}$-module, on which some representation φ of $\mathfrak{g}$ operates. That is, φ is a homomorphism of $\mathfrak{g}$ into $\mathrm{Hom}(V, V)$, being considered as Lie algebras, for which $\varphi(\xi)L$ is continuous in ξ and L ($\xi \in \mathfrak{g}$, $L \in V$). A continuous p-multilinear alternating map from $\mathfrak{g}$ to V is called a p-cochain of $\mathfrak{g}$ with value in V, the totality of which is denoted by $C^p(\mathfrak{g}, V)$. We set $C^0(\mathfrak{g}, V)=V$, $C^{-1}(\mathfrak{g}, V)=0$. For $L \in C^p(\mathfrak{g}, V)$ ($p \geqq 1$), put

$$\begin{aligned} (2.1) \qquad & dL(\xi_1, \cdots, \xi_{p+1}) \\ &= \sum_{s=1}^{p+1}(-1)^{s-1}\varphi(\xi_s)L(\xi_1, \cdots, \check{\xi}_s, \cdots, \xi_{p+1}) \\ &\quad + \sum_{1 \leqq s<t \leqq p+1}(-1)^{s+t}L([\xi_s, \xi_t], \xi_1, \cdots, \check{\xi}_s, \cdots, \check{\xi}_t, \cdots, \xi_{p+1}) , \end{aligned}$$

and for $f \in C^0(\mathfrak{g}, V)=V$, put $df(\xi)=\varphi(\xi)f$. Then we have $dL \in C^{p+1}(\mathfrak{g}, V)$ and $d \circ d=0$, whence we get a cochain complex $\{C^p(\mathfrak{g}, V), d\}$. The cohomology group of this complex is denoted by

$$H^*(\mathfrak{g}, V)=\sum H^p(\mathfrak{g}, V) .$$

Suppose that there is a bilinear map

$$\cup : V \times V \longrightarrow V$$

such that

$$\varphi(\xi)(u \cup v)=\varphi(\xi)u \cup v+u \cup \varphi(\xi)v .$$

For $L \in C^p(\mathfrak{g}, V)$ and $L_1 \in C^q(\mathfrak{g}, V)$ we set

$$L \cup L_1(\xi_1, \cdots, \xi_{p+q})=\sum_S(-1)^{\nu(S)}L(\xi_{s_1}, \cdots, \xi_{s_p}) \cup L_1(\xi_{t_1}, \cdots, \xi_{t_q})$$

where the summation is extended over all ordered subsets $S=\{s_1, \cdots, s_p\}$ of $\{1, 2, \cdots, p+q\}$; $T=\{t_1, \cdots, t_q\}$ denotes the ordered complement of S, and $\nu(S)=\sum_{j=1}^{q} S(j)$, $S(j)$ standing for the number of indices i for which s_i is greater than t_j. The pairing $L\cup L_1$ induces a linear map

$$C^p(\mathfrak{g}, V)\otimes C^q(\mathfrak{g}, V) \longrightarrow C^{p+q}(\mathfrak{g}, V)$$

and we have

$$d(L\cup L_1)=dL\cup L_1+(-1)^p L\cup dL_1\,,$$

so that $\{C^p(\mathfrak{g}, V), d\}$ becomes a multiplicative complex. In particular, we have a cup product

$$\cup: H^p(\mathfrak{g}, V)\otimes H^q(\mathfrak{g}, V) \longrightarrow H^{p+q}(\mathfrak{g}, V)\,.$$

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Then we can introduce a filtration to $C^p(\mathfrak{g}, V)$ associated to $\mathfrak{k}$ as follows: For $s\leqq 0$, put $C_s^p=C^p(\mathfrak{g}, V)$; for $s>0$, put

$$C_s^p=\{L \mid L\in C^p(\mathfrak{g}, V),\ L(\xi_1, \cdots, \xi_p)=0$$
$$\text{whenever } p-s+1 \text{ of } \xi_1, \cdots, \xi_p \text{ belong to } \mathfrak{k}\}\,.$$

We have

$$C^p(\mathfrak{g}, V)=C_0^p\supset C_1^p\supset \cdots \supset C_s^p\supset \cdots \supset C_p^p\supset C_{p+1}^p=0\,,$$

$$d(C_s^p)\subset C_s^{p+1}\,.$$

From this filtration a spectral sequence $\{E_r^{s,t}, d_r\}$ is induced in a canonical way, which converges as $r\to\infty$ to a graded module associated to some filtration of $H^*(\mathfrak{g}, V)$. We cite this spectral sequence as *Hochschild-Serre's spectral sequence* associated to $\mathfrak{k}$. Note that

$$E_0^{s,t}=C_s^{s+t}/C_{s+1}^{s+t}\,.$$

In the sequel we shall utilize the cohomology theory of Lie algebras as a basic tool. But as far as the global aspects are concerned, we do not use the total cochain space $C^p(\mathfrak{g}, V)$ itself. Roughly speaking, we restrict our considerations only to Lie algebras and representations with the "support-preserving" property, and take the space of differential p-cochains to be pertinent to our theory.

Let E be a vector bundle over M.

DEFINITION 2.1. Assume there is a $\Phi\in C^2[E, E]$ such that, if we put

$$[\xi, \eta]=\Phi(\xi, \eta)\,,$$

we have the Jacobi identity

$$[\xi, [\eta, \zeta]]+[\eta, [\zeta, \xi]]+[\zeta, [\xi, \eta]]=0\,, \qquad \xi, \eta, \zeta\in\Gamma(E)\,,$$

then $\Gamma(E)$ is called a *Lie algebra over* M.

In other words, $\Gamma(E)$ is called a Lie algebra over M, if $\Gamma(E)$ has a bracket operation $[\xi, \eta]$ as Lie algebra, which satisfies the continuity condition and $\operatorname{supp}[\xi, \eta]\subset \operatorname{supp}\xi\cap\operatorname{supp}\eta$. Note that the bracket operation is then expressed as a differential operator on each compact set of M.

We shall mention some examples. We denote by $\tau(M)$ the tangent bundle of M and by $\tau^*(M)$ the cotangent bundle of M.

EXAMPLE 1. The space of vector fields, being identified with $\Gamma(\tau(M))$, becomes a Lie algebra over M under the usual bracket operation. This Lie algebra is denoted by $\mathfrak{A}(M)$. The complexification of $\mathfrak{A}(M)$ also forms a Lie algebra over M which will be denoted by $\mathfrak{A}(M)^C$.

EXAMPLE 2. Consider the vector bundle $\tau(M)\oplus\varepsilon^1$. Then any cross-section of this vector bundle is expressed as $\xi\oplus f$ ($\xi\in\mathfrak{A}(M)$, $f\in\mathcal{E}$), whence $\Gamma(\tau(M)\oplus\varepsilon^1)$ is canonically identified with the space $\boldsymbol{D}(1)$ of the first order differential operators on M. The bracket operation as differential operators permits us to endow $\boldsymbol{D}(1)$ with a structure of Lie algebra over M. Actually, the bracket operation is explicitly given by the formula

$$[\xi\oplus f,\ \eta\oplus g]=[\xi,\ \eta]\oplus(\xi g-\eta f)\,.$$

EXAMPLE 3. More generally, $\Gamma(\tau(M)\oplus\varepsilon^h)$ $(h=1, 2, \cdots)$ becomes a Lie algebra over M under the bracket operation

$$\Big[\xi\oplus\sum_{i=1}^{h}f_i,\ \eta\oplus\sum_{i=1}^{h}g_i\Big]=[\xi,\ \eta]\oplus\sum_{i=1}^{h}(\xi g_i-\eta f_i)\,.$$

EXAMPLE 4. Let E be a line bundle over M. $\boldsymbol{D}(1\,;E)$ denotes the space of the differential operators with the first order, from $\Gamma(E)$ to $\Gamma(E)$. We can identify

$$\boldsymbol{D}(1\,;E)=\Gamma(\mathrm{Hom}\,(J^1(E),\ E))\,.$$

$\boldsymbol{D}(1\,;E)$ becomes a Lie algebra over M, as in Example 2. In fact, Example 2 may be regarded as a special case of this example where we set $E=\varepsilon^1$.

EXAMPLE 5. Put

$$\tilde{\wedge}(M)=\Gamma\Big(\sum_{p=0}^{[\frac{n}{2}]}\wedge^{2p+1}(\tau^*(M))\Big)\,;$$

that is, $\tilde{\wedge}(M)$ is the space of the differential forms with odd degree. For $\xi, \eta\in\tilde{\wedge}(M)$, set

$$[\xi,\ \eta]=d(\xi\wedge\eta)=d\xi\wedge\eta-\xi\wedge d\eta\,.$$

As is easily verified, $\tilde{\wedge}(M)$ becomes a Lie algebra over M under this operation. Observe that $\tilde{\wedge}(M)$ is nilpotent and has a functorial property in the sense that, if $f: M\to N$ is a smooth map, then $f^*: \tilde{\wedge}(N)\to\tilde{\wedge}(M)$ naturally gives a homomorphism.

EXAMPLE 6. For any vector bundle E, if we set

$$[\xi,\ \eta]=0\,,\qquad \xi,\ \eta\in\Gamma(E)\,,$$

we have trivially a Lie algebra over M.

EXAMPLE 7. Let M be a complex manifold with the complex dimension n. Then $\tau(M)\otimes \boldsymbol{C}$ is decomposed as

$$\tau(M)\otimes \boldsymbol{C}=T\oplus \bar{T}\,, \tag{2.2}$$

where the local basis of T and $\bar{T}$ are given by $\{\partial/\partial z^1, \cdots, \partial/\partial z^n\}$, $\{\partial/\partial \bar{z}^1, \cdots, \partial/\partial \bar{z}^n\}$, respectively. $\Gamma(\bar{T})$ then becomes a Lie algebra over M as a subalgebra $\mathfrak{A}(M)^C$. We write $\mathfrak{A}_{\bar{\partial}}(M)$ for this Lie algebra.

EXAMPLE 8. Let M be a complex manifold. Then we have the decomposition

$$\tau^*(M)\otimes \boldsymbol{C}=T^*\oplus \bar{T}^*$$

dual to (2.2). For $p=0, 1, \cdots, n$ put

$$\tilde{\wedge}^p(M)=\begin{cases}\Gamma\Big(\wedge^p T^*\wedge \sum\limits_{q=0}^{[\frac{n}{2}]} \wedge^{2q+1}\bar{T}^*\Big)\,, & \text{if } p \text{ is even}\\[2ex] \Gamma\Big(\wedge^p T^*\wedge \sum\limits_{q=0}^{[\frac{n}{2}]} \wedge^{2q}\bar{T}^*\Big)\,, & \text{if } p \text{ is odd}\,,\end{cases}$$

and define

$$[\xi, \eta]=\bar{\partial}(\xi\wedge\eta)\,, \qquad \xi, \eta\in \tilde{\wedge}^p(M)\,.$$

Then $\tilde{\wedge}^p(M)$ $(p=0, 1, \cdots, n)$ has a structure of Lie algebra over M.

EXAMPLE 9. Assume that M admits a foliation. Then associated to this foliation, there exists a subbundle E of $\tau(M)$ such that $\Gamma(E)$ becomes a Lie subalgebra of $\mathfrak{A}(M)$.

Let $\Gamma(E)$ be a Lie algebra over M and W be a vector bundle over M. Notice that $\mathrm{Hom}\,(\Gamma(W), \Gamma(W))$ has a structure of Lie algebra with the bracket operation $[L, L_1]=L\circ L_1-L_1\circ L$.

DEFINITION 2.2. A Lie algebra representation $\varphi\colon \Gamma(E)\to \mathrm{Hom}\,(\Gamma(W), \Gamma(W))$ is called a *differential representation* if φ satisfies the conditions

i) the map $(\xi, L)\to\varphi(\xi)L$ is continuous from $\Gamma(E)\times\Gamma(W)$ to $\Gamma(W)$;

ii) $\mathrm{supp}\,\varphi(\xi)L\subset \mathrm{supp}\,\xi\cap \mathrm{supp}\,L$.

If φ is a differential representation, then a slight modification of the proof of Proposition 1.4 yields that there is a unique element $\tilde{\Phi}\in\Gamma(\varinjlim_{k,k'}\mathrm{Hom}\,(J^k(E)\otimes J^{k'}(W), W))$ such that

$$(\varphi(\xi)L)(x)=\tilde{\Phi}(j(\xi)(x), j(L)(x))\,.$$

Assume that a differential representation φ of $\Gamma(E)$ on $\Gamma(W)$ be given. According to (2.1), for any $L\in C^p[E, W]$ put

$$dL(\xi_1, \cdots, \xi_{p+1})$$
$$=\sum_{s=1}^{p+1}(-1)^{s-1}\varphi(\xi_s)L(\xi_1, \cdots, \check{\xi}_s, \cdots, \xi_{p+1})$$
$$+\sum_{1\leqq s<t\leqq p+1}(-1)^{s+t}L([\xi_s, \xi_t], \xi_1, \cdots, \check{\xi}_s, \cdots, \check{\xi}_t, \cdots, \xi_{p+1}).$$

Then we have

PROPOSITION 2.1. *dL belongs to $C^{p+1}[E, W]$.*

PROOF. It is clear that dL gives an alternating $(p+1)$-linear map from $\Gamma(E)$ to $\Gamma(W)$. Take any $\xi_1, \cdots, \xi_{p+1}\in\Gamma(E)$. Since $L\in C^p[E, W]$ and $\operatorname{supp}\varphi(\xi_1)L\subset\operatorname{supp}\xi_1\cap\operatorname{supp}L$, we have

$$\operatorname{supp}(\varphi(\xi_1)L(\xi_2, \cdots, \xi_{p+1}))\subset\operatorname{supp}\xi_1\cap\operatorname{supp}\xi_2\cap\cdots\cap\operatorname{supp}\xi_{p+1}.$$

On the other hand, we have

$$\operatorname{supp}L([\xi_1, \xi_2], \xi_3, \cdots, \xi_{p+1})$$
$$\subset\operatorname{supp}[\xi_1, \xi_2]\cap\operatorname{supp}\xi_3\cap\cdots\cap\operatorname{supp}\xi_{p+1}$$
$$\subset\operatorname{supp}\xi_1\cap\operatorname{supp}\xi_2\cap\cdots\cap\operatorname{supp}\xi_{p+1}.$$

From these it follows that dL has the support-preserving property. Since the continuity of dL is clear, this completes the proof.

Now we wish to show that d gives rise to a continuous map from $C^p[E, W]$ to $C^{p+1}[E, W]$. It is easy to verify that $\operatorname{supp}dL\subset\operatorname{supp}L$. Hence by Proposition 1.7 the continuity of d becomes a consequence of the following

LEMMA. *d gives rise to a continuous map from $C^p[E, W]$ to $C^{p+1}[E, W]$ in the distributional topology.*

The proof of Lemma reduces to establishing the continuity of

$$d: C^p[E, W]|K \longrightarrow C^{p+1}[E, W]|K \qquad \text{(distributional topology)}$$

for each compact set $K\subset M$, because of the definition of the distributional topology of $C^{p+1}[E, W]$ and the support-preserving property of d. But then Lemma is easily deduced from the following two facts:

(i) When ξ ranges over a bounded set of $\Gamma(E|K)$, the maps $L\rightarrow\varphi(\xi)L$ form an equi-continuous family;

(ii) If ξ and η range over a bounded set of $\Gamma(E|K)$, then $[\xi, \eta]$ form a bounded set of $\Gamma(E|K)$.

To summarize:

THEOREM 2.1. *Let $\Gamma(E)$ be a Lie algebra over M and φ a differential representation of $\Gamma(E)$ on $\Gamma(W)$. Then, putting*

$$dL(\xi_1, \cdots, \xi_{p+1})$$
$$=\sum_{s=1}^{p+1}(-1)^{s-1}\varphi(\xi_s)L(\xi_1, \cdots, \check{\xi}_s, \cdots, \xi_{p+1})$$
$$+\sum_{1\leqq s<t\leqq p+1}(-1)^{s+t}L([\xi_s, \xi_t], \xi_1, \cdots, \check{\xi}_s, \cdots, \check{\xi}_t, \cdots, \xi_{p+1})$$

for $L \in C^p[E, W]$, *we obtain a complex* $\{C^p[E, W], d\}$. d *gives rise to a continuous map from* $C^p[E, W]$ *to* $C^{p+1}[E, W]$ $(p=0, 1, 2, \cdots)$.

The resulting complex $\{C^p[E, W], d\}$ is called the *differential complex* of the Lie algebra $\Gamma(E)$ associated to the differential representation φ. It is clear that the adjoint representation of $\Gamma(E)$ gives a differential representation to which a differential complex $\{C^p[E, E], d\}$ is associated. We call this the *adjoint complex* of $\Gamma(E)$.

Let φ be a differential representation of $\Gamma(E)$ on $\Gamma(W)$ and $\{C^p[E, W], d\}$ the differential complex associated to φ. We write $H^*(E, W)=\sum H^p(E, W)$ for its cohomology group. Henceforth, we shall be mainly concerned with the investigation of the complex $\{C^p[E, W], d\}$ from the local and the global viewpoints. To begin the study, we first observe that the inductive vector bundle $C^p(E, W)$ admits many filtrations; of course, among them the jet filtration $\varinjlim C^p_k(E, W)=C^p(E, W)$ is considered as a natural one. We say that a filtration

$$C^p(E, W)=\varinjlim {}_hC^p(E, W)$$

simultaneously given for $p=0, 1, 2, \cdots$ is *admissible* if, for $h=0, 1, 2, \cdots$, the sequence

$$\cdots \xrightarrow{d} \Gamma({}_hC^p(E, W)) \xrightarrow{d} \Gamma({}_hC^{p+1}(E, W)) \xrightarrow{d} \cdots \tag{2.3}$$

is well-defined, so that $\{\Gamma({}_hC^{p+1}(E, W)), d\}$ gives rise to a subcomplex of $\{C^p[E, W], d\}$ for $h=0, 1, 2, \cdots$. For brevity, set ${}_hC^{p+1}[E, W]=\Gamma({}_hC^{p+1}(E, W))$.

DEFINITION 2.3. We say that $\{C^p[E, W], d\}$ has a *stable range* if there is an admissible filtration $C^p(E, W)=\varinjlim {}_hC^p(E, W)$ such that the inclusion map as a subcomplex

$${}_h\iota : \{{}_hC^p[E, W], d\} \longrightarrow \{C^p[E, W], d\}$$

induces an isomorphism on the cohomology level for each $h=0, 1, 2, \cdots$. In particular, if there is a non-negative integer k_0 such that the jet filtration for $k \geqq k_0$

$$C^p_{k0}(E, W) \subset C^p_{k0+1}(E, W) \subset \cdots \longrightarrow C^p(E, W), \qquad p=0, 1, 2, \cdots$$

gives a stable range, then we say that $\{C^p[E, W], d\}$ has the *stable jet range* $k \geqq k_0$.

Note that ${}_hC^p[E, W]$ is the cross-section space of a "finite-dimensional" vector bundle ${}_hC^p(E, W)$. Although in some case the sequence (2.3) continues infinitely in the right direction, the following definition is meaningful if we slightly extend the definition on elliptic complexes so as to be applicable to this case.

DEFINITION 2.4. We say that $\{C^p[E, W], d\}$ has an *elliptic range* if there is an admissible filtration $C^p(E, W) = \lim_h {}_hC^p(E, W)$ such that each subcomplex $\{{}_hC^p(E, W), d\}$ $(h=0, 1, 2, \cdots)$ is an elliptic complex over M. In particular, if the jet filtration for $l \geqq l_0$ gives an elliptic range, then we say that $\{C^p[E, W], d\}$ has the *elliptic jet range* $l \geqq l_0$.

We shall mention a typical example of a differential complex on $\mathfrak{A}(M)$ (cf. Example 1), first considered by M. V. Losik [**12**]. Any $\xi \in \mathfrak{A}(M)$ operates on $\xi = \Gamma(\varepsilon^1)$ in a natural way, being locally expressed as $(\xi, f) \to \sum_{\mu=1}^{n} \xi^\mu \partial f/\partial x^\mu$, where $\xi = \sum \xi^\mu \partial/\partial x^\mu$ (locally) and $f \in \mathcal{E}$. This operation induces a differential representation of $\mathfrak{A}(M)$. The differential complex on $\mathfrak{A}(M)$ obtained from this representation is called the *Losik complex*, which is, for the sake of reference, written by

$$\mathcal{L} = \{C^p[\tau(M), \varepsilon^1], d\} .$$

The structure of cohomology group $H^*(\mathcal{L})$ of the Losik complex has been completely determined by Losik [**11**], which can be summarized as follows:

LOSIK'S THEOREM. The Losik complex has the stable jet range $k \geqq 1$, and admits a canonical isomorphism

$$H^*(\mathcal{L}) = H^*(B(\tau^C), \boldsymbol{R}) ,$$

where $B(\tau^C)$ is the principal $U(n)$-bundle over M, associated to the vector bundle $\tau(M) \otimes \boldsymbol{C}$. Besides, the sheaf of cohomology $H^*(\mathcal{L})$ is a constant sheaf, isomorphic to $M \times H^*(U(n), \boldsymbol{R})$. In particular, the Euler characteristic of the Losik complex is zero: $\chi(\mathcal{L}) = 0$.

We shall show later in a more general setting that the Losik complex has the elliptic jet range $l \geqq 0$.

§ 3. Local Lie algebras.

Gelfand and Fuks [**7**] generalized Losik's result cited at the end of Section 2 to the case where $\mathfrak{A}(M)$ has the space of differential forms on M as the representation space, the representation being given by Lie differentiation. The viewpoint there, taken up by them, seems to be crucial in the whole theory, because the deep relation between the local and the global aspects has been considerably clarified by them. Although we follow essentially the similar lines, we wish to make this relation more explicit from a general point of view. In this section we shall only deal with local situation. Transfer to global situation will be carried out in the next section.

Let A be a Noetherian local $\boldsymbol{R}$-algebra with the unit 1. Let $\mathfrak{m}$ be the maximal ideal of A and ι a natural injection of $\boldsymbol{R}$ in A such that $\iota(\alpha) = \alpha \cdot 1$. We shall always assume that A satisfies the following two conditions:

i) The composition of maps $\boldsymbol{R} \xrightarrow{\iota} A \xrightarrow{\pi} A/\mathfrak{m}$ is bijection.

ii) $\mathfrak{m}$ is finitely generated over A.

From this follows directly that $\mathfrak{m}^k$ is also finitely generated over A and that $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is finitely generated over $A/\mathfrak{m} \cong \boldsymbol{R}$. Hence we have $\dim_{\boldsymbol{R}} (A/\mathfrak{m}^k) < +\infty$ for $k=1, 2, \cdots$. Let B be a finitely generated A-module. Then B is a Noetherian A-module so that, setting $B_k = \mathfrak{m}^{k+1}B$, from Nakayama's lemma we have $\cap B_k = \{0\}$. We shall endow B with the Krull topology; namely, the sequence of submodules

$$B \supset B_0 \supset B_1 \supset \cdots \supset B_k \supset \cdots$$

gives a fundamental system of neighborhoods of 0. Note that $\dim_{\boldsymbol{R}} B/B_k < +\infty$ $(k=1, 2, \cdots)$.

Assume that a finite-dimensional vector space V over $\boldsymbol{R}$ be given. We denote by $C^p(B, V)$ the set of alternating continuous p-linear maps from $\overbrace{B \times \cdots \times B}^{p}$ to V. Let $\pi_k : B \to B/B_k$ be the canonical projection map, and put $C_k^p(B, V) = \pi_k^* C^p(B/B_k, V)$, where $C^p(B/B_k, V) = \wedge^p (B/B_k)^* \otimes V$ (* denotes the dual space). Hence any element of $C_k^p(B, V)$ is obtained as a lifting of some alternating p-linear map from $\overbrace{B/B_k \times \cdots \times B/B_k}^{p}$ to V. We have a canonical inclusion $C_k^p(B, V) \subset C_{k+1}^p(B, V)$.

Proposition 3.1. $C^p(B, V) = \bigcup_{k=0}^{\infty} C_k^p(B, V)$.

Proof. First observe that for $L \in C^p(B, V)$ we have

$$L \in C_k^p \Longleftrightarrow L(\xi_1, \cdots, \xi_p) = 0 \text{ if some } \xi_i \ (i=1, \cdots, p) \text{ belongs to } B_k .$$

Also, note that any small neighborhood of $0 \in V$ does not contain non-trivial subspace of V. To avoid some notational complications, we shall only prove Proposition in case $p=3$. Take an $L \in C^3(B, V)$. By the continuity of L, we have $L | B_{k_0} \times B_{k_0} \times B_{k_0} \equiv 0$ for some k_0. Let N_{k_0} be a complementary subspace of B_{k_0} to B and $\{e_1, \cdots, e_s\}$ be a basis of N_{k_0}; $B = B_{k_0} \oplus N_{k_0}$. Then the continuity of the bilinear map $(\xi_2, \xi_3) \mapsto L(e_i, \xi_2, \xi_3)$ for each $i=1, 2, \cdots, s$ implies that for some k_1' we have $L | N_{k_0} \times B_{k_1}' \times B_{k_1}' \equiv 0$. Thus, putting $k_1 = \operatorname{Max}(k_0, k_1')$, we have $L | B \times B_{k_1} \times B_{k_1} \equiv 0$. Next, take a complementary subspace N_{k_1} of B_{k_1} and let $\{e_1', \cdots, e_t'\}$ be a basis of N_{k_1}. The continuity of the map $\xi_3 \mapsto L(e_i', e_j', \xi_3)$ for each $i, j=1, \cdots, t$ implies that for some k_2' we have $L | N_{k_1} \times N_{k_1} \times B_{k_2'} \equiv 0$. If we set $k = \operatorname{Max}(k_1, k_2')$, then it can be easily shown that $L \in C_k^p(B, V)$, which completes the proof.

By virtue of Proposition 3.1, we can and do give the inductive limit topology to $C^p(B, V)$: $C^p(B, V) = \varinjlim C_k^p(B, V)$ $(= \varinjlim (\wedge (B/B_k)^* \otimes V))$. Set $C^0(B, V) = V$.

Definition 3.1. B is said to be a *local Lie algebra* (over A), if B is a

Lie algebra over $\boldsymbol{R}$ and the bracket operation satisfies the following condition: for any k, there is an l such that $[B_l, B] \subset B_k$.

We remark that in this definition if we can choose l with $l \leqq k$, then B_k necessarily becomes an ideal of B. Assume that a local Lie algebra B over A be given. Consider a finite-dimensional representation φ of B on V. The continuity of φ is always assumed. This implies that there exists an integer h with $\varphi(B_h)=0$. Hence, for the existence of non-trivial finite-dimensional representation, it is necessary and sufficient that there is an ideal I such that $B \supsetneqq I \supset B_h$ for some h. According to the cohomology theory of Lie algebras, given a representation φ, we can obtain a complex $\{C^p(B, V), d\}$ associated to φ. Note that if $L \in C_k^p(B, V)$, then we have $dL \in C^{p+1}_{\mathrm{Max}(h,k,l)}(B, V)$ where $[B_l, B] \subset B_k$ and $\varphi(B_h)=0$.

It is clear that d gives rise to a continuous map from $C^p(B, V)$ to $C^{p+1}(B, V)$. By Proposition 3.1 $C^p(B, V)$ admits a canonical filtration. Thus, by the analogy of Definition 2.3, we can define that $\{C^p(B, V), d\}$ has the stable range $k \geqq k_0$, if for each $k \geqq k_0$ the subcomplex $\cdots \longrightarrow C_k^p(B, V) \xrightarrow{d} C_k^{p+1}(B, V) \longrightarrow \cdots$ is well-defined and the inclusion map $\{C_k^p(B, V), d\} \to \{C^{p+1}(B, V), d\}$ induces an isomorphism on cohomology level.

One might hope that there is a general theory which clarifies the relation between the cohomology groups of local Lie algebras and those of Lie algebras over a manifold. But such an extensive possibility of the theory is far beyond our scope. In the sequel, we shall only investigate the relationship between local and global aspects in some cases. Really, in the rest of this section we shall only deal with a local Lie algebra related to the Lie algebra of vector fields; global passage will be discussed in the next section. Some of other cases will be treated in the subsequent papers.

Let $\mathfrak{a}_n$ be the Lie algebra of formal vector fields with n indeterminates. That is,

$$\mathfrak{a}_n = \boldsymbol{R}[[x_1, \overbrace{\cdots, x_n]] \oplus \cdots \oplus \boldsymbol{R}[[x_1,}^{n} \cdots, x_n]]$$

with the bracket rule

$$\Big[\sum_{\mu=1}^{n} a^\mu(x_1, \cdots, x_n)\frac{\partial}{\partial x_\mu}, \sum_{\mu=1}^{n} b^\mu(x_1, \cdots, x_n)\frac{\partial}{\partial x_\mu}\Big]$$
$$= \sum_{\mu=1}^{n}\sum_{\nu=1}^{n}\Big(a^\nu\frac{\partial b^\mu}{\partial x_\nu} - b^\nu\frac{\partial a^\mu}{\partial x_\nu}\Big)\frac{\partial}{\partial x_\mu};$$

here $\boldsymbol{R}[[x_1, \cdots, x_n]]$ denotes the ring of formal power series in $x_1, \cdots, x_n$ and element of $\mathfrak{a}_n$ is formally expressed as $\sum a^\mu(x_1, \cdots, x_n)\dfrac{\partial}{\partial x_\mu}$ $(a^\mu \in \boldsymbol{R}[[x_1, \cdots, x_n]])$. $\mathfrak{a}_n$ is a local Lie algebra over $\boldsymbol{R}[[x_1, \cdots, x_n]]$; it is known that $\mathfrak{a}_n$ is simple if we regard $\mathfrak{a}_n$ as a Lie algebra over $\boldsymbol{R}$. We denote by

$$\mathfrak{a}_n \supset L_0 \supset L_1 \supset \cdots \supset L_k \supset \cdots$$

the canonical filtration of $\mathfrak{a}_n$. Then L_0 is a subalgebra of $\mathfrak{a}_n$ and each L_k is an ideal of L_0. $L_0/L_1 \cong \mathfrak{gl}(n\,;\boldsymbol{R})$, and if we identify $\sum \alpha_\nu^\mu x_\nu \dfrac{\partial}{\partial x_\mu} \in L_0$ with $(\alpha_\nu^\mu) \in \mathfrak{gl}(n\,;\boldsymbol{R})$, then we have a canonical splitting $L_0 = \mathfrak{gl}(n\,;\boldsymbol{R}) \oplus L_1$. Our main interest here lies in the study of the local Lie algebra L_0. (For convenience' sake, we set $C_k^p(L_0, V) = \wedge^p(L_0/L_k)^* \otimes V$ in this case.)

For $\mu = 1, 2, \cdots, n$ and any multi-index A, let θ_A^μ be the element of L_0^* defined by

$$\theta_A^\mu\Bigl(\sum_{\nu=1}^n \sum_{|B|\geqq 1} \gamma_B^\nu x^B \frac{\partial}{\partial x_\nu}\Bigr) = A\,!\gamma_A^\mu\,,$$

where $x^B = x_1^{b_1} \cdots x_n^{b_n}$ for $B = (b_1, \cdots, b_n)$. Then $\{\theta_A^\mu \mid \mu = 1, 2, \cdots, n\,;\ 1 \leqq |A| \leqq k\}$ forms a base of $(L_0/L_k)^*$. Hence if we take a base $\{e_1, \cdots, e_v\}$ of V, then $C^p(L_0, V)$ is spanned by the elements

$$\theta_{A_1}^{\mu_1} \wedge \cdots \wedge \theta_{A_p}^{\mu_p} \otimes e_\gamma\,, \qquad \mu_1, \cdots, \mu_p = 1, \cdots, n\,;\ |A_i| \geqq 1\,;\ \gamma = 1, \cdots, v\,.$$

Now for any family of integers $(f_1, \cdots, f_m)$ with $f_1 \geqq \cdots \geqq f_m > 0$, we denote by $[f_1, \cdots, f_m]$ the irreducible representation of $\mathfrak{gl}(n\,;\boldsymbol{R})$ corresponding to the Young diagram $f_1 \geqq \cdots \geqq f_m > 0$. Moreover, for any $\delta \in \boldsymbol{R}$, write $\delta\,\mathrm{Trace}$ for the one-dimensional representation of $\mathfrak{gl}(n\,;\boldsymbol{R})$, which assigns $\delta\,\mathrm{Trace}\,A$ to A ($A \in \mathfrak{gl}(n\,;\boldsymbol{R})$). Set

$$[f_1, \cdots, f_m\,;\ \delta] = [f_1, \cdots, f_m] \otimes 1 + 1 \otimes \delta\,\mathrm{Trace}\,.$$

Then $[f_1, \cdots, f_m\,;\ \delta]$ is an irreducible representation of $\mathfrak{gl}(n\,;\boldsymbol{R})$ with the same degree as the one of $[f_1, \cdots, f_m]$.

Let φ be a representation of L_0 on V. Then, as remarked above, φ can be regarded as a lifting of some representation of L_0/L_h on V for a suitable h.

DEFINITION 3.2. φ is said to be *decomposable* if $\varphi|\mathfrak{gl}(n\,;\boldsymbol{R})$ is decomposed in irreducible representations with the form $[f_1, \cdots, f_m\,;\ \delta]$.

REMARK. This means that $\varphi|\mathfrak{gl}(n\,;\boldsymbol{R})$ is completely reducible and that the restriction of φ to the center of $\mathfrak{gl}(n\,;\boldsymbol{R})$ has real eigenvalues.

Assume that a decomposable representation φ be given. Then $\varphi|\mathfrak{gl}(n\,;\boldsymbol{R})$ is decomposed in irreducible constituents, which is explicitly expressed as

$$\varphi|\mathfrak{gl}(n\,;\boldsymbol{R}) = \sum \oplus [f_1, \cdots, f_m\,;\ \delta]\,.$$

Among the irreducible representations occurring in the right side, we shall observe the irreducible representations $[f_1, \cdots, f_m\,;\ \delta]$ satisfying

(i) $f_1 + \cdots + f_m + n\delta \geqq 0$,

(ii) $n\delta$ is an integer,

and denote by Δ the subfamily formed by these $[f_1, \cdots, f_m\,;\ \delta]$.

THEOREM 3.1. *Assume that a decomposable representation* φ *of* L_0 *on* V *be given:* $\varphi|\mathfrak{gl}(n;\boldsymbol{R})=\sum\oplus[f_1,\cdots,f_m;\delta]$. *Then the complex* $\{C^p(L_0,V),d\}$ *associated to* φ *has the following property.*

i) $\{C^p(L_0,V),d\}$ *has the stable range* $k\geqq k_0$, *where*

$$k_0=\mathrm{Max}\left\{h,\ \mathop{\mathrm{Max}}_{[f_1,\cdots,f_m;\delta]\in\Delta}\left(\sum_{\mu=1}^{m}f_\mu+n\delta+1\right)\right\}.$$

ii) *If* $\Delta=\emptyset$, *then the cohomology group* $H^*(L_0,V)$ *vanishes.*

PROOF. Recall first that the irreducible representation $[f_1,\cdots,f_m]$ is realized as an invariant subspace of the tensor space $\boldsymbol{R}^n\otimes\cdots\otimes\boldsymbol{R}^n$ under the tensorial representation, where $\boldsymbol{R}^n$ appears in $(f_1+\cdots+f_m)$-times. A base of this invariant subspace can be obtained in such a way that we apply the Young symmetrizer associated with $\{f_1,\cdots,f_m\}$ to each $\tilde{e}_{i_1}\otimes\cdots\otimes\tilde{e}_{i_s}$ $(i_1,\cdots,i_s=1,\cdots,n;\ s=f_1+\cdots+f_m)$, $\tilde{e}_i$ being the i-th unit vector of $\boldsymbol{R}^n$, and then pick out of them a maximal set of linearly independent family. Corresponding to each $[f_1,\cdots,f_m;\delta]$ appearing in the decomposition of $\varphi|\mathfrak{gl}(n;\boldsymbol{R})$, take such a base for $[f_1,\cdots,f_m]$ and then arrange all of them in order. Thus we can obtain a base $\{e_1,\cdots,e_v\}$ of V. In what follows, we shall fix this base of V. Let $\{e^1,\cdots,e^v\}$ be the dual base to $\{e_1,\cdots,e_v\}$.

Let J be the set of those pairs (μ,A) that $\mu=1,\cdots,n$ and that for each μ A runs over the multi-indices with $|A|\geqq 1$ and $A\neq(0,\cdots,0,\overset{\mu}{1},0,\cdots,0)$. Then φ can be explicitly written in the form

$$\varphi\left(\sum\gamma_A^\mu x^A\frac{\partial}{\partial x_\mu}\right)=\sum_{\alpha=1}^{v}\left(\sum_{\mu=1}^{n}c_\mu^\alpha\gamma_\mu^\mu\right)e_\alpha\otimes e^\alpha+\sum_{\alpha,\beta=1}^{v}\sum_{(\mu,A)\in J}\Psi_{\beta\mu}^{\alpha A}\gamma_A^\mu e_\alpha\otimes e^\beta,$$

where c_μ^α and $\Psi_{\beta\mu}^{\alpha A}$ are some constants and $\Psi_{\beta\mu}^{\alpha A}=0$ for $|A|>h$. More precisely, if an e_α is chosen from the base of the representation space associated to $[f_1,\cdots,f_m;\delta]$, then it is easy to see that

$$\sum_{\mu=1}^{n}c_\mu^\alpha=f_1+\cdots+f_m+n\delta\,. \tag{3.1}$$

Denote by $\hat{d}$ the coboundary operator

$$\hat{d}:\ C^p(L_0,\boldsymbol{R})\longrightarrow C^{p+1}(L_0,\boldsymbol{R})$$

associated to the trivial representation of L_0 on $\boldsymbol{R}$:

$$\hat{d}L(\xi_1,\cdots,\xi_{p+1})=\sum_{s<t}(-1)^{s+t}L([\xi_s,\xi_t],\xi_1,\cdots,\check{\xi}_s,\cdots,\check{\xi}_t,\cdots,\xi_{p+1})\,.$$

It is easy to verify that $\{C^p(L_0,\boldsymbol{R}),\hat{d}\}$ is a multiplicative complex and that

$$\hat{d}\theta_A^\mu=\sum_{\substack{B\leqq A\\|B|\neq 0}}\sum_{\lambda=1}^{n}\binom{A}{B}\theta_{A-B+\lambda}^\mu\wedge\theta_B^\lambda\,. \tag{3.2}$$

(Here we adopt the following notations: For multi-indices $A=(\alpha_1,\cdots,\alpha_n)$

and $B=(\beta_1,\cdots,\beta_n)$ $A\geqq B$ means $\alpha_\mu\geqq\beta_\mu$ $(\mu=1,\cdots,n)$; and if $A\geqq B$, $A-B+\lambda$ denotes the multi-index

$$(\alpha_1-\beta_1,\cdots,\alpha_{\lambda-1}-\beta_{\lambda-1},\alpha_\lambda-\beta_\lambda+1,\alpha_{\lambda+1}-\beta_{\lambda+1},\cdots,\alpha_n-\beta_n).$$

Any $L\in C^p(L_0,V)$ is written as $L=\sum_{\alpha=1}^{v}L^\alpha\otimes e_\alpha$, where $L^\alpha\in C^p(L_0,\boldsymbol{R})$. Making use of $\hat{d}$ and the above expression of φ, we can give the explicit form of d in the following way:

$$(3.3)\qquad d\Big(\sum_{\alpha=1}^{v}L^\alpha\otimes e_\alpha\Big)=\sum_{\alpha=1}^{v}\hat{d}L^\alpha\otimes e_\alpha +\sum_{\alpha=1}^{v}\Big(\sum_{\mu=1}^{n}c_\mu^\alpha\theta_\mu^\mu\wedge L^\alpha\Big)\otimes e_\alpha+\sum_{\alpha,\beta=1}^{v}\sum_{(\mu,A)\in J}(\Psi_{\beta\mu}^{\alpha A}\theta_A^\mu\wedge L^\beta)\otimes e_\alpha .$$

Let $\mathfrak{h}$ be the abelian subalgebra of $\mathfrak{gl}(n;\boldsymbol{R})$ which consists of the diagonal matrices. Let $\{E_r^{p,q},d_r\}$ be Hochshild-Serre's spectral sequence of $\{C^p(L_0,V),d\}$ associated to the subalgebra $\mathfrak{h}$. It is possible to identify $E_r^{p,q}$ with a subspace of $C^{p+q}(L_0,V)$; in fact, any element $\eta\in E_r^{p,q}$ is expressed as $\eta=\sum_{\alpha=1}^{v}\eta^\alpha\otimes e_\alpha$, where

$$(3.4)\qquad \eta^\alpha=\sum_{\nu_1<\cdots<\nu_q}\sum_{(\mu,B)\in J}Y^{\alpha}_{\nu_1\cdots\nu_q}{}^{B_1\cdots B_p}_{\mu_1\cdots\mu_p}\theta_{\nu_1}^{\nu_1}\wedge\cdots\wedge\theta_{\nu_q}^{\nu_q}\wedge\theta_{B_1}^{\mu_1}\wedge\cdots\wedge\theta_{B_p}^{\mu_p}.$$

Since $d_0:E_0^{p,q}\to E_0^{p,q+1}$, by (3.3) we have

$$d_0\eta=\sum_{\alpha=1}^{v}\hat{d}_0\eta^\alpha\otimes e_\alpha+\sum_{\alpha=1}^{v}\Big(\sum_{\mu=1}^{n}c_\mu^\alpha\theta_\mu^\mu\wedge\eta^\alpha\Big)\otimes e_\alpha ,$$

where $\hat{d}_0$ means the differentiation of the 0-th terms of Hochshild-Serre's spectral sequence associated to $\mathfrak{h}$ which is obtained from the complex $\{C^p(L_0,\boldsymbol{R}),\hat{d}\}$. From (3.2), we can obtain

$$\hat{d}_0\theta_A^\mu=\sum_{\lambda=1}^{n}\{\delta_\lambda^\mu-(A)_\lambda\}\theta_\lambda^\lambda\wedge\theta_A^\mu ;$$

here δ_λ^μ is the Kronecker index and $(A)_\lambda=\alpha_\lambda$ if $A=(\alpha_1,\cdots,\alpha_n)$.

Therefore, setting

$$(3.5)\qquad Q^{\alpha B_1\cdots B_p}_{\lambda\mu_1\cdots\mu_p}=\sum_{j=1}^{p}\{\delta_\lambda^{\mu_j}-(B_j)_\lambda\}+c_\lambda^\alpha ,$$

and introducing the forms

$$(3.6)\qquad \tilde{Q}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}=\sum_{\lambda=1}^{n}\tilde{Q}^{\alpha B_1\cdots B_p}_{\lambda\mu_1\cdots\mu_p}\theta_\lambda^\lambda$$

and

$$\tilde{\eta}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}=\sum_{\nu_1<\cdots<\nu_q}Y^{\alpha}_{\nu_1\cdots\nu_q}{}^{B_1\cdots B_p}_{\mu_1\cdots\mu_p}\theta_{\nu_1}^{\nu_1}\wedge\cdots\wedge\theta_{\nu_q}^{\nu_q}$$

we can simply express $d_0\eta$ in the form

$$d_0\eta=\sum_{\alpha=1}^{v}\Big(\sum_{(\mu,B)\in J}\tilde{Q}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}\wedge\tilde{\eta}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}\wedge\theta^{\mu_1}_{B_1}\wedge\cdots\wedge\theta^{\mu_p}_{B_p}\Big)\otimes e_\alpha\,.$$

Hence we have $d_0\eta=0$ if and only if for each indices $\alpha,(\mu_1,B_1),\cdots,(\mu_p,B_p)$ occurring in the terms of the summation (3.4) it holds

$$\tilde{Q}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}\wedge\eta^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}=0\,. \tag{3.7}$$

Making use of $d_0\circ d_0=0$ and a well-known lemma due to E. Cartan, we can deduce from (3.7) that $\operatorname{Im} d_0(E_0^{p,q-1})$ consists of such closed η's that are expressible as

$$\eta=\sum_\alpha\sum_{(\mu,B)}\sum_{\nu_1<\cdots<\nu_q}Y^{\alpha\quad\ B_1\cdots B_p}_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p}\theta^{\nu_1}_{\nu_1}\wedge\cdots\wedge\theta^{\nu_q}_{\nu_q}\wedge\theta^{\mu_1}_{B_1}\wedge\cdots\wedge\theta^{\mu_p}_{B_p}\otimes e_\alpha$$

where the summation only ranges over the indices $\alpha,(\mu_1,B_1),\cdots,(\mu_p,B_p)$ satisfying

$$\tilde{Q}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}\neq 0\,.$$

From this it follows directly that

$$E_1^{p,q}=\{\eta\mid\eta=\sum_\alpha\sum_{(\mu,B)}\sum_{\nu_1<\cdots<\nu_q}Y^{\alpha\quad\ B_1\cdots B_p}_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p}\theta^{\nu_1}_{\nu_1}\wedge\cdots\wedge\theta^{\nu_q}_{\nu_q}\wedge\theta^{\mu_1}_{B_1}\wedge\cdots\wedge\theta^{\mu_p}_{B_p}\otimes e_\alpha\ \text{where}\ \tilde{Q}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}=0\}\,. \tag{3.8}$$

We wish to obtain further informations about the structure of $E_1^{p,q}$. To this aim, fixing a family of indices $\alpha,(\mu_1,B_1),\cdots,(\mu_p,B_p)$ with

$$\tilde{Q}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}=0\,,$$

we shall study to what extent this condition imposes restrictions on the indices $\alpha,(\mu_1,B_1),\cdots,(\mu_1,B_p)$. By (3.6), $\tilde{Q}^{\alpha B_1\cdots B_p}_{\mu_1\cdots\mu_p}=0$ is equivalent to the fact

$$\tilde{Q}^{\alpha B_1\cdots B_p}_{\lambda\mu_1\cdots\mu_p}=0\qquad\text{for}\quad\lambda=1,2,\cdots,n\,,$$

whence by (3.5) we can simply express the conditions by

$$\sum_{j=1}^{p}\{\delta^{\mu_j}_\lambda-(B_j)_\lambda\}+c^\alpha_\lambda=0\,,\qquad\lambda=1,2,\cdots,n\,. \tag{3.9}$$

Since $(B_j)_\lambda$ are all integers, (3.9) immediately yields the following necessary condition on c^α_λ.

(3.10) In order to hold $\tilde{Q}_\lambda=0$ $(\lambda=1,2,\cdots,n)$, it is necessary that $c^\alpha_\lambda\,(\lambda=1,\cdots,n)$ are all integers.

Now, we shall only consider the value of α for which all c^α_λ $(\lambda=1,\cdots,n)$ are integers. Then we have

$$\sum_{\lambda=1}^{n}c^\alpha_\lambda\geqq|B_j|-1\qquad j=1,2,\cdots,p\,. \tag{3.11}$$

Actually, in view of (3.9) we have

$$\sum_{\lambda=1}^{n} c_\lambda^\alpha = \sum_{\lambda=1}^{n}\sum_{j=1}^{p}\{(B_j)_\lambda - \delta_\lambda^{\mu_j}\} = \sum_{j=1}^{p}(|B_j|-1) \geqq |B_j|-1 .$$

In particular, we have

$$\sum_{\lambda=1}^{n} c_\lambda^\alpha \geqq 0 . \tag{3.12}$$

We shall complete the proof of Theorem 3.1. Note that, if $k \geqq h$, the subcomplex $\{C_k^p(L_0, V), d\}$ is well-defined. In fact, this follows immediately from $[L_0, L_k] \subset L_k$ and $\varphi(L_h)=0$. Take any integer k which satisfies

$$k \geqq \mathrm{Max}\Big\{\underset{(f_1,\cdots,f_m;\,\delta)\in\Delta}{\mathrm{Max}}\Big(\sum_{\mu=1}^{m} f_\mu + n\delta + 1\Big),\ h\Big\} .$$

Let $\{E_r^{p,q}(k), d_r\}$ be Hochshild-Serre's spectral sequence of $\{C_k^p(L_0, V), d\}$ associated to $\mathfrak{h}$. Then the injection map $\iota(k)$ from $C_k^p(L_0, V)$ to $C^p(L_0, V)$ induces the homomorphism of the spectral sequence

$$\iota_r(k) : \{E_r^{p,q}(k), d_r\} \longrightarrow \{E_r^{p,q}, d_r\}$$

for $r=0, 1, 2, \cdots$. From (3.1), (3.8) and (3.11), we can easily deduce that $\iota_r(k)$ gives an isomorphism

$$\{E_r^{p,q}(k), d_r\} \cong \{E_r^{p,q}, d_r\} \qquad \text{for} \quad r \geqq 1 ,$$

so that $\iota(k)$ induces an isomorphism of cohomology groups of $\{C_k^p(L_0, V), d\}$ and $\{C^p(L_0, V), d\}$, which completes the proof of i). If $\Delta=\emptyset$, then by (3.1), (3.10) and (3.12) we have $E_1^{p,q}=0$ for all p, q, whence ii) follows. This completes the proof of Theorem.

COROLLARY 1. $\dim H^*(L_0, W) < +\infty$.

COROLLARY 2. *In the case where all $n\delta$ are not integers, we have* $H^*(L_0, W)=0$.

COROLLARY 3. $\sum(-1)^p \dim H^p(L_0, W)=0$.

PROOF OF COROLLARY 3. It is well-known that $\sum(-1)^p \dim H^p(L_0/L_k, W) = \sum(-1)^p \dim C^p(L_0/L_k, W)$, which in turn equals $\sum(-1)^p \dim \wedge^p(L_0/L_k)\cdot\dim W = 0$. But in the stable range we have $\dim H^p(L_0, W) = \dim H^p(L_0/L_k, W)$, whence the assertion follows. AN ALTERNATIVE PROOF: We deform δ arbitrarily small so as to satisfy the conditions stated in Corollary 2. This induces the deformation of the coboundary operator d. But, in the stable range, $\sum(-1)^p \dim H^p(L_0/L_k, W)$ is homotopy invariant, whence this must vanish.

§ 4. Passage from local aspects to global ones.

In this section we shall describe how to obtain a global version of Theorem 3.1. Let $\mathrm{Diff}_n(0)$ be the group germ of local diffeomorphisms of

$\boldsymbol{R}^n$ around 0. For $\varphi \in \mathrm{Diff}_n(0)$, we have $\varphi(0)=0$ and denote by $[\varphi]_h$ the h-jet of φ at 0. Then the totality of $[\varphi]_h$ becomes a Lie group $G(h)$ under the composition rule $[\varphi]_h[\psi]_h=[\varphi\circ\psi]_h$. It is known that $G(h)$ is obtained as successive extensions of $GL(n;\boldsymbol{R})$ by vector groups and possesses L_0/L_h as its Lie algebra. Let Diff_n be the pseudo-group of local diffeomorphisms of $\boldsymbol{R}^n$. When φ ranges over Diff_n, the differentials $d\varphi$ form a pseudo-group of local diffeomorphisms of the tangent space $\tau(\boldsymbol{R}^n)$. If $\varphi(x_0)=y_0$, the $(h-1)$-jet of $d\varphi$ at $(x_0, 0)$ $(\in \tau(\boldsymbol{R}^n)_{x_0})$ gives rise to an isomorphism from $J^{h-1}(\tau(\boldsymbol{R}^n))_{x_0}$ to $J^{h-1}(\tau(\boldsymbol{R}^n))_{y_0}$. It should be noted that, for fixed x_0 and y_0, the group consisting of those isomorphisms is canonically isomorphic to $G(h)$, but not to $G(h-1)$. To any vector field ξ of $\boldsymbol{R}^n$ assign a formal Taylor series of ξ at x_0. This assignment induces an isomorphism

$$J^{\infty}(\tau(\boldsymbol{R}^n))_{x_0}\cong \mathfrak{a}_n\,,$$

where $J^{\infty}(\tau(\boldsymbol{R}^n))=\varinjlim J^h(\tau(\boldsymbol{R}^n))$, so that we have

$$J^{h-1}(\tau(\boldsymbol{R}^n))_{x_0}\cong \mathfrak{a}_n/L_{h-1}\,.$$

Let M be a smooth manifold. Then it follows from the above that the structural group of $J^{h-1}(\tau(M))$ is reducible to $G(h)$. Let $P(h)$ be the principal $G(h)$-bundle associated to $J^{h-1}(\tau(M))$. Alternatively, $P(h)$ can be obtained as the principal $G(h)$-bundle with those transition functions which consist of the h-jets without constant terms induced from coordinates transformations. Thus we have

$$J^{h-1}(\tau(M))=P(h)\times_{\sigma}(\mathfrak{a}_n/L_{h-1})\,,$$

where σ is the natural representation of $G(h)$ on $\mathfrak{a}_n/L_{h-1}$ as explained above. Set

$$Ad_h(\tau(M))=P(h)\times_{Ad}(L_0/L_h)$$

for the associated vector bundle of $P(h)$ to the adjoint representation of $G(h)$. Recall that we put $\mathfrak{A}(M)=\Gamma(\tau(M))$. Let

$$\mathfrak{A}(M)(x_0)=\{\tilde{\xi}\mid\tilde{\xi}\in\mathfrak{A}(M),\ \tilde{\xi}(x_0)=0\}$$

and denote by $[\tilde{\xi}]_h(x_0)$ the h-jet of $\tilde{\xi}$ at x_0.

PROPOSITION 4.1. *For any x_0, the assignment $\tilde{\xi}\to[\tilde{\xi}]_h(x_0)$ gives rise to a well-defined surjective map from $\mathfrak{A}(M)(x_0)$ to $Ad_h(\tau(M))_{x_0}$.*

PROOF. Since what we must prove is of local character, it suffices to consider only on a neighborhood of 0 in $\boldsymbol{R}^n$ (x_0 corresponds to $0\in\boldsymbol{R}^n$). Then $\tilde{\xi}(x_1,\cdots,x_n)=\sum_{\mu=1}^{n}\tilde{\xi}^{\mu}(x_1,\cdots,x_n)\dfrac{\partial}{\partial x_\mu}$ and $[\tilde{\xi}]_h(0)$ is expressed as

$$\sum_{\mu}\sum_{|\alpha|\geqq 1}^{h}D^{\alpha}\tilde{\xi}^{\mu}(0)\frac{\partial}{\partial x_\mu}\,,$$

which is identified with an element of L_0/L_h. Let $\varphi \in \mathrm{Diff}_n(0)$. Then the operation rule of φ to $\tilde{\xi}$ is given by

$$\tilde{\xi} \longrightarrow \frac{d}{dt}(\varphi \circ \exp t\tilde{\xi} \circ \varphi^{-1})_{t=0}\,.$$

But, since $\tilde{\xi}(0)=0$, it is easily verified that if $[\varphi]_h=[\psi]_h$ and $[\tilde{\xi}]_h(0)=[\tilde{\eta}]_h(0)$, we obtain

$$\Big[\frac{d}{dt}(\varphi \circ \exp t\tilde{\xi} \circ \varphi^{-1})_{t=0}\Big]_h(0)=\Big[\frac{d}{dt}(\psi \circ \exp t\tilde{\eta} \circ \psi^{-1})_{t=0}\Big]_h(0)\,.$$

By virtue of the definition of $G(h)$ this implies that $[\varphi]_h$ operates on $[\tilde{\xi}]_h$ as the adjoint representation, which completes the proof.

Now let a finite-dimensional representation ρ of $G(h)$ on V be given. Consider the Lie algebra representation $d\rho$ associated to ρ, so that $d\rho$ gives a representation of L_0/L_h on V. Expressing the elements of L_0/L_h by their components, we can write $d\rho$ explicitly as

$$d\rho(\{\eta_A^\mu\})=\sum C_{\beta\mu}^{\alpha A}\eta_A^\mu e^\beta \otimes e_\alpha\,, \qquad C_{\beta\mu}^{\alpha A} \in \boldsymbol{R}\,, \tag{4.1}$$

where $1 \leqq |A| \leqq h$, $\mu=1, \cdots, n$ and α, $\beta=1, \cdots, v$ $(v=\dim V)$.

Let

$$W=P(h)\times_\rho V$$

be the vector bundle over M, associated to the representation ρ. Take an element of W, which is, say, represented by an equivalence class containing (p, α) $(p \in P(h), \alpha \in V)$. Let $\pi(p)=x_0$. By Proposition 4.1 any $\tilde{\xi} \in \mathfrak{A}(M)(x_0)$ defines an element of $Ad_h(\tau(M))_{x_0}$, which is assumed to be represented by an equivalence class containing $(p, \tilde{\xi}')$ $(\tilde{\xi}' \in L_0/L_h)$. Then we have

PROPOSITION 4.2. *Set*

$$d\tilde{\rho}(\tilde{\xi})(p, \alpha)=(p, d\rho(\tilde{\xi}')\alpha)\,.$$

Then $d\tilde{\rho}$ gives rise to a well-defined map from $\mathfrak{A}(M)(x_0)$ to $\mathrm{Hom}(W_{x_0}, W_{x_0})$.

PROOF. Notice that

$$(p, \tilde{\xi}')\sim(pg^{-1}, Ad(g)\tilde{\xi}')$$

and

$$(p, \alpha)\sim(pg^{-1}, \rho(g)\alpha)\,.$$

But we have

$$(pg^{-1}, d\rho(Ad(g)\tilde{\xi}')\rho(g)\alpha)=(pg^{-1}, \rho(g)d\rho(\tilde{\xi}')\alpha)\sim(p, d\rho(\tilde{\xi}')\alpha)\,.$$

From this the assertion follows.

To any local coordinates $(U; x^1, \cdots, x^n)$ with $\sum|x^\mu|^2<1$, we can associate a local triviality of $P(h)$ on U in a canonical way, which in turn induces local trivialities of the vector bundles $\tau(M)$ and W on U. Thus when we fix a base $\{e_1, \cdots, e_v\}$ of V, we can obtain a local base $\{e_1, \cdots, e_v\}$ of W on U. We wish to define a differential representation $\rho^\#$ of $\mathfrak{A}(M)$ on $\Gamma(W)$. To this aim, using (4.1) we first define this locally on U, by setting

$$\rho^\#(\xi)\sigma = \sum_{\mu=1}^{n} \xi^\mu \frac{\partial \sigma^\alpha}{\partial x^\mu} e_\alpha + \sum_{\alpha, \beta, A, \mu} C_{\beta\mu}^{\alpha A} \frac{\partial^{|A|} \xi^\mu}{\partial x^A} \sigma^\beta e_\alpha \tag{4.2}$$

where, on U, we write $\xi = \sum_{\mu=1}^{n} \xi^\mu(x) \frac{\partial}{\partial x^\mu}$ $(\in \mathfrak{A}(M))$ and $\sigma = \sum_{\alpha=1}^{v} \sigma^\alpha(x) e_\alpha$ $(\in \Gamma(W))$.

PROPOSITION 4.3. *$\rho^\#$ gives rise to a well-defined differential representation of $\mathfrak{A}(M)$ on $\Gamma(W)$.*

PROOF. Let $\mathfrak{a}_n'$ be the subalgebra of $\mathfrak{a}_n$ consisting of formal vector fields with polynomial coefficients: $\mathfrak{a}_n' \cong \boldsymbol{R}[x_1, \cdots, x_n] \otimes \boldsymbol{R}^n$. Put $L_0' = L_0 \cap \mathfrak{a}_n'$. Denote by $[\mathfrak{a}_n']$ and $[L_0']$ the universal enveloping algebras of $\mathfrak{a}_n'$ and L_0'. We regard $[\mathfrak{a}_n']$ and V as left $[L_0']$-modules, where the latter module structure is given by the representation $d\rho$. Denote by $\mathrm{Hom}_{[L_0']}([\mathfrak{a}_n'], V)$ the space of $[L_0']$-homomorphism from $[\mathfrak{a}_n']$ to V. Then $\mathrm{Hom}_{[L_0']}([\mathfrak{a}_n'], V)$ has the $[\mathfrak{a}_n']$-module structure, whence the induced representation of $\mathfrak{a}_n'$ is obtained. The resulting representation of $\mathfrak{a}_n'$ on $\mathrm{Hom}_{[L_0']}([\mathfrak{a}_n'], V)$ is referred to as $d\rho^\#$. Since $\mathfrak{a}_n'$ is spanned by $\boldsymbol{R}^n$ and L_0', the Poincaré-Witt theorem immediately yields that a left $[L_0']$-base of $[\mathfrak{a}_n']$ consists of the set of monomials with n-indeterminates (cf. [**3**]). Thus $\mathrm{Hom}_{[L_0']}([\mathfrak{a}_n'], V)$ is identified with the n-variables polynomial ring with value in V: $\mathrm{Hom}_{[L_0']}([\mathfrak{a}_n'], V) \cong \boldsymbol{R}[x_1, \cdots, x_n] \otimes V$. The completion of $\mathrm{Hom}_{[L_0']}([\mathfrak{a}_n'], V)$ is denoted by $\mathrm{Hom}_{[L_0]}([\mathfrak{a}_n], V)$, whence we have $\mathrm{Hom}_{[L_0]}([\mathfrak{a}_n], V) \cong \boldsymbol{R}[[x_1, \cdots, x_n]] \otimes V$ and any element in it is expressed as $\gamma(x_1, \cdots, x_n) \otimes \alpha$ with $\gamma \in \boldsymbol{R}[[x_1, \cdots, x_n]]$ and $\alpha \in V$. The representation $d\rho^\#$ naturally extends to the representation of $\mathfrak{a}_n$ on $\mathrm{Hom}_{[L_0]}([\mathfrak{a}_n], V)$. Explicitly, for $\eta \in \mathfrak{a}_n$ we have

$$d\rho^\#(\eta)(\gamma \otimes \alpha) = \eta(\gamma) \otimes \alpha + \sum_{A \geq 0} \gamma(x_1, \cdots, x_n) x^A \otimes \frac{1}{A!} \left(\widetilde{\frac{\partial^{|A|} \eta}{\partial x^A}} \right) \alpha,$$

where $\eta(\gamma)$ means the application of the formal vector field η to each term of γ and $(\widetilde{\partial^{|A|}\eta/\partial x^A})$ denotes the formal vector field deleting the constant term from $\partial^{|A|}\eta/\partial x^A$. This is checked by a direct calculation. Note that $d\rho^\#$ is uniquely determined by $d\rho$.

Now let $x_0 \in M$ and take local coordinates $\{U; x^1, \cdots, x^n\}$ around x_0. Then $\tau(M)$ and W canonically admit local trivialities on U, whence to any vector field ξ and any $\sigma \in \Gamma(W)$ we can associate their formal Taylor series $[\xi]$ and $[\sigma]$ at x_0 with reference to these trivialities. We regard $[\xi]$ and $[\sigma]$ as the elements of $[\mathfrak{a}_n]$ and $\mathrm{Hom}_{[L_0]}([\mathfrak{a}_n], V)$, respectively. The above formula then implies

$$d\rho^{\#}([\xi])[\sigma]=[\rho^{\#}(\xi)\sigma]$$

$$(=\text{the formal Taylor series of } \rho^{\#}(\xi)\sigma \text{ at } x_0).$$

From this first follows that $\rho^{\#}$ gives a differential representation of $\Gamma(\tau(M)|U)$ on $\Gamma(W|U)$, because $d\rho^{\#}$ is a representation and two cross-sections coincide if their formal Taylor series coincide at each point of U. Secondly, ρ is well-defined independent of the choice of local coordinates. In fact, if a point x_0 has two local coordinates $(x^1, \cdots, x^n)$ and $(y^1, \cdots, y^n)$ and if we carry over the above construction from $(x^1, \cdots, x^n)$ to $(y^1, \cdots, y^n)$, then, by virtue of Proposition 4.2, $d\rho$ and thus also $d\rho^{\#}$ obey the transition rule relevant to the fibre W_{x_0}. Hence the same holds for $[\rho^{\#}(\xi)\sigma]$. But this is valid at each point in a neighborhood of x_0, from which the assertion follows. This completes the proof.

DEFINITION 4.1. $\rho^{\#}$ is called the *induced differential representation* of ρ.

Therefore a representation ρ of $G(h)$ on V brings about two representations $d\rho$ and $\rho^{\#}$: the former concerns L_0/L_h while the latter $A(M)$. In what follows, the lifting of $d\rho$ to L_0 is also denoted by the same notation, which we believe will cause no confusion. We shall apply the discussions in Sections 2 and 3 to these representations. Then, corresponding to each representation, we obtain the "local" complex $\{C^p(L_0, V), d\}_{d\rho}$ and the "global" one $\{C^p[\tau(M), W], d\}_{\rho^{\#}}$. (Here the subscripts indicate the representations which are used to construct the complexes.) Their cohomology groups are denoted respectively by $H^*(L_0, V)$ and $H^*(\tau(M), W)$. These two complexes, however, are closely related.

THEOREM 4.1. *There is a spectral sequence $\{E_r^{p,q}, d_r\}$ which converges to a graded module associated to $H^*(\tau(M), W)$ with some filtration. The E_2-terms of this spectral sequence have the form*

$$E_2^{p,q} \cong H^p(M, \underline{H}^q(L_0, V)),$$

where $\underline{H}^q(L_0, V)$ means a locally constant sheaf each stalk of which is isomorphic to $H^q(L_0, V)$.

Before entering into the proof, we recall some basic facts about the theory of double complexes. Let Σ^p be the sheaf of germs of elements belonging to $C^p[\tau(M), W]$. Note that Σ^p is fine, because $C^p[\tau(M), W]$ is the cross-section space of inductive vector bundle. Since d preserves the supports of cochains, it follows that, by localizing $\{C^p[\tau(M), W], d\}$, we can get the sheaf of complex

$$\Sigma : 0 \longrightarrow \Sigma^0 \xrightarrow{d} \Sigma^1 \xrightarrow{d} \cdots \xrightarrow{d} \Sigma^p \xrightarrow{d} \cdots.$$

The sheaf of cohomology associated to Σ is denoted by

$$\mathcal{H}^*(\Sigma)=\sum \mathcal{H}^p(\Sigma)\,.$$

Let

$$\mathcal{R}:0\longrightarrow \boldsymbol{R}\longrightarrow \mathcal{R}^0\xrightarrow{'d}\mathcal{R}^1\xrightarrow{'d}\cdots\xrightarrow{'d}\mathcal{R}^n\longrightarrow 0$$

be the resolution sheaf of $\boldsymbol{R}$. (We can take the sheaf of the de Rham complex as $\mathcal{R}$.) Consider the double complex

$$\mathcal{D}=\{\Gamma(\mathcal{R}\otimes\Sigma),\bar{d}\}\,,$$

where $\bar{d}='d+''d$ and $''d=(-1)^p d$ on $\Gamma(\mathcal{R}^p\otimes\Sigma^q)$. To this double complex canonically associate the first and the second spectral sequences $\{'E_r^{p,q},'d_r\}$ and $\{''E_r^{p,q},''d_r\}$. We know that

$$'E_2^{p,q}='H^p(''H^q(\mathcal{D}))$$
$$''E_2^{p,q}=''H^p('H^q(\mathcal{D}))\,,$$

where the meaning of the right side may be inferable from the context in the proof of the next lemma (as to the precise definition, see [**9**]). Specifically we have

LEMMA.

$$'E_2^{p,q}=H^p(M,\mathcal{H}^q(\Sigma))\,, \tag{4.3}$$

$$''E_2^{p,q}=\begin{cases} H^p(\tau(M),W)\,, & \textit{if } q=0\\ 0\,, & \textit{otherwise}\,.\end{cases} \tag{4.4}$$

PROOF. We shall first prove (4.3). Note that $''H^q(\mathcal{D})$ is the q-th cohomology group of the complex

$$\cdots\xrightarrow{''d}\Gamma(\sum_p\mathcal{R}^p\otimes\Sigma^q)\xrightarrow{''d}\Gamma(\sum_p\mathcal{R}^p\otimes\Sigma^{q+1})\xrightarrow{''d}\cdots.$$

Hence there is a natural homomorphism

$$\Phi:\ ''H^q(\mathcal{D})\longrightarrow\Gamma(\sum_p\mathcal{R}^p\otimes\mathcal{H}^p(\Sigma))\,.$$

It is necessary to show that Φ is bijective. We only prove that Φ is surjective, since the injectivity of Φ can be similarly proved. Take $\sigma\in\Gamma(\sum_p\mathcal{R}^p\otimes\mathcal{H}^q(\Sigma))$. Then, by a suitable choice of locally finite open covering $\{U_i\}$ $(i=1,2,\cdots)$ of M, σ can be represented by $\{s_{u_i}\}$ $(i=1,2,\cdots)$. That is, each s_{u_i} belongs to $\Gamma(U_i,\sum_p\mathcal{R}^p\otimes\Sigma^q)$ and satisfies $''ds_{u_i}=0$; besides, the cohomology class of s_{u_i} coincides with σ on U_i. Let ρ_i $(i=1,2,\cdots)$ be homomorphisms from $\sum_p\mathcal{R}^p$ to itself with $\operatorname{supp}\rho_i\subset U_i$ and $\sum_i\rho_i=1$. Setting $\tilde{s}=\sum(\rho_i\otimes 1)(s_{u_i})\in\Gamma(\sum_p\mathcal{R}^p\otimes\Sigma^q)$, we have $''d\tilde{s}=0$. Moreover, if we denote by $[\tilde{s}]$ the cohomology represented by $\tilde{s}$, $\Phi([\tilde{s}])=\sigma$ holds, which shows the surjectivity of Φ. Once the bijectivity of Φ is established, $'H^p(''H^q(\mathcal{D}))$ turns out to be the p-th cohomology group of

$$\cdots \xrightarrow{'d} \Gamma(\mathcal{R}^p \otimes \mathcal{H}^q(\Sigma)) \xrightarrow{'d} \Gamma(\mathcal{R}^{p+1} \otimes \mathcal{H}^q(\Sigma)) \xrightarrow{'d} \cdots .$$

Then (4.3) immediately follows, since the sequence of sheaves

$$0 \longrightarrow \mathcal{H}^q(\Sigma) \longrightarrow \mathcal{R}^1 \otimes \mathcal{H}^q(\Sigma) \xrightarrow{'d} \cdots \xrightarrow{'d} \mathcal{R}^p \otimes \mathcal{H}^p(\Sigma) \xrightarrow{'d} \cdots$$

gives a resolution of $\mathcal{H}^q(\Sigma)$.

The similar reasoning applies to $''E_2$-terms and then (4.4) becomes an immediate consequence of Poincaré's lemma. This completes the proof.

Proof of Theorem. Let $\{E_r^{p,q}, d_r\}$ be the first spectral sequence constructed in the above. Then we have

$$E_2^{p,q} \cong H^p(M, \mathcal{H}^q(\Sigma)) .$$

Moreover, this spectral sequence converges to the graded module associated to $H^*(\tau(M), W)$ with some filtration, because the second spectral sequence is degenerate in $H^*(\tau(M), W)$ and the first and the second spectral sequences both approximate the cohomology group of the double complex. Thus it suffices to show that $\mathcal{H}^q(\Sigma)$ is a locally constant sheaf and each stalk is isomorphic to $H^q(L_0, V)$.

Let a local coordinates neighborhood $(U; x^1, \cdots, x^n)$ with $\sum|x^\mu|^2<1$ be fixed. Assume that the representation $d\rho$ be explicitly written in the form of (4.1). Consequently, $\rho^\#$ is given by (4.2) on U. According to (1.4), any $L \in C^p[\tau(M), W]|U$ is expressible as

$$L=\sum_{\alpha=1}^{v} L^\alpha \otimes \tilde{e}_\alpha ,$$

where $L^\alpha \in C^p[\tau(M), \varepsilon^1]|_U$ and each L^α has the form

$$L^\alpha=\sum L^{\alpha A_1 \cdots A_p}_{\mu_1 \cdots \mu_p}(x) \omega^{\mu_1}_{A_1} \wedge \cdots \wedge \omega^{\mu_p}_{A_p} .$$

Let $\mathcal{L}=\{C^p[\tau(M), \varepsilon^1], \hat{d}\}$ be the Losik complex. Since the Losik complex is multiplicative, the operation rule of $\hat{d}$ is uniquely determined by the following formulas:

$$\begin{aligned} &\hat{d}f=\sum_{\mu=1}^{n} \frac{\partial f}{\partial x^\mu} \omega^\mu_0 \qquad (f\colon \text{smooth function on } M), \\ &\hat{d}\omega^\mu_0=0 , \\ &\hat{d}\omega^\mu_A=\sum_{\substack{B \leqq A \\ |B| \neq 0}} \sum_{\lambda=1}^{n} \binom{A}{B} \omega^\mu_{A-B+\lambda} \wedge \omega^\lambda_B \qquad (|A| \geqq 1) . \end{aligned} \tag{4.5}$$

Note that in the customary usage of notation we have $\omega_0^\mu = dx^\mu$. Hence, in case where $L \in C^p[\tau(M), \varepsilon^1]$ coincides with a usual differential form, $\hat{d}L$ is nothing but the exterior differentiation of L. Using $\hat{d}$, we can write down the explicit form of d as follows:

$$d\Big(\sum_{\alpha=1}^{v} L^\alpha \otimes \tilde{e}_\alpha\Big) = \sum_{\alpha=1}^{v} \hat{d}L^\alpha \otimes \tilde{e}_\alpha + \sum_{\alpha,\beta,\mu,A} C_{\beta\mu}^{\alpha A} \omega_A^\mu \wedge L^\beta \otimes \tilde{e}_\alpha . \tag{4.6}$$

Here $C_{\beta\mu}^{\alpha A}$ denote the coefficients of the representation $d\rho$ as given in (4.1). Observe that in the second summation of the right side the multi-indices A range over only a finite set of multi-indices with $|A| \geqq 1$.

Now consider the abelian subalgebra $\mathfrak{k}$ of $\mathfrak{A}(M)|U$, generated by $\{\partial/\partial x^1, \cdots, \partial/\partial x^n\}$. Let $\{E_r^{p,q}, d_r\}$ be Hochshild-Serre's spectral sequence of $\{C^p[\tau(M), W]|U, d\}$ associated to $\mathfrak{k}$. Then, as is easily seen, $E_0^{p,q}$ is identified with the subspace of $C^{p+q}[\tau(M), W]|U$, consisting of the elements

$$L = \sum_{\alpha=1}^{v} L^\alpha \otimes \tilde{e}_\alpha \qquad \text{with}$$

$$L^\alpha = \sum_{\nu,\mu,A} L_{\nu_1\cdots\nu_q}^{\alpha}{}_{\mu_1\cdots\mu_p}^{A_1\cdots A_p} \omega_0^{\nu_1} \wedge \cdots \wedge \omega_0^{\nu_q} \wedge \omega_{A_1}^{\mu_1} \wedge \cdots \wedge \omega_{A_p}^{\mu_p} ,$$

where $\mu, \nu = 1, \cdots, n$ and $|A_i| \geqq 1$. To find the explicit form of d_0, it is useful to consider Hochshild-Serre's spectral sequence $\{\hat{E}_r^{p,q}, \hat{d}\}$ of the Losik complex on U, associated to $\mathfrak{k}$. Then, since $d_0 : E_0^{p,q} \to E_0^{p,q+1}$, it follows from (4.6) that

$$d_0 L = \sum_{\alpha=1}^{v} \hat{d}_0 L^\alpha \otimes e_\alpha .$$

$\hat{d}_0 L^\alpha$ is calculated as follows:

$$\begin{aligned}\hat{d}_0 L^\alpha = &\sum_{\nu,\mu,A} \hat{d}_0 (L_{\nu_1\cdots\nu_q}^{\alpha}{}_{\mu_1\cdots\mu_p}^{A_1\cdots A_p}(x) \omega_0^{\nu_1} \wedge \cdots \wedge \omega_0^{\nu_q}) \wedge \omega_{A_1}^{\mu_1} \wedge \cdots \wedge \omega_{A_p}^{\mu_p} \\ &+ (-1)^q \sum_{\nu,\mu,A} (L_{\nu_1\cdots\nu_q}^{\alpha}{}_{\mu_1\cdots\mu_p}^{A_1\cdots A_p}(x) \omega_0^{\nu_1} \wedge \cdots \wedge \omega_0^{\nu_q} \wedge \hat{d}_0 (\omega_{A_1}^{\mu_1} \wedge \cdots \wedge \omega_{A_p}^{\mu_p}) .\end{aligned}$$

Observe the right side. By virtue of (4.5), $\hat{d}_0$ occurring in the first summation is found to be the exterior differentiation, while $\hat{d}_0$ in the second summation becomes the zero map. Thus, passing from E_0-terms to E_1-terms, we can apply Poincaré's lemma. Actually we obtain

$$E_1^{p,q} = 0 , \qquad q > 0$$

$$E_1^{p,0} = C^p[\tau(M), W]_{\mathrm{const}} .$$

Here $C^p[\tau(M), W]_{\mathrm{const}}$ denotes the subspace of $C^p[\tau(M), W]|U$, which consists of the elements having the form $\sum L^\alpha \otimes \tilde{e}_\alpha$ with

$$L^\alpha = \sum_{\substack{|A| \geqq 1 \\ \mu}} Y^{\alpha A_1\cdots A_p}_{\mu_1\cdots\mu_p} \omega_{A_1}^{\mu_1} \wedge \cdots \wedge \omega_{A_p}^{\mu_p}$$

(the coefficients Y are constants).

It is easy to see that $\{C^p[\tau(M), W]_{\mathrm{const}}, d\}$ becomes a subcomplex of $\{C^p[\tau(M), W]|U, d\}$, and that the above construction of Hochshild-Serre's spectral sequence remains valid for this subcomplex. This involves that the injection

$$\{C^p[\tau(M), W]_{\mathrm{const}}, d\} \subsetneq \{C^p[\tau(M), W]|U, d\}$$

induces an isomorphism on E_1-level and hence on the cohomology level. Finally, referring to § 3, we can verify that the assignment $\omega_A^\mu \to \theta_A^\mu$ induces an isomorphism

$$\{C^p[\tau(M), W]_{\mathrm{const}}, d\} \cong \{C^p(L_0, V), d\}\,.$$

Hence, setting $\mathcal{H}^q(\Sigma)_x$ for the stalk of $\mathcal{H}^q(\Sigma)$ at x, we have a canonical isomorphism

$$\mathcal{H}^q(\Sigma)_x \cong H^q(L_0, V)$$

for each point $x \in U$, which completes the proof.

Let $\{C^p[\tau(M), W], d\}$ be the same as above. Hence there is given a representation ρ of $G(h)$ on V and $\{C^p[\tau(M), W], d\}$ is obtained from the induced differential representation $\rho^{\#}$. We say that M is of *finite type* if $\dim H^*(M, \mathfrak{S}) < +\infty$ is valid for any locally constant sheaf $\mathfrak{S}$ over M, with $\dim \mathfrak{S}_x < +\infty$ $(x \in M)$.

It is well-known that in the following two cases M is of finite type:

(i) M admits a simple finite open covering;

(ii) $\dim H^*(M; \boldsymbol{R}) < +\infty$ and $\pi_1(M)$ is finite.

Then we have

THEOREM 4.2. i) a) *If M is of finite type and* $\dim H^*(L_0, V) < +\infty$, *then* $\dim H^*(\tau(M), W) < +\infty$.

b) *In addition, if M is simply connected, we have*

$$\dim H^p(\tau(M), W) \leqq \sum_{q+r=p} \dim H^q(M; \boldsymbol{R}) \times \dim H^r(L_0, V)\,.$$

ii) *If $\{C^p(L_0, V), d\}$ has the stable range $k \geqq k_0$, then $\{C^p[\tau(M), W], d\}$ has the stable jet range $k \geqq k_0$.*

iii) *$\{C^p[\tau(M), W], d\}$ has the elliptic jet range $l \geqq h$.*

PROOF. i) a): Immediate from Theorem 4.1.

i) b): We have only to notice that, if M is simply connected, a locally constant sheaf necessarily becomes a constant sheaf.

ii): First note that if $\{C^p(L_0, V), d\}$ has the stable range $k \geqq k_0$, then, at least for $k \geqq k_0$, the subcomplex $\{C_k^p[\tau(M), W], d\}$ is well-defined. This follows from a careful comparison of the coboundary operators in the two complexes. For $k \geqq k_0$, the injection

$$\{C_k^p[\tau(M), W], d\} \subsetneq \{C^p[\tau(M), W], d\}$$

induces a homomorphism between the corresponding spectral sequences which we have introduced in Theorem 4.1. But this homomorphism becomes an isomorphism on E_2-level, whence the assertion follows.

iii): For $l \geqq h$, the subcomplex $\{C^p[\tau(M), W], d\}$ is well-defined and the coboundary operator is given by the differential operator with the first order. At least locally, the principal part of this differential operator is essentially expressed as a direct sum of exterior differentiations. Hence the usual proof on the ellipticity of the de Rham complex applies to this case, which completes the proof.

It is a fundamental theorem on elliptic complexes that, if M is compact, any elliptic complex has the finite-dimensional cohomology group (cf. [1]). Hence we have the following

COROLLARY. *If M is compact and $\{C^p[\tau(M), W], d\}$ has the stable jet range, then* $\dim H^*(\tau(M), W) < +\infty$.

This proposes us a problem: If M is compact, are the three conditions below equivalent? (i) $\dim H^*(\tau(M), W) < +\infty$; (ii) $\dim H^*(L_0, V) < +\infty$; (iii) $\{C^p[\tau(M), W], d\}$ has the stable jet range.

Combining Theorem 4.2 with Theorem 3.1, we obtain

THEOREM 4.3. *Assume that $d\rho|\mathfrak{gl}(n; \boldsymbol{R})$ is decomposable. Then the assertions* i) *and* ii) *of Theorem* 4.2 *are valid without any assumption on $\{C^p(L_0, V), d\}$. Moreover, we have*

iv) *If $\Delta = \emptyset$, then* $H^*(\tau(M), W) = 0$ *(as to the definition of Δ, see the preceding paragraph of Theorem* 3.1).

v) *If M is simply connected and* $\dim H^*(M; \boldsymbol{R}) < +\infty$, *then we have* $\sum(-1)^p \dim H^p(\tau(M), W) = 0$.

COROLLARY 1. *If $d\rho|\mathfrak{gl}(n; \boldsymbol{R})$ is an irreducible rational representation on a contravariant tensor space, then the cohomology group of $\{C^p[\tau(M), W], d\}$ associated to $\rho^{\#}$ becomes zero.*

This follows from the observation that in this case $d\rho|\mathfrak{gl}(n; \boldsymbol{R}) = [f_1, \cdots, f_m; \delta]$ has to satisfy $f_1 + \cdots + f_m + n\delta < 0$. Hence, in particular, we have

COROLLARY 2. *The cohomology group of the adjoint complex $\{C^p[\tau(M), \tau(M)], d\}$ vanishes.*

COROLLARY 3. *Let $T(a, b)$ be the tensor bundle with contravariant type a and covariant type b. Let $\{C^p[\tau(M), T(a, b)], d\}$ be the differential complex associated to the differential representation by means of Lie differentiation. Then its cohomology group vanishes when $a > b$. In other case, a stable jet range is given by* $k \geqq \mathrm{Max}\,\{1, b-a+1\}$.

PROOF. The Lie differentiation can be obtained as the induced differential representation from the canonical tensorial representation ρ of $\mathfrak{gl}(n; \boldsymbol{R})$ on the tensor space with type (a, b). From this we can directly check that ρ

is decomposed in the form

$$\rho=\sum[f_1, \cdots, f_m\,;\,\delta]$$

with $f_1+\cdots+f_m=(n-1)a+b$, $\delta=-a$. Hence the assertion follows.

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