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On a bound for periods of solutions of a certain nonlinear differential equation (I)

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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§ 0. Introduction.

As is shown in [6], the nonlinear differential equation

(E) $$nx(1-x^{2})\frac{d^{2}x}{dt^{2}}+\left(\frac{dx}{dt}\right)^{2}+(1-x^{2})(nx^{2}-1)=0,$$

where n is an integer $\geqq 2$, is the equation for the support function $x(t)$ of a geodesic in the 2-dimensional Riemannian manifold $\dot{O}_{n}^{2}$ with the metric:

(0.1) $$ds^{2}=(1-u^{2}-v^{2})^{n-2}\{(1-v^{2})du^{2}+2uv\,du\,dv+(1-u^{2})dv^{2}\}$$

in the unit disk: $u^{2}+v^{2}<1$. Another geometric meaning of (E) is given in [4]. Any non constant solution $x(t)$ of (E) such that

$$x^{2}+\left(\frac{dx}{dt}\right)^{2}<1$$

is periodic and its period T is given by the improper integral:

(0.2) $$T=2\int_{a_{0}}^{a_{1}}\frac{dx}{\sqrt{1-x^{2}-C\left(\frac{1}{x^{2}}-1\right)^{\alpha}}},$$

where

(0.3) $$C=(a_{0}^{2})^{\alpha}(1-a_{0}^{2})^{1-\alpha}=(a_{1}^{2})^{\alpha}(1-a_{1}^{2})^{1-\alpha}$$
$$(0<a_{0}<\sqrt{\alpha}<a_{1}<1,\ \alpha=1/n)$$

is the integral constant of (E) and $0<C<A=\alpha^{\alpha}(1-\alpha)^{1-\alpha}$.

Regarding T as a function of C, the following is known in [4]:

(i) T is differentiable and $T>\pi$,

(ii) $\lim_{C\to 0}T=\pi$ and $\lim_{C\to A}T=\sqrt{2}\pi$.

By means of a numerical analysis and observation about (E) in [5] and [7], M. Urabe conjectures the inequality

(U) $$T<\sqrt{2}\pi.$$

The author however wanted originally to have the inequality

(0.4) $$T<2\pi$$

from the standpoint of a geometrical problem related with the existence of compact minimal hypersurfaces of a certain type in the spheres. S. Furuya gave firstly an answer to it by proving the inequality

(0.5) $$T<\sqrt{1-\alpha}\cdot 2\pi$$

in [2] and the author proved a little sharper inequality

(0.6) $$T<\left(\frac{1}{\sqrt{2}}+\sqrt{1-\alpha}\right)\cdot\pi$$

in [5]. (U) is true by (0.5) or (0.6) when $n=2$ and S. Furuya proved also that (U) is true when $n \doteqdot 3$.

The equation (E) however may be considered for any real number $n \geqq 2$. In the present paper the author will prove (U) for any real number $n \geqq 3$.

§ 1. Period function $T_n(x_0)$.

Replacing nx^2 and nC by x and C respectively, the period T given by (0.2) can be written as

(1.1) $$T=T_n(x_0):=\int_{x_0}^{x_1}\frac{dx}{\sqrt{x(n-x)-Cx^{1-\alpha}(n-x)^{\alpha}}},$$

where

(1.2) $$C=x_0^{\alpha}(n-x_0)^{1-\alpha}=x_1^{\alpha}(n-x_1)^{1-\alpha}$$

and

(1.3) $$0<x_0<1<x_1<n.$$

LEMMA 1.1. *The function* $\varphi(x):=x^{\alpha}(n-x)^{1-\alpha}$ $(0\leqq x\leqq n)$ *is monotone increasing in* $[0, 1]$ *and decreasing in* $[1, n]$ *and we have*

$$\varphi'(x)=\frac{1-x}{x(n-x)}\varphi(x), \qquad \varphi''(x)=-\frac{n-1}{x^2(n-x)^2}\varphi(x),$$

$$\varphi'''(x)=\frac{(n-1)(2n-1-3x)}{x^3(n-x)^3}\varphi(x)$$

and

$$\varphi^{(4)}(x)=-\frac{(n-1)\{(3n-1)(2n-1)-8(2n-1)x+12x^2\}}{x^4(n-x)^4}\varphi(x).$$

PROOF. We get easily $\varphi'(x)$, $\varphi''(x)$ and $\varphi'''(x)$, from which

$$\varphi^{(4)}(x)=(n-1)\Big[\frac{-3}{x^3(n-x)^3}-\frac{3(2n-1-3x)(n-2x)}{x^4(n-x)^4}+\frac{(2n-1-3x)(1-x)}{x^4(n-x)^4}\Big]\varphi(x)$$

$$=-\frac{(n-1)\{(3n-1)(2n-1)-8(2n-1)x+12x^2\}}{x^4(n-x)^4}\varphi(x).$$

Since $\varphi(x)>0$ in $(0, n)$, $\varphi(x)$ is monotone in $[0, 1]$ and $[1, n]$. Q. E. D.

Now, using $\varphi(x)$ and putting $B=\varphi(1)=nA$, we have

$$\int_{x_0}^{1}\frac{dx}{\sqrt{x(n-x)-Cx^{1-\alpha}(n-x)^{\alpha}}}$$

$$=\int_{x_0}^{1}\frac{x(n-x)d\varphi(x)}{(1-x)\varphi(x)\sqrt{x(n-x)}\{1-C/\varphi(x)\}}$$

$$=\int_{x_0}^{1}\frac{\sqrt{x(n-x)(B-\varphi(x))}}{(1-x)\sqrt{\varphi(x)}}\cdot\frac{d\varphi(x)}{\sqrt{(B-\varphi(x))(\varphi(x)-C)}}$$

and

$$\int_{1}^{x_1}\frac{dx}{\sqrt{x(n-x)-Cx^{1-\alpha}(n-x)^{\alpha}}}$$

$$=\int_{1}^{x_1}\frac{\sqrt{x(n-x)(B-\varphi(x))}}{(1-x)\sqrt{\varphi(x)}}\cdot\frac{d\varphi(x)}{\sqrt{(B-\varphi(x))(\varphi(x)-C)}}.$$

Now, define a function $X=X_n(x)$ $(0\leqq x\leqq 1)$ by

$$x(n-x)^{n-1}=X(n-X)^{n-1},\qquad 1\leqq X\leqq n, \tag{1.4}$$

then we have $\varphi(x)=\varphi(X)$. Hence, the last integral can be written as

$$\int_{x_0}^{1}\frac{\sqrt{X(n-X)(B-\varphi(x))}}{(X-1)\sqrt{\varphi(x)}}\cdot\frac{d\varphi(x)}{\sqrt{(B-\varphi(x))(\varphi(x)-C)}}.$$

Thus, we get a formula for T as follows:

$$T_n(x_0)=\int_{x_0}^{1}\Big\{\frac{\sqrt{x(n-x)}}{1-x}+\frac{\sqrt{X_n(x)(n-X_n(x))}}{X_n(x)-1}\Big\}\sqrt{\frac{B-\varphi(x)}{\varphi(x)}}\times\frac{d\varphi(x)}{\sqrt{(B-\varphi(x))(\varphi(x)-C)}}. \tag{1.5}$$

LEMMA 1.2.

$$\int_{x_0}^{1}\frac{d\varphi(x)}{\sqrt{(B-\varphi(x))(\varphi(x)-C)}}=\pi.$$

PROOF. Since $\varphi(x)$ is monotone increasing in $[0, 1]$, putting $u=\varphi(x)$, we have

$$\int_{x_0}^{1}\frac{d\varphi(x)}{\sqrt{(B-\varphi(x))(\varphi(x)-C)}}=\int_{C}^{B}\frac{du}{\sqrt{(B-u)(u-C)}}=\pi.$$ Q. E. D.

Thus from (1.5) and Lemma 1.2 we shall have the inequality $T<\sqrt{2}\pi$, if we have the inequality:

$$\left\{\frac{\sqrt{x(n-x)}}{1-x}+\frac{\sqrt{X_n(x)(n-X_n(x))}}{X_n(x)-1}\right\}\sqrt{\frac{B-\varphi(x)}{\varphi(x)}}<\sqrt{2} \qquad (0<x<1). \tag{1.6}$$

LEMMA 1.3. *The function* $F(x):=\frac{x(n-x)}{(1-x)^2}\cdot\frac{B-\varphi(x)}{\varphi(x)}$ $(0\leqq x\leqq n,\ x\neq 1)$ *and* $:=1/2$ $(x=1)$, *is smooth and positive in* $(0, n)$.

PROOF. Since $\varphi(x)$ is analytic in $(0, n)$ and

$$\varphi(1)=(n-1)^{1-\alpha}=B,\quad \varphi'(1)=0,\quad \varphi''(1)=-\frac{1}{n-1}B,\quad \varphi'''(1)=\frac{2(n-2)}{(n-1)^2}B,$$
$$\varphi^{(4)}(1)=-\frac{3(2n^2-7n+7)}{(n-1)^3}B$$

by Lemma 1.1, we have

$$\varphi(x)=B\Big\{1-\frac{1}{2(n-1)}(x-1)^2+\frac{n-2}{3(n-1)^2}(x-1)^3 -\frac{2n^2-7n+7}{8(n-1)^3}(x-1)^4+\cdots\Big\}.$$

Hence we have

$$B-\varphi(x)=B(x-1)^2\Big\{\frac{1}{2(n-1)}-\frac{n-2}{3(n-1)^2}(x-1) +\frac{2n^2-7n+7}{8(n-1)^3}(x-1)^2-\cdots\Big\} \tag{1.7}$$

near $x=1$, which shows that $F(x)$ is analytic in x near 1. Q. E. D.

Using the function $F(x)$, (1.6) can be written as

$$\sqrt{F(x)}+\sqrt{F(X_n(x))}<\sqrt{2} \qquad (0<x<1). \tag{1.8}$$

§ 2. Properties of $F(x)$.

In $(0, n)$, for $x\neq 1$ we have

$$\frac{F'(x)}{F(x)}=\frac{1}{x}-\frac{1}{n-x}+\frac{2}{1-x}-\Big\{\frac{1}{B-\varphi(x)}+\frac{1}{\varphi(x)}\Big\}\varphi'(x)$$
$$=\frac{n+(n-2)x}{x(n-x)(1-x)}-\frac{B}{B-\varphi(x)}\cdot\frac{1-x}{x(n-x)},$$

that is

$$\frac{F'(x)}{F(x)}=\frac{\{n+(n-2)x\}\{B-\varphi(x)\}-B(1-x)^2}{x(n-x)(1-x)\{B-\varphi(x)\}}. \tag{2.1}$$

From (2.1), we have

$$(\sqrt{F(x)})'=\frac{\{n+(n-2)x\}\{B-\varphi(x)\}-B(1-x)^2}{2(1-x)^2\sqrt{x(n-x)\varphi(x)}\{B-\varphi(x)\}^{1/2}}, \tag{2.2}$$

where $\{B-\varphi(x)\}^{1/2}$ denotes the function:

$$\{B-\varphi(x)\}^{1/2}=(1-x)\sqrt{\frac{B-\varphi(x)}{(x-1)^2}}\,. \tag{2.3}$$

LEMMA 2.1. *Let* $g_0(x)$ *be the function:*

$$g_0(x):=\frac{x(n-x)^{n-1}\{n+(n-2)x\}^n}{(n-1+nx-x^2)^n}\,.$$

$g_0(x)$ *is monotone increasing in* $[0, n/2]$ *and decreasing in* $[n/2, n]$.

PROOF. We have $n-1+nx-x^2>0$ in $[0, n]$, since $n\gneqq 2$. Therefore $g_0(x)>0$ in $(0, n)$. In $(0, n)$, we have

$$\frac{g_0'(x)}{g_0(x)}=\frac{1}{x}-\frac{n-1}{n-x}+\frac{n(n-2)}{n+(n-2)x}-\frac{n(n-2x)}{n-1+nx-x^2}$$

$$=\frac{n(n-1)(1-x)^2(n-2x)}{x(n-x)\{n+(n-2)x\}\{n-1+nx-x^2\}}\,.$$

Hence we have

$$g_0'(x)=\frac{n(n-1)(1-x)^2(n-x)^{n-2}\{n+(n-2)x\}^{n-1}(n-2x)}{(n-1+nx-x^2)^{n+1}}\,, \tag{2.4}$$

from which we see that $g_0(x)$ is monotone increasing in $[0, n/2]$ and decreasing in $[n/2, n]$. Q. E. D.

We get easily $g_0(1)=(n-1)^{n-1}$. Let Λ be the unique value such that

$$g_0(\Lambda)=(n-1)^{n-1} \qquad 1<\Lambda<n\,. \tag{2.5}$$

This is assured by Lemma 2.1 which implies furthermore $n/2<\Lambda<n$.

LEMMA 2.2. $n/2<\Lambda<n-1$ *for* $n\geqq 3$.

PROOF. By Lemma 2.1, it is sufficient to prove that

$$g_0(n-1)<(n-1)^{n-1}\,.$$

Since we have

$$g_0(n-1)=\frac{(n^2-2n+2)^n}{2^n(n-1)^{n-1}}\,,$$

the above inequality is equivalent to

$$2\cdot(n-1)^{-2/n}>\frac{1}{(n-1)^2}+1\,. \tag{2.6}$$

For the function $L(n):=2\cdot(n-1)^{-2/n}-\frac{1}{(n-1)^2}$, we have

$$L'(n)=\frac{4}{n^2}(n-1)^{-2/n}\Big[\log(n-1)-\frac{n}{n-1}+\frac{n^2}{2(n-1)^3}\cdot(n-1)^{2/n}\Big].$$

For $n>2$, we have

$$\log(n-1)-\frac{n}{n-1}+\frac{n^2}{2(n-1)^3}\cdot(n-1)^{2/n}$$
$$>\log(n-1)-\frac{n}{n-1}+\frac{n^2}{2(n-1)^3}$$
$$=\log(n-1)-\frac{n(n-2)(2n-1)}{2(n-1)^3}\ .$$

Denote the right-hand side by $R(n)$, then putting $\tau=\frac{1}{n-1}$, we get easily

$$R'(n)=\frac{d}{dn}\Big[\log(n-1)-(1+\tau)\Big\{1-\frac{1}{2}(\tau+\tau^2)\Big\}\Big]$$
$$=\frac{\tau}{2}(2+\tau-4\tau^2-3\tau^3)\,.$$

Since we have

$$\frac{d}{d\tau}(2+\tau-4\tau^2-3\tau^3)=(1+\tau)(1-9\tau)$$

and $(2+\tau-4\tau^2-3\tau^3)_{\tau=0}=2>0$, $(2+\tau-4\tau^2-3\tau^3)_{\tau=1/2}=9/8>0$, it must be that

$$2+\tau-4\tau^2-3\tau^3>0 \qquad \text{for}\quad 0<\tau\leqq 1/2\,,$$

from which we have

$$R'(n)>0 \qquad \text{for}\quad n\geqq 3\,.$$

On the other hand, we have

$$R(2)=0\,,\qquad R(3)=\log_e 2-\frac{15}{16}=-0.24435\cdots\,,$$
$$R(4)=\log_e 3-\frac{28}{27}=0.06157\cdots>0\,,$$

hence

$$R(n)>0 \qquad \text{for}\quad n\geqq 4\,.$$

Therefore we get

(2.7) $$L'(n)>0 \qquad \text{for}\quad n\geqq 4\,.$$

Next, we have

$$\frac{d}{dn}(n-1)^{2/n}=\frac{2}{n^2}(n-1)^{2/n}\Big[-\log(n-1)+\frac{n}{n-1}\Big]$$

and

$$\frac{d}{dn}\Big[-\log(n-1)+\frac{n}{n-1}\Big]=-\tau(1+\tau)<0 \qquad \text{for}\quad \tau>0\,.$$

Hence the function $-\log(n-1)+\frac{n}{n-1}$ is monotone decreasing in $(1,\infty)$ and

$$\Big(-\log(n-1)+\frac{n}{n-1}\Big)_{n=4}=-\log 3+\frac{4}{3}=-1.09861\cdots+\frac{4}{3}>0\,.$$

Hence we have

$$\frac{d}{dn}(n-1)^{2/n} > 0 \qquad \text{for} \quad 1 < n \leqq 4,$$

consequently we have

$$(n-1)^{2/n} > 2^{2/3} = 1.58740 \cdots > \frac{3}{2} \qquad \text{for} \quad 3 \leqq n \leqq 4.$$

Therefore, in the interval $3 \leqq n \leqq 4$, we have

$$\log(n-1) - \frac{n}{n-1} + \frac{n^2}{2(n-1)^3}(n-1)^{2/n}$$
$$> \log(n-1) - \frac{n}{n-1} + \frac{3n^2}{4(n-1)^3}$$
$$= \log(n-1) - \frac{n(4n^2-11n+4)}{4(n-1)^3}.$$

Denote the right-hand side by $R_1(n)$, then we obtain

$$R_1'(n) = \frac{d}{dn}\Big[\log(n-1) - (1+\tau)\Big\{1 - \frac{3}{4}(\tau+\tau^2)\Big\}\Big]$$
$$= \frac{\tau}{4}(4+\tau-12\tau^2-9\tau^3).$$

Since we have

$$\frac{d}{d\tau}(4+\tau-12\tau^2-9\tau^3) = 1-24\tau-27\tau^2$$

and the positive root of the equation: $27\tau^2+24\tau-1=0$ is less than 1/3 and $(4+\tau-12\tau^2-9\tau^3)_{\tau=\frac{1}{2}} = \frac{3}{8} > 0$, it must be that

$$4+\tau-12\tau^2-9\tau^3 > 0 \qquad \text{for} \quad \frac{1}{3} \leqq \tau \leqq \frac{1}{2},$$

from which

$$R_1'(n) > 0 \qquad \text{for} \quad 3 \leqq n \leqq 4.$$

On the other hand we have

$$R_1(3) = \log 2 - \frac{21}{32} = 0.03689 \cdots > 0.$$

Hence, we get

$$R_1(n) > 0 \qquad \text{for} \quad 3 \leqq n \leqq 4,$$

from which we get

$$L'(n) > 0 \qquad \text{for} \quad 3 \leqq n \leqq 4. \tag{2.8}$$

By means of (2.7) and (2.8), $L(n)$ is monotone increasing for $n \geqq 3$. On the other hand we have

$$L(3)=2\cdot 2^{-2/3}-\frac{1}{4}=2^{1/3}-\frac{1}{4}=1.25992\cdots-\frac{1}{4}>1\,.$$

Consequently we get

$$L(n)>1 \qquad \text{for} \quad n\geqq 3\,.$$

Thus (2.6) has been proved. Q. E. D.

THEOREM 2.3. *The function $F(x)$ is monotone increasing in $(0, \Lambda]$ and decreasing in $[\Lambda, n)$, and*

$$F(1)=\frac{1}{2}\,, \qquad F'(1)=\frac{n-2}{6(n-1)}\,.$$

PROOF. Near $x=1$, from (1.7) we have

$$F(x)=\frac{Bx(n-x)}{\varphi(x)}\left\{\frac{1}{2(n-1)}-\frac{n-2}{3(n-1)^2}(x-1)+\frac{2n^2-7n+7}{8(n-1)^3}(x-1)^2+\cdots\right\}, \tag{2.9}$$

from which we get easily

$$F(1)=\frac{1}{2}$$

and

$$F'(1)=(n-2)\cdot\frac{1}{2(n-1)}-(n-1)\cdot\frac{n-2}{3(n-1)^2}=\frac{n-2}{6(n-1)}>0\,.$$

Then, since $F(x)>0$, (2.2) implies that $F'(x)>0$ if and only if

$$\{n+(n-2)x\}\{B-\varphi(x)\}>B(1-x)^2 \qquad \text{for} \quad 0<x<1$$

and

$$\{n+(n-2)x\}\{B-\varphi(x)\}<B(1-x)^2 \qquad \text{for} \quad 1<x<n\,.$$

These are equivalent to

$$(n-1+nx-x^2)B>\{n+(n-2)x\}\varphi(x)$$

and

$$(n-1+nx-x^2)B<\{n+(n-2)x\}\varphi(x)$$

respectively. Since $B=(n-1)^{1-1/n}$ and $\varphi(x)=x^{1/n}(n-x)^{1-1/n}$, the above inequalities become

$$(n-1)^{n-1}>\frac{x(n-x)^{n-1}\{n+(n-2)x\}^n}{(n-1+nx-x^2)^n} \qquad \text{for} \quad 0<x<1$$

and

$$(n-1)^{n-1}<\frac{x(n-x)^{n-1}\{n+(n-2)x\}^n}{(n-1+nx-x^2)^n} \qquad \text{for} \quad 1<x<n\,.$$

The right-hand sides of these inequalities are $g_0(x)$ in Lemma 2.1, which implies

$$g_0(x)<g_0(1)=(n-1)^{n-1} \qquad \text{for} \quad 0<x<1 \text{ and } \Lambda<x<n\,,$$

$$g_0(x)>(n-1)^{n-1} \qquad \text{for} \quad 1<x<\Lambda\,.$$

Therefore, $F(x)$ is monotone increasing in $(0, A)$ and decreasing in (A, n).

Q. E. D.

Now, we shall give an estimation on the maximum of the function $F(x)$ in the interval $(0, n)$.

LEMMA 2.4. *The function* $\frac{x(n-x)}{n-1+nx-x^2}$ *is monotone decreasing in* $[n/2, n]$.

PROOF. $n-1+nx-x^2>0$ in $[0, n]$ and

$$\left\{\frac{x(n-x)}{n-1+nx-x^2}\right\}'=\frac{(n-1)(n-2x)}{(n-1+nx-x^2)^2},$$

which implies immediately this lemma. Q. E. D.

THEOREM 2.5.

$$F(x)<\frac{n^2}{n^2+4n-4} \qquad (n\geqq 3).$$

PROOF. By Theorem 2.3, the maximum value of $F(x)$ in $(0, n)$ is $F(A)$. Then, by (2.1) we have

$$\{n+(n-2)A\}\{B-\varphi(A)\}-B(1-A)^2=0, \tag{2.10}$$

which implies

$$\frac{B-\varphi(A)}{\varphi(A)}=\frac{(A-1)^2}{n-1+nA-A^2},$$

hence

$$F(A)=\frac{A(n-A)}{(A-1)^2}\cdot\frac{B-\varphi(A)}{\varphi(A)}=\frac{A(n-A)}{n-1+nA-A^2}.$$

Then, by Lemma 2.2 and Lemma 2.4, we obtain

$$F(A)<\left.\frac{x(n-x)}{n-1+nx-x^2}\right|_{x=n/2}=\frac{n^2}{n^2+4n-4}. \qquad \text{Q. E. D.}$$

REMARK. Since $\frac{n^2}{n^2+4n-4}<\frac{n-1}{n}$ for $n>2$, we get a more sharper inequality on the period T than (0.6) as follows:

$$T<\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{1+4\alpha-4\alpha^2}}\right)\cdot\pi \qquad (n\geqq 3) \tag{2.11}$$

by means of (1.5), Theorem 2.3 and Theorem 2.5.

§3. Properties of $f(x)$.

On the function $X=X_n(x)$ defined by (1.4), we have

$$\frac{dX}{dx}=\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{1-X}. \tag{3.1}$$

From (2.2) we get in the interval (0.1)

$$\frac{d}{dx}\sqrt{F(x)}+\frac{d}{dx}\sqrt{F(X(x))}$$

$$=\frac{\{n+(n-2)x\}\{B-\varphi(x)\}-B(1-x)^2}{2(1-x)^2\sqrt{x(n-x)\varphi(x)}\,\{B-\varphi(x)\}^{1/2}}$$

$$+\frac{\{n+(n-2)X\}\{B-\varphi(X)\}-B(1-X)^2}{2(1-X)^2\sqrt{X(n-X)\varphi(X)}\,\{B-\varphi(X)\}^{1/2}}\cdot\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{1-X}.$$

Since $\varphi(X)=\varphi(x)$ and $\{B-\varphi(X)\}^{1/2}=-\{B-\varphi(x)\}^{1/2}$ by (2.3), the above equality can be written as

$$\frac{d}{dx}\sqrt{F(x)}+\frac{d}{dx}\sqrt{F(X(x))} \tag{3.2}$$

$$=\frac{1-x}{2x(n-x)\sqrt{\varphi(x)\{B-\varphi(x)\}}}$$

$$\times\left[\frac{\sqrt{x(n-x)}\,M(x)}{(1-x)^3}-\frac{\sqrt{X(n-X)}\,M(X)}{(1-X)^3}\right] \qquad (0<x<1),$$

where

$$M(x):=\{n+(n-2)x\}\{B-\varphi(x)\}-B(1-x)^2. \tag{3.3}$$

From (3.2), we have

LEMMA 3.1. *$\sqrt{F(x)}+\sqrt{F(X(x))}$ is increasing at x $(0<x<1)$, if and only if*

$$\frac{\sqrt{x(n-x)}\,M(x)}{(1-x)^3}>\frac{\sqrt{X(n-X)}\,M(X)}{(1-X)^3}, \qquad X=X_n(x).$$

Let $f(x)$ be the function defined by

$$f(x):=\frac{\sqrt{x(n-x)}\,M(x)}{(1-x)^3} \tag{3.4}$$

$$=\frac{\sqrt{x(n-x)}}{(1-x)^3}\left[\{n+(n-2)x\}\{B-\varphi(x)\}-B(1-x)^2\right].$$

LEMMA 3.2. *$f(x)>0$ in $(0, \Lambda)$ and $f(x)<0$ in (Λ, n).*

PROOF. As is shown in the proof of Theorem 2.3,

$$g_0(x)<g_0(1)=B^n \qquad \text{in } (0, 1) \text{ and } (\Lambda, n)$$

and

$$g_0(x)>B^n \qquad \text{in } (1, \Lambda).$$

The first inequality implies

$$\frac{\varphi(x)\{n+(n-2)x\}}{n-1+nx-x^2}<B, \quad \text{i.e.} \quad M(x)>0.$$

The second one implies $M(x)<0$. Now as is seen from (1.7), $f(1)=\dfrac{(n-2)B}{6\sqrt{n-1}}$.

Hence $f(x)>0$ in $(0, A)$ and $f(x)<0$ in (A, n). Q. E. D.

Now, we compute $f'(x)$ in $(0, A)$. We have

$$\frac{d}{dx}\log f(x)$$
$$=\frac{1}{2}\left(\frac{1}{x}-\frac{1}{n-x}\right)+\frac{3}{1-x}$$
$$+\frac{1}{M(x)}\left[(n-2)\{B-\varphi(x)\}+2B(1-x)-\{n+(n-2)x\}\frac{1-x}{x(n-x)}\varphi(x)\right]$$
$$=\frac{6x(n-x)M(x)+(1-x)(n-2x)[M(x)+2\{1+(n-2)x\}\{B-\varphi(x)\}-2B(1-x)^2]}{2(1-x)x(n-x)M(x)},$$

from which we get

$$f'(x)=\frac{1}{2(1-x)^4\sqrt{x(n-x)}}\times[\{n(n+2)+2(4n^2-5n-2)x$$
$$+(3n^2-16n+16)x^2\}\{B-\varphi(x)\}-3B(1-x)^2\{n+(n-2)x\}]. \tag{3.5}$$

For simplicity, putting

$$P(x):=n(n+2)+2(4n^2-5n-2)x+(3n^2-16n+16)x^2, \tag{3.6}$$

we obtain from (3.5) the following

LEMMA 3.3. *$f(x)$ is decreasing at x $(0<x<n)$, if and only if*

$$[P(x)-3(1-x)^2\{n+(n-2)x\}]B<P(x)\varphi(x).$$

LEMMA 3.4. *$P(x)-3(1-x)^2\{n+(n-2)x\}>0$ in $[0, n]$.*

PROOF. From (3.6) we get

$$P(x)-3(1-x)^2\{n+(n-2)x\}$$
$$=n(n-1)+(8n^2-7n+2)x+(3n^2-13n+4)x^2-3(n-2)x^3.$$

For $n>2$, we have $8n^2-7n+2>0$, $-3(n-2)<0$ and

$$\{8n^2-7n+2+(3n^2-13n+4)x-3(n-2)x^2\}_{x=n}=(n-2)(n-1)>0.$$

Hence

$$8n^2-7n+2+(3n^2-13n+4)x-3(n-2)x^2>0$$

for $0\leqq x\leqq n$ and so $P(x)-3(1-x)^2\{n+(n-2)x\}>0$ there. Q. E. D.

By virtue of Lemma 3.4, we consider an auxiliary function:

$$g(x):=\frac{P(x)\varphi(x)}{P(x)-3(1-x)^2\{n+(n-2)x\}}\qquad(0<x<n). \tag{3.7}$$

Next, we compute $g'(x)$ in $(0, n)$. We have

$$\frac{d}{dx}\log g(x)$$

$$=\frac{P'(x)}{P(x)}-\frac{P'(x)+3(1-x)\{n+2+3(n-2)x\}}{P(x)-3(1-x)^2\{n+(n-2)x\}}+\frac{1-x}{x(n-x)}$$

$$=\frac{1-x}{x(n-x)P(x)[P(x)-3(1-x)^2\{n+(n-2)x\}]}$$

$$\times[-6x(n-x)(1-x)\{n+(n-2)x\}\{4n^2-5n-2+(3n^2-16n+16)x\}$$

$$+P(x)\{P(x)-3(1-x)^2(n+(n-2)x)-3x(n-x)(n+2+3(n-2)x)\}].$$

The polynomial of x in the brackets of the above equality becomes

$$(n-1)(1-x)^2\{n^2(n+2)-n(9n^2-2n+8)x+4(3n^2-2n+2)x^2\}.$$

Hence, we get

$$g'(x)=\frac{(n-1)(1-x)^3\{n^2(n+2)-n(9n^2-2n+8)x+4(3n^2-2n+2)x^2\}}{x^{1-\alpha}(n-x)^{\alpha}[P(x)-3(1-x)^2\{n+(n-2)x\}]^2}. \tag{3.8}$$

LEMMA 3.5. *$g'(x)=0$ $(0<x<n)$ has unique roots γ in $(0, 1)$ and $\bar{\gamma}$ in $(1, n)$ and $n/2<\bar{\gamma}<n$.*

PROOF. For the quadratic polynomial of x:

$$y=n^2(n+2)-n(9n^2-2n+8)x+4(3n^2-2n+2)x^2,$$

we have

$$(y)_{x=0}=n^2(n+2)>0,$$

$$(y)_{x=1}=-8(n^3-2n^2+2n-1)=-8(n-1)(n^2-n+1)<0,$$

$$(y)_{x=n/2}=-\frac{3}{2}n^4<0,$$

$$(y)_{x=n}=n^2(3n^2-5n+2)=n^2(3n-2)(n-1)>0.$$

These relations easily imply the lemma. Q. E. D.

Using Lemma 3.5 and (3.8), we obtain immediately the following

LEMMA 3.6. *$g(x)$ is monotone increasing in $(0,\gamma]$ and $[1,\bar{\gamma}]$ and decreasing in $[\gamma, 1]$ and $[\bar{\gamma}, n)$.*

Since $g(1)=\varphi(1)=B$, $g(x)=B$ has a unique solution in $(0, 1)$ and $(1, n)$ respectively by means of Lemma 3.6. We denote them by σ and $\bar{\sigma}$ respectively, i. e. they are solutions of the equation:

$$[P(x)-3(1-x)^2\{n+(n-2)x\}]B=P(x)\varphi(x), \quad 0<x<n, \quad x\neq 1 \tag{3.9}$$

and

$$0<\sigma<\gamma \quad \text{and} \quad \bar{\gamma}<\bar{\sigma}<n. \tag{3.10}$$

Now as is seen from (1.7) and (3.4)

$$f'(1) = -\frac{n^2-n+1}{12(n-1)^{3/2}} B < 0 .$$

Hence, Lemma 3.3 and Lemma 3.6 imply the following

PROPOSITION 3.7. *The function $f(x)$ is monotone decreasing in $(\sigma, \bar{\sigma})$ and increasing in $(0, \sigma]$ and $[\bar{\sigma}, n)$ and*

$$f(\sigma) \geqq f(x) \geqq f(\bar{\sigma}) .$$

THEOREM 3.8. *The function $\sqrt{F(x)} + \sqrt{F(X(x))}$ is monotone increasing and less than $\sqrt{2}$ in $[\sigma, 1)$.*

PROOF. By Lemma 3.2 and Proposition 3.7 we have

$$f(\Lambda) = 0 > f(\bar{\sigma})$$

and

$$f(x) > f(X(x)) \qquad \text{for} \quad \sigma \leqq x < 1 .$$

Hence, by Lemma 3.1, $\sqrt{F(x)} + \sqrt{F(X(x))}$ is monotone increasing in $[\sigma, 1)$. Since $F(1) = \frac{1}{2}$ by Theorem 2.3, we obtain

$$\sqrt{F(x)} + \sqrt{F(X(x))} < \sqrt{2} \qquad \text{for} \quad \sigma \leqq x < 1 . \qquad \text{Q.E.D.}$$

§ 4. Proof of $T < \sqrt{2}$ for $3 \leqq n \leqq 14$.

LEMMA 4.1. *$\gamma < 1/5$ for $n \geqq 3$.*

PROOF. γ is the smallest root of the equation of x:

$$n^2(n+2) - n(9n^2-2n+8)x + 4(3n^2-2n+2)x^2 = 0$$

according to Lemma 3.5. Substituting $x = 1/5$ in the left hand side and multiplying it by 25, we get

$$\begin{aligned} &25n^2(n+2) - 5n(9n^2-2n+8) + 4(3n^2-2n+2) \\ &= -20n^3 + 72n^2 - 48n + 8 \\ &= -4\{n(n-3)(5n-3) + 3n - 2\} \leqq -4(3n-2) < 0 , \end{aligned}$$

which implies $\gamma < 1/5$. Q.E.D.

(3.10) and Lemma 4.1 yield immediately the following:

PROPOSITION 4.2. *$\sigma < 1/5$ for $n \geqq 3$.*

LEMMA 4.3. *When $n \geqq 3$, for $0 < x \leqq \sigma$, we have*

$$\sqrt{F(x)} + \sqrt{F(X(x))} < \frac{1}{4}\sqrt{5(n-1)\left(\frac{5n-1}{n-1}\right)^{1/n} - (5n-1)} + \frac{n}{\sqrt{n^2+4n-4}} .$$

PROOF. By Theorem 2.5, we have

$$\sqrt{F(X(x))} < \frac{n}{\sqrt{n^2+4n-4}} .$$

By Theorem 2.3 and Proposition 4.2, we have

$$\sqrt{F(x)} < \sqrt{F\left(\frac{1}{5}\right)} \quad \text{for} \quad 0 < x \leqq \sigma$$

in the present case of n. By the definition of $F(x)$,

$$F\left(\frac{1}{5}\right) = \left[\frac{x(n-x)}{(1-x)^2} \cdot \frac{B-\varphi(x)}{\varphi(x)}\right]_{x=1/5}$$

$$= \frac{1}{16}\left\{5(n-1)\left(\frac{5n-1}{n-1}\right)^{1/n} - (5n-1)\right\}.$$

Thus we obtain the following:

$$\text{(4.1)} \qquad \sqrt{F(x)} + \sqrt{F(X(x))}$$

$$< \frac{1}{4}\sqrt{5(n-1)\left(\frac{5n-1}{n-1}\right)^{1/n} - (5n-1)} + \frac{n}{\sqrt{n^2+4n-4}}. \qquad \text{Q. E. D.}$$

In the following, we shall estimate the right hand side of (4.1).

Now, putting $n = \frac{1}{t}$, $x = \frac{1}{a}$ in $F(x) = \frac{x(n-x)}{(1-x)^2} \cdot \frac{B-\varphi(x)}{\varphi(x)}$, we get

$$\text{(4.2)} \qquad F(x) = \frac{x}{(1-x)^2}\left[\left\{\frac{n-x}{(n-1)x}\right\}^{1/n}(n-1) - (n-x)\right]$$

$$= \frac{a}{(a-1)^2}\left[\left(\frac{a-t}{1-t}\right)^t\left(\frac{1}{t}-1\right) - \frac{1}{t} + \frac{1}{a}\right].$$

We shall investigate the following auxiliary function of t:

$$\text{(4.3)} \qquad G_a(t) := \left(\frac{a-t}{1-t}\right)^t\left(\frac{1}{t}-1\right) - \frac{1}{t} \qquad (0 < t < 1 < a).$$

Differentiating $G_a(t)$, we get easily

$$\text{(4.4)} \qquad G_a'(t) = \frac{1}{t^2} + \left(-\frac{1}{t^2} + \frac{a-1}{a-t} + \frac{1-t}{t}\log\frac{a-t}{1-t}\right)\left(\frac{a-t}{1-t}\right)^t.$$

Putting

$$\text{(4.5)} \qquad u = \log\frac{a-t}{1-t},$$

(4.4) can be written as

$$G_a'(t) = \frac{1}{t^2} + \left(-\frac{1}{t^2} + \frac{a-1}{a-t} + \frac{1-t}{t}u\right)e^{tu}$$

$$= \frac{1}{t^2} + \left(-\frac{1}{t^2} + \frac{a-1}{a-t} + \frac{1-t}{t}u\right)\left(1 + tu + \frac{t^2u^2}{2} + \sum_{m>2}\frac{t^m}{m!}u^m\right),$$

that is

$$\text{(4.6)} \qquad G_a'(t) = -u + \left(\frac{1}{2} - t\right)u^2 + \frac{a-1}{a-t} + \frac{a-1}{a-t}tu + \frac{1}{2}tu^2\left\{(1-t)u + \frac{a-1}{a-t}t\right\}$$

$$+\sum_{m>2}\frac{u^m t^{m-2}}{m!}\left\{-1+t(1-t)u+\frac{a-1}{a-t}t^2\right\}.$$

LEMMA 4.4. *$G_5(t)$ is monotone increasing in* $\left(0, \frac{1}{3}\right]$.

PROOF. For $G(t)=G_5(t)$, from (4.6) we obtain

$$G'(t)=-u+\left(\frac{1}{2}-t\right)u^2+\frac{4}{5-t}+\frac{4}{5-t}tu+\frac{1}{2}tu^2\left\{(1-t)u+\frac{4t}{5-t}\right\} \tag{4.7}$$
$$+\sum_{m>2}\frac{u^m t^{m-2}}{m!}\left\{-1+t(1-t)u+\frac{4}{5-t}t^2\right\}.$$

We show $G'(t)>0$ by dividing the interval $\left(0, \frac{1}{3}\right]$ into four subintervals as follows.

I. $\frac{1}{4}\leqq t\leqq\frac{1}{3}$. For such t, we have

$$\frac{19}{3}\leqq\frac{5-t}{1-t}\leqq 7, \qquad \log_e\frac{19}{3}\leqq u\leqq\log_e 7,$$

$$-1+t(1-t)u+\frac{4}{5-t}t^2\geqq-\frac{18}{19}+\frac{3}{16}\log_e\frac{19}{3}=-b_1$$

and

$$\sum_{m>2}\frac{u^m t^{m-2}}{m!}<\frac{u^3 t}{6}e^{ut}=\frac{u^3 t}{6}\left(\frac{5-t}{1-t}\right)^t\leqq\frac{u^3 t}{6}7^{1/3}.$$

Hence, from (4.7) we obtain

$$G'(t)>-u+\frac{1}{6}u^2+\frac{16}{19}+\frac{4}{19}u+\frac{1}{2}u^2\left(\frac{3}{16}u+\frac{1}{19}\right)-\frac{1}{18}7^{1/3}b_1u^3,$$

i. e.

$$G'(t)>\frac{16}{19}-\frac{15}{19}u+\left(\frac{1}{6}+\frac{1}{38}\right)u^2+\left(\frac{3}{32}-\frac{1}{18}7^{1/3}b_1\right)u^3. \tag{4.8$_1$}$$

Since we have

$$\log_e\frac{19}{3}\doteqdot 1.84583, \qquad \frac{3}{16}\log_e\frac{19}{3}\doteqdot 0.34609, \qquad \frac{18}{19}\doteqdot 0.94737,$$

and $b_1\doteqdot 0.60128$;

$$7^{1/3}\doteqdot 1.91293, \qquad \frac{1}{18}7^{1/3}b_1\doteqdot 0.06390, \qquad \frac{3}{32}\doteqdot 0.09375$$

and $\frac{3}{32}-\frac{1}{18}7^{1/3}b_1\doteqdot 0.02985>0$, we obtain from (4.8$_1$)

$$G'(t)>\frac{16}{19}-\frac{15}{19}\log_e 7+\frac{11}{57}\left(\log_e\frac{19}{3}\right)^2+\left(\frac{3}{32}-\frac{1}{18}7^{1/3}b_1\right)\left(\log_e\frac{19}{3}\right)^3.$$

Since we have

$$\frac{16}{19}\doteqdot 0.84211, \qquad \frac{11}{57}\left(\log_e\frac{19}{3}\right)^2\doteqdot 0.65751,$$

$$\left(\frac{3}{32}-\frac{1}{18}7^{1/3}b_1\right)\left(\log_e\frac{19}{3}\right)^3 \doteqdot 0.18772,$$

$$\log_e 7 \doteqdot 1.94591, \qquad \frac{15}{19}\log_e 7 \doteqdot 1.53624,$$

the right-hand side of the above inequality $\doteqdot 0.15109$, hence

$$G'(t)>0 \qquad \text{for } \frac{1}{4}\leqq t\leqq\frac{1}{3}.$$

II. $\frac{1}{8}\leqq t\leqq\frac{1}{4}$. For such t, we have

$$\frac{39}{7}\leqq\frac{5-t}{1-t}\leqq\frac{19}{3}, \qquad \log_e\frac{39}{7}\leqq u\leqq\log_e\frac{19}{3},$$

$$-1+t(1-t)u+\frac{4}{5-t}t^2\geqq-\frac{77}{78}+\frac{7}{64}\log_e\frac{39}{7}=-b_2$$

and

$$\sum_{m>2}\frac{u^m t^{m-2}}{m!}<\frac{u^3t}{6}\left(e^{ut}-\frac{3}{4}ut\right)=\frac{u^3t}{6}\left\{\left(\frac{5-t}{1-t}\right)^t-\frac{3}{4}ut\right\}<c_2u^3t,$$

where

$$c_2=\frac{1}{6}\left\{\left(\frac{19}{3}\right)^{1/4}-\frac{3}{32}\log_e\frac{39}{7}\right\}.$$

On the constants b_2 and c_2, we have

$$\log_e\frac{39}{7}\doteqdot 1.71765, \qquad \frac{7}{64}\log_e\frac{39}{7}\doteqdot 0.18787, \qquad \frac{77}{78}\doteqdot 0.98718$$

and $b_2\doteqdot 0.79931$;

$$\left(\frac{19}{3}\right)^{1/4}\doteqdot 1.58638, \qquad \frac{3}{32}\log_e\frac{39}{7}\doteqdot 0.16103$$

and $c_2\doteqdot 0.23756$. From (4.7) and these inequalities we obtain

$$G'(t)>-u+\left(\frac{1}{2}-t\right)u^2+\frac{4}{5-t}+\frac{4t}{5-t}u+\frac{1}{2}tu^2\left\{(1-t)u+\frac{4t}{5-t}\right\}-b_2c_2tu^3$$

$$=\frac{4}{5-t}+\left[-1+\frac{4t}{5-t}\right]u+\left[\frac{1}{2}-t+\frac{2t^2}{5-t}\right]u^2+\left[\frac{t(1-t)}{2}-b_2c_2t\right]u^3.$$

Since $\frac{1}{8}\leqq t\leqq\frac{1}{4}$ and the function $-t+\frac{2t^2}{5-t}$ is decreasing in $\left(-\infty,\frac{1}{4}\right]$,

$$\frac{1}{2}-t+\frac{2t^2}{5-t}\geqq\frac{1}{2}-\frac{1}{4}+\frac{1}{38}=\frac{21}{76}, \qquad b_2c_2\doteqdot 0.18988,$$

$$\frac{t(1-t)}{2}-b_2c_2t\geqq\frac{1}{8}\left(\frac{7}{16}-b_2c_2\right)\doteqdot 0.03095>0,$$

we obtain

$$G'(t) > \frac{32}{39} - u\left[\frac{35}{39} - \frac{21}{76}u - \frac{1}{8}\left(\frac{7}{16} - b_2c_2\right)u^2\right]. \tag{4.8$_2$}$$

On the other hand, we have

$$\frac{35}{39} - \frac{21}{76}u - \frac{1}{8}\left(\frac{7}{16} - b_2c_2\right)u^2$$

$$< \frac{35}{39} - \frac{21}{76}\log_e \frac{39}{7} - \frac{1}{8}\left(\frac{7}{16} - b_2c_2\right)\left(\log_e \frac{39}{7}\right)^2,$$

hence

$$G'(t) > \frac{32}{39} - \log_e \frac{19}{3}\left[\frac{35}{39} - \frac{21}{76}\log_e \frac{39}{7} - \frac{1}{8}\left(\frac{7}{16} - b_2c_2\right)\left(\log_e \frac{39}{7}\right)^2\right].$$

Since we have

$$\frac{32}{39} \doteqdot 0.82051, \qquad \frac{35}{39} \doteqdot 0.89744, \qquad \frac{21}{76}\log_e \frac{39}{7} \doteqdot 0.47461,$$

$$\frac{1}{8}\left(\frac{7}{16} - b_2c_2\right)\left(\log_e \frac{39}{7}\right)^2 \doteqdot 0.09132,$$

the right hand side of the above inequality $\doteqdot 0.20858$, hence

$$G'(t) > 0 \qquad \text{for} \quad \frac{1}{8} \leqq t \leqq \frac{1}{4}.$$

III. $\frac{1}{16} \leqq t \leqq \frac{1}{8}$. For such t, we have

$$\frac{79}{15} \leqq \frac{5-t}{1-t} \leqq \frac{39}{7}, \qquad \log_e \frac{79}{15} \leqq u \leqq \log_e \frac{39}{7}.$$

$$-1 + t(1-t)u + \frac{4}{5-t}t^2 \geqq -\frac{315}{316} + \frac{15}{256}\log_e \frac{79}{15} = -b_3,$$

and

$$\sum_{m \geq 2} \frac{u^m t^{m-2}}{m!} < \frac{u^3 t}{6}\left\{\left(\frac{5-t}{1-t}\right)^t - \frac{3}{4}ut\right\} < c_3 u^3 t,$$

where

$$c_3 = \frac{1}{6}\left\{\left(\frac{39}{7}\right)^{1/8} - \frac{3}{64}\log_e \frac{79}{15}\right\}.$$

On the constants b_3 and c_3, we have

$$\log_e \frac{79}{15} \doteqdot 1.66140, \qquad \frac{15}{256}\log_e \frac{79}{15} \doteqdot 0.09735, \qquad \frac{315}{316} \doteqdot 0.99684$$

and $b_3 \doteqdot 0.89949$;

$$\left(\frac{39}{7}\right)^{1/8} \doteqdot 1.23950, \qquad \frac{3}{64}\log_e \frac{79}{15} \doteqdot 0.07788$$

and $c_3 \doteqdot 0.19360$. From (4.7) and these inequalities, we obtain

$$G'(t)>\frac{4}{5-t}+\Big[-1+\frac{4t}{5-t}\Big]u+\Big[\frac{1}{2}-t+\frac{2t^2}{5-t}\Big]u^2+\Big[\frac{t(1-t)}{2}-b_3c_3t\Big]u^3.$$

Since $\frac{1}{16}\leqq t\leqq\frac{1}{8}$ and

$$\frac{1}{2}-t+\frac{2t^2}{5-t}\geqq\frac{1}{2}-\frac{1}{8}+\frac{1}{156}=\frac{119}{312}\doteqdot 0.38141,$$

$$b_3c_3\doteqdot 0.17414, \qquad \frac{t(1-t)}{2}-b_3c_3t\geqq\frac{1}{16}\Big(\frac{15}{32}-b_3c_3\Big)\doteqdot 0.01841,$$

we obtain

$$G'(t)>\frac{64}{79}-u\Big[\frac{75}{79}-\frac{119}{312}u-\frac{1}{16}\Big(\frac{15}{32}-b_3c_3\Big)u^2\Big], \tag{4.8$_3$}$$

which yields

$$G'(t)>\frac{64}{79}-\log_e\frac{39}{7}\Big[\frac{75}{79}-\frac{119}{312}\log_e\frac{79}{15}-\frac{1}{16}\Big(\frac{15}{32}-b_3c_3\Big)\Big(\log_e\frac{79}{15}\Big)^2\Big].$$

Since we have

$$\frac{64}{79}\doteqdot 0.81013, \qquad \frac{75}{79}\doteqdot 0.94937, \qquad \frac{119}{312}\log_e\frac{79}{15}\doteqdot 0.63367,$$

$$\frac{1}{16}\Big(\frac{15}{32}-b_3c_3\Big)\Big(\log_e\frac{79}{15}\Big)^2\doteqdot 0.05082,$$

the right-hand side of the above inequality $\doteqdot 0.35517$, hence

$$G'(t)>0 \qquad \text{for} \quad \frac{1}{16}\leqq t\leqq\frac{1}{8}.$$

IV. $0<t\leqq\frac{1}{16}$. For such t, we have

$$5<\frac{5-t}{1-t}\leqq\frac{79}{15}, \qquad \log_e 5<u\leqq\log_e\frac{79}{15},$$

$$-1+t(1-t)u+\frac{4}{5-t}t^2>-1, \qquad \sum_{m>2}\frac{u^m t^{m-2}}{m!}<c_4u^3t,$$

where

$$c_4=\frac{1}{6}\Big(\frac{79}{15}\Big)^{1/16}\doteqdot 0.18490.$$

From (4.7) and these inequalities, we obtain analogously as before

$$G'(t)>\frac{4}{5}-u+\frac{555}{1264}u^2. \tag{4.8$_4$}$$

Since $\log_e 5\doteqdot 1.60944>\frac{632}{555}\doteqdot 1.13874$, we have

$$G'(t)>\frac{4}{5}-\log_e 5+\frac{555}{1264}(\log_e 5)^2\doteqdot 0.32791>0 \qquad \text{for} \quad 0<t\leqq\frac{1}{16}.$$

Thus, putting together the above arguments we obtain

$$G'(t)>0 \quad \text{for} \quad 0<t\leqq \frac{1}{3}.$$

Therefore $G(t)$ must be monotone increasing in this interval. Q.E.D.

THEOREM 4.5. *When* $3\leqq n\leqq 14$, *we have*

$$\sqrt{F(x)}+\sqrt{F(X(x))}<\sqrt{2} \quad \textit{for} \quad 0<x<1.$$

PROOF. By Theorem 3.8, it suffices to prove the above inequality for $0<x\leqq\sigma$. By Lemma 4.3, we have

$$\sqrt{F(x)}+\sqrt{F(X(x))}<\frac{1}{4}\sqrt{5(n-1)\Big(\frac{5n-1}{n-1}\Big)^{1/n}-(5n-1)}+\frac{n}{\sqrt{n^2+4n-4}}$$

$$\text{for} \quad 0<x\leqq\sigma, \quad n\geqq 3.$$

When $3\leqq n\leqq 4$, by Lemma 4.4 we have

$$\frac{1}{4}\sqrt{5(n-1)\Big(\frac{5n-1}{n-1}\Big)^{1/n}-(5n-1)}\leqq\frac{1}{4}\sqrt{5\cdot 2\cdot 7^{1/3}-14}\doteqdot 0.56620,$$

$$\frac{n}{\sqrt{n^2+4n-4}}\leqq\frac{4}{\sqrt{28}}=\frac{2}{\sqrt{7}}\doteqdot 0.75593$$

and

$$\frac{1}{4}\sqrt{5(n-1)\Big(\frac{5n-1}{n-1}\Big)^{1/n}-(5n-1)}+\frac{n}{\sqrt{n^2+4n-4}}<1.32213<\sqrt{2}.$$

When $4\leqq n\leqq 10$, by Lemma 4.4 we have

$$\frac{1}{4}\sqrt{5(n-1)\Big(\frac{5n-1}{n-1}\Big)^{1/n}-(5n-1)}\leqq\frac{1}{4}\sqrt{5\cdot 3\cdot\Big(\frac{19}{3}\Big)^{1/4}-19}\doteqdot 0.54748,$$

$$\frac{n}{\sqrt{n^2+4n-4}}\leqq\frac{10}{\sqrt{136}}=\frac{5}{\sqrt{34}}\doteqdot 0.85749$$

and

$$\frac{1}{4}\sqrt{5(n-1)\Big(\frac{5n-1}{n-1}\Big)^{1/n}-(5n-1)}+\frac{n}{\sqrt{n^2+4n-4}}<1.40498<\sqrt{2}.$$

When $8\leqq n\leqq 14$, we have analogously

$$\frac{1}{4}\sqrt{5(n-1)\Big(\frac{5n-1}{n-1}\Big)^{1/n}-(5n-1)}\leqq\frac{1}{4}\sqrt{5\cdot 7\cdot\Big(\frac{39}{7}\Big)^{1/8}-39}\doteqdot 0.52336,$$

$$\frac{n}{\sqrt{n^2+4n-4}}\leqq\frac{14}{\sqrt{248}}=\frac{7}{\sqrt{62}}\doteqdot 0.88900$$

and

$$\frac{1}{4}\sqrt{5(n-1)\Big(\frac{5n-1}{n-1}\Big)^{1/n}-(5n-1)}+\frac{n}{\sqrt{n^2+4n-4}}<1.41236<\sqrt{2}.$$

Thus we have proved the inequality:

$$\sqrt{F(x)}+\sqrt{F(X(x))}<\sqrt{2} \qquad \text{for} \quad 0<x\leqq\sigma. \qquad \text{Q. E. D.}$$

Theorem 4.5, Lemma 1.2 and (1.5) imply the inequality:

$$T<\sqrt{2}\pi \qquad \text{for} \quad 3\leqq n\leqq 14. \tag{U}$$

§ 5. An estimation of σ for $n\geqq 14$.

In this section, we shall show that $\sigma<\frac{1}{11}$ for $n\geqq 14$.

By Lemma 3.6 and (3.7), $\sigma<\frac{1}{11}$ is equivalent to $g\left(\frac{1}{11}\right)>B=(n-1)^{1-1/n}$ By (3.6) and (3.7), we have

$$P\left(\frac{1}{11}\right)=\frac{4}{121}(53n^2+29n-7),$$

$$P\left(\frac{1}{11}\right)-3\left(1-\frac{1}{11}\right)^2\left(n+\frac{n-2}{11}\right)=\frac{4}{1331}(583n^2-581n+73),$$

$$\varphi\left(\frac{1}{11}\right)=\frac{1}{11}(11n-1)^{1-1/n},$$

hence

$$g\left(\frac{1}{11}\right)=\frac{(53n^2+29n-7)(11n-1)^{1-1/n}}{583n^2-581n+73}. \tag{5.1}$$

Therefore $g\left(\frac{1}{11}\right)>B$ is equivalent to

$$\left(\frac{11n-1}{n-1}\right)^{1-1/n}>\frac{583n^2-581n+73}{53n^2+29n-7}. \tag{5.2}$$

Putting $\frac{1}{n}=t$, the above inequality can be written as follows:

$$\left(\frac{11-t}{1-t}\right)^{1-t}>\frac{583-581t+73t^2}{53+29t-7t^2},$$

which is also equivalent to

$$(1-t)\log\frac{11-t}{1-t}>\log\frac{583-581t+73t^2}{53+29t-7t^2}. \tag{5.3}$$

On the other hand, we have

$$\frac{11-t}{1-t}=11\left\{1+\frac{10t}{11(1-t)}\right\}$$

and

$$0<\frac{10t}{11(1-t)}\leqq\frac{1}{11} \qquad \text{for} \quad 0<t\leqq\frac{1}{11}. \tag{5.4}$$

Using these relations, we obtain

$$(5.5)\qquad (1-t)\log\frac{11-t}{1-t}$$

$$=(1-t)\log 11+\frac{10}{11}t-\frac{1}{2}\left(\frac{10}{11}\right)^2\frac{t^2}{1-t}+\frac{1}{3}\left(\frac{10}{11}\right)^3\frac{t^3}{(1-t)^2}-\cdots$$

$$+(-1)^{m-1}\frac{1}{m}\left(\frac{10}{11}\right)^m\frac{t^m}{(1-t)^{m-1}}+\cdots.$$

Next, we have

$$\frac{583-581t+73t^2}{53+29t-7t^2}=11(1-Q),$$

where

$$(5.6)\qquad Q:=\frac{150t(6-t)}{11(53+29t-7t^2)}.$$

Since for $0<t\leqq\frac{1}{11}$ we have

$$0<Q<\frac{150}{11\cdot 53}\cdot\frac{1}{11}\cdot\left(6-\frac{1}{11}\right)=\frac{150\cdot 65}{11^3\cdot 53}<1,$$

we obtain

$$(5.7)\qquad \log\frac{583-581t+73t^2}{53+29t-7t^2}=\log 11-Q-\frac{1}{2}Q^2-\frac{1}{3}Q^3-\cdots-\frac{1}{n}Q^n-\cdots.$$

From (5.5) and (5.7), we obtain the following:

$$(5.8)\qquad (1-t)\log\frac{11-t}{1-t}-\log\frac{583-581t+73t^2}{53+29t-7t^2}$$

$$=-t\log 11+\frac{10}{11}t-\frac{1}{2}\left(\frac{10}{11}\right)^2\frac{t^2}{1-t}+\frac{1}{3}\left(\frac{10}{11}\right)^3\frac{t^3}{(1-t)^2}-\cdots$$

$$+(-1)^{m-1}\frac{1}{m}\left(\frac{10}{11}\right)^m\frac{t^m}{(1-t)^{m-1}}+\cdots$$

$$+Q+\frac{1}{2}Q^2+\frac{1}{3}Q^3+\cdots+\frac{1}{m}Q^m+\cdots.$$

LEMMA 5.1. *$Q^m>\left(\frac{10}{11}\right)^m\frac{t^m}{(1-t)^{m-1}}$ for $0<t\leqq\frac{1}{11}$* $(m=1, 2, 3, \cdots)$.

PROOF. This inequality is equivalent to

$$\frac{150t(6-t)}{11(53+29t-7t^2)}>\frac{10}{11}\frac{t}{(1-t)^{1-1/m}},$$

that is

$$(5.9)\qquad \frac{15(1-t)(6-t)}{53+29t-7t^2}>(1-t)^{1/m}.$$

Since the left-hand side of (5.9) is monotone decreasing in $\left[0, \frac{1}{11}\right]$ and

$$\frac{15(1-t)(6-t)}{53+29t-t^2}\bigg|_{t=1/11}=\frac{390}{269}>1>(1-t)^{1/m},$$

(5.9) is true. Q. E. D.

THEOREM 5.2. $\sigma<\frac{1}{11}$ *for* $n\geqq 14$.

PROOF. By means of (5.8), it suffices to prove that

$$\begin{aligned}(5.10)\qquad &\frac{10}{11}+\frac{1}{3}\Big(\frac{10}{11}\Big)^3\frac{t^2}{(1-t)^2}+\frac{1}{5}\Big(\frac{10}{11}\Big)^5\frac{t^4}{(1-t)^4}+\cdots\\ &+\frac{1}{t}\Big(Q+\frac{1}{3}Q^3+\frac{1}{5}Q^5+\cdots\Big)\\ &+\frac{1}{2}\Big\{\frac{Q^2}{t}-\Big(\frac{10}{11}\Big)^2\frac{t}{1-t}\Big\}+\frac{1}{4}\Big\{\frac{Q^4}{t}-\Big(\frac{10}{11}\Big)^4\frac{t^3}{(1-t)^3}\Big\}+\cdots\\ &>\log_e 11 \fallingdotseq 2.39790\,.\end{aligned}$$

By Lemma 5.1, every term in the left-hand side of (5.10) is positive.

When $n\geqq 20$, i. e. $t\leqq\frac{1}{20}$, we have

$$\frac{Q}{t}=\frac{150(6-t)}{11(53+29t-7t^2)}\geqq\Big(\frac{Q}{t}\Big)_{t=1/20}=\frac{357000}{239503}\fallingdotseq 1.49059\,,$$

hence

$$\frac{10}{11}+\frac{Q}{t}\geqq\frac{10}{11}+\frac{357000}{239503}\fallingdotseq 2.39968\,,$$

which implies the following:

$$\frac{10}{11}+\frac{Q}{t}>\log_e 11\,.$$

Consequently (5.10) is true for $0<t\leqq\frac{1}{20}$.

In the following, putting

$$\begin{aligned}(5.11)\qquad \Psi(t):=&\frac{10}{11}+\frac{1}{3}\Big(\frac{10}{11}\Big)^3\Big(\frac{t}{1-t}\Big)^2+\frac{1}{5}\Big(\frac{10}{11}\Big)^5\Big(\frac{t}{1-t}\Big)^4+\cdots\\ &+\frac{1}{t}\Big(Q+\frac{1}{3}Q^3+\frac{1}{5}Q^5+\cdots\Big)+\frac{1}{2}\Big(\frac{Q^2}{t}-\Big(\frac{10}{11}\Big)^2\frac{t}{1-t}\Big),\end{aligned}$$

we shall prove that

$$\Psi(t)>\log_e 11\qquad\text{for }\ \frac{1}{20}\leqq t\leqq\frac{1}{14}.$$

First, we have

$$\begin{aligned}&\frac{10}{11}+\frac{1}{3}\Big(\frac{10}{11}\Big)^3\Big(\frac{t}{1-t}\Big)^2+\frac{1}{5}\Big(\frac{10}{11}\Big)^5\Big(\frac{t}{1-t}\Big)^4+\cdots\\ &+\frac{Q}{t}\Big(1+\frac{1}{3}Q^2+\frac{1}{5}Q^4+\cdots\Big)\end{aligned}$$

$$> \frac{10}{11}\Big[1+\Big(\frac{10t}{11\sqrt{3}(1-t)}\Big)^2+\Big(\frac{10t}{11\sqrt{3}(1-t)}\Big)^4+\cdots\Big]$$

$$+\frac{Q}{t}\Big[1+\Big(\frac{1}{\sqrt{3}}Q\Big)^2+\Big(\frac{1}{\sqrt{3}}Q\Big)^4+\cdots\Big]$$

$$=\frac{10}{11}\cdot\frac{1}{1-\dfrac{100t^2}{363(1-t)^2}}+\frac{Q}{t}\cdot\frac{1}{1-\dfrac{Q^2}{3}}.$$

Hence we obtain the following:

$$\Psi(t) > \frac{10}{11}\cdot\frac{1}{1-\dfrac{100t^2}{363(1-t)^2}}+\frac{Q}{t}\cdot\frac{3}{3-Q^2}+\frac{1}{2}\Big\{\frac{Q^2}{t}-\frac{100t}{121(1-t)}\Big\}. \tag{5.12}$$

In the interval $\Big[\frac{1}{20}, \frac{1}{14}\Big]$ of t, we have

$$\frac{100t^2}{363(1-t)^2} \geqq \frac{100t^2}{363(1-t)^2}\Big|_{t=1/20} = \frac{100}{363\cdot 19^2}$$

and hence

$$\frac{10}{11}\cdot\frac{1}{1-\dfrac{100t^2}{363(1-t)^2}} \geqq \frac{119130}{130943} \doteqdot 0.90979;$$

$$\frac{Q}{t}=\frac{150(6-t)}{11(53+29t-7t^2)} \geqq \frac{Q}{t}\Big|_{t=1/14} = \frac{150\cdot 83\cdot 2}{11\cdot 1541} = \frac{24900}{16951} \doteqdot 1.46894.$$

Here we need the following

LEMMA 5.3. *Q and $\frac{Q^2}{t}$ are monotone increasing in* $(0, 1)$.

PROOF. First we have

$$\Big(\frac{t(6-t)}{53+29t-7t^2}\Big)'=\frac{318-106t+13t^2}{(53+29t-7t^2)^2}>0 \qquad \text{for} \quad 0<t<1$$

and

$$\Big(\frac{t(6-t)^2}{(53+29t-7t^2)^2}\Big)'=\frac{(6-t)(318-333t+97t^2-7t^3)}{(53+29t-7t^2)^3}>0 \qquad \text{for} \quad 0<t<1,$$

because

$$318-333t+97t^2-7t^3 > 311-333t+97t^2 > 0.$$

Hence it follows that Q' and $\Big(\frac{Q^2}{t}\Big)'$ are positive in $(0, 1)$. Consequently Q and $\frac{Q^2}{t}$ are monotone increasing there. Q. E. D.

Now, we go back to the proof of Theorem 5.2. Using Lemma 5.3, for $\frac{1}{20} \leqq t \leqq \frac{1}{14}$ we have

$$\frac{1}{2}\left\{\frac{Q^2}{t}-\frac{100t}{121(1-t)}\right\}>\frac{1}{2}\left\{\left(\frac{Q^2}{t}\right)_{t=1/20}-\frac{100}{121}\left(\frac{t}{1-t}\right)_{t=1/14}\right\}$$

$$=\frac{50\cdot 15^2}{11^2}\cdot\frac{\frac{1}{20}\cdot\left(6-\frac{1}{20}\right)^2}{\left(53+\frac{29}{20}-\frac{7}{400}\right)^2}-\frac{50}{11^2}\cdot\frac{1}{13}$$

$$=\frac{15^2\cdot 119^2\cdot 10^3}{11^2\cdot 21773^2}-\frac{50}{11^2\cdot 13}$$

$$=\frac{3186225000}{57361687009}-\frac{50}{1573}\doteqdot 0.02376$$

and

$$\frac{Q}{t}\cdot\frac{3}{3-Q^2}\geqq\left(\frac{Q}{t}\right)_{t=1/14}\cdot\frac{3}{3-(Q^2)_{t=1/20}}$$

$$=\frac{24900}{16951}\cdot\frac{3}{3-\left(\frac{17850}{239503}\right)^2}$$

$$=\frac{3\cdot 249\cdot 239503^2\cdot 10^2}{16951\cdot 171766438527}\doteqdot 1.47166\,.$$

Using these inequalities, from (5.12) we obtain

$$\Psi(t)>\frac{119130}{130943}+\frac{3\cdot 249\cdot 239503^2\cdot 10^2}{16951\cdot 171766438527}+\frac{3186225000}{57361687009}-\frac{50}{1573} \tag{5.13}$$

$$\doteqdot 2.40521>\log_e 11 \quad (\doteqdot 2.39790)\,.$$

Consequently, (5.10) is true for $\frac{1}{20}\leqq t\leqq\frac{1}{14}$. Q. E. D.

REMARK. We have proved $\sigma<\frac{1}{11}$ for any real number $n\geqq 14$. However it may be also true for $6\leqq n<14$, because we can show that it is true for the integers $n=6, 7, 8, 9, 10, 11, 12$ and 13 by means of the following inequality equivalent to (5.2):

$$A_n:=\left(\frac{11n-1}{n-1}\right)^{1/n}<\frac{(11n-1)(53n^2+29n-7)}{(n-1)(583n^2-581n+73)}:=E_n\,. \tag{5.14}$$

In fact

$$A_6=\left(\frac{65}{5}\right)^{1/6}\doteqdot 1.53341\,,\qquad E_6=\frac{1079}{703}\doteqdot 1.53485\,;$$

$$A_7=\left(\frac{76}{6}\right)^{1/7}\doteqdot 1.43722\,,\qquad E_7=\frac{106134}{73719}\doteqdot 1.43971\,;$$

$$A_8=\left(\frac{87}{7}\right)^{1/8}\doteqdot 1.37026\,,\qquad E_8=\frac{314679}{229159}\doteqdot 1.37319\,;$$

$$A_9=\left(\frac{98}{8}\right)^{1/9}\doteqdot 1.32100\,,\qquad E_9=\frac{222803}{168268}\doteqdot 1.32410\,;$$

$$A_{10}=\left(\frac{109}{9}\right)^{1/10}\doteqdot 1.28327, \qquad E_{10}=\frac{202849}{157689}\doteqdot 1.28639;$$

$$A_{11}=\left(\frac{120}{10}\right)^{1/11}\doteqdot 1.25345, \qquad E_{11}=\frac{3228}{2569}\doteqdot 1.25652;$$

$$A_{12}=\left(\frac{131}{11}\right)^{1/12}\doteqdot 1.22930, \qquad E_{12}=\frac{1044463}{847583}\doteqdot 1.23228;$$

$$A_{13}=\left(\frac{142}{12}\right)^{1/13}\doteqdot 1.20934, \qquad E_{13}=\frac{220739}{182094}\doteqdot 1.21223.$$

However

$$A_5=\left(\frac{54}{4}\right)^{1/5}\doteqdot 1.68293, \qquad E_5=\frac{39501}{23486}\doteqdot 1.68190.$$

Hence $\sigma>\frac{1}{11}$ for $n=5$.

PROPOSITION 5.3. $\sigma<\frac{1}{11}$ *for any integer* $n\geqq 6$.

§ 6. Proof of $T<\sqrt{2}\pi$ for $n\geqq 14$.

In this section, we shall prove the inequality $T<\sqrt{2}\pi$ for any real number $n\geqq 14$, by the same method used for the case $3\leqq n\leqq 14$.

LEMMA 6.1. *When* $n\geqq 14$, *for* $0<x\leqq\sigma$ *we have*

$$\sqrt{F(x)}+\sqrt{F(X(x))}$$
$$<\frac{1}{10}\sqrt{11(n-1)\left(\frac{11n-1}{n-1}\right)^{1/n}-(11n-1)}+\frac{n}{\sqrt{n^2+4n-4}}.$$

PROOF. By Theorem 5.2 and Theorem 2.3, for $0<x\leqq\sigma$ we have

$$F(x)<F\left(\frac{1}{11}\right)=\left[\frac{x(n-x)}{(1-x)^2}\cdot\frac{B-\varphi(x)}{\varphi(x)}\right]_{x=1/11}$$
$$=\frac{1}{100}\left\{11(n-1)\left(\frac{11n-1}{n-1}\right)^{1/n}-(11n-1)\right\}.$$

This inequality and Theorem 2.5 imply this lemma. Q. E. D.

LEMMA 6.2. $G_{11}(t)$ *is monotone increasing in* $\left(0,\frac{1}{11}\right]$.

PROOF. For $G(t)=G_{11}(t)$, from (4.6) we obtain

$$G'(t)=-u+\left(\frac{1}{2}-t\right)u^2+\frac{10}{11-t}+\frac{10}{11-t}tu \tag{6.1}$$
$$+\frac{1}{2}tu^2\left\{(1-t)u+\frac{10}{11-t}t\right\}$$
$$+\sum_{m>2}\frac{u^m t^{m-2}}{m!}\left\{-1+t(1-t)u+\frac{10}{11-t}t^2\right\},$$

where

(6.2) $$u=\log_e \frac{11-t}{1-t}.$$

Since for $0<t\leqq\frac{1}{11}$

$$11<\frac{11-t}{1-t}\leqq 12, \qquad \log_e 11<u\leqq \log_e 12;$$

$$-1+t(1-t)u+\frac{10}{11-t}t^2>-1,$$

$$\sum_{m>2}\frac{u^m t^{m-2}}{m!}<\frac{u^3 t}{6}e^{ut}=\frac{u^3 t}{6}\left(\frac{11-t}{1-t}\right)^t<\frac{u^3 t}{6}\cdot 12^{1/11},$$

(6.1) implies

(6.3) $$G'(t)>\frac{10}{11-t}+\left[-1+\frac{10t}{11-t}\right]u+\left[\frac{1}{2}-t+\frac{5t^2}{11-t}\right]u^2 +\left[\frac{t(1-t)}{2}-\frac{t}{6}\cdot 12^{1/11}\right]u^3.$$

However for $0<t\leqq\frac{1}{11}$ we have

$$\frac{10}{11-t}>\frac{10}{11}, \qquad -1+\frac{10t}{11-t}>-1,$$

$$\frac{1}{2}-t+\frac{5t^2}{11-t}>\frac{1}{2}-t+\frac{5}{11}t^2\geqq\frac{1}{2}-\frac{1}{11}+\frac{5}{11^3}=\frac{1099}{2\cdot 11^3}$$

and

$$\frac{t(1-t)}{2}-\frac{t}{6}\cdot 12^{1/11}=\frac{t}{2}\left(1-\frac{1}{3}\cdot 12^{1/11}-t\right)>0$$

since $1-\frac{1}{3}\cdot 12^{1/11}\fallingdotseq 1-\frac{1.25345}{3}>\frac{1}{11}$. From these and (6.3), we obtain the following inequality:

(6.4) $$G'(t)>\frac{10}{11}-u+\frac{1099}{2\cdot 11^3}u^2.$$

Since

$$\frac{11^3}{1099}\fallingdotseq 1.21110<\log_e 11\fallingdotseq 2.39790,$$

it is seen easily that

$$\frac{10}{11}-u+\frac{1099}{2\cdot 11^3}u^2>\frac{10}{11}-\log_e 11+\frac{1099}{2\cdot 11^3}(\log_e 11)^2.$$

However

$$\frac{1099}{2\cdot 11^3}(\log_e 11)^2\fallingdotseq 2.37384, \qquad \frac{10}{11}\fallingdotseq 0.90909,$$

hence the right-hand side of the above inequality $\fallingdotseq 0.88503$. Consequently, we obtain

$$G'(t)>0,$$

which implies that $G_{11}(t)=G(t)$ is monotone increasing in $\left(0, \frac{1}{11}\right]$. Q. E. D.

THEOREM 6.3. *When* $n \geqq 14$, *we have*

$$\sqrt{F(x)}+\sqrt{F(X(x))}<\sqrt{2} \quad \textit{for} \quad 0<x<1.$$

PROOF. By Theorem 3.8, it suffices to prove the above inequality for $0<x \leqq \sigma$. By Theorem 5.2 and Lemma 6.1, we have for $0<x \leqq \sigma$ the following:

$$\text{(6.5)} \qquad \sqrt{F(x)}+\sqrt{F(X(x))}$$
$$<\frac{1}{10}\sqrt{11(n-1)\left(\frac{11n-1}{n-1}\right)^{1/n}-(11n-1)}+\frac{n}{\sqrt{n^2+4n-4}}.$$

The first term of the right-hand side of (6.5) is decreasing for $n \geqq 14$ by Lemma 6.2 and (4.3) and the second term is increasing for $n \geqq 2$. Making use of these facts, we obtain:

i) When $14 \leqq n \leqq 21$,

$$\frac{1}{10}\sqrt{11(n-1)\left(\frac{11n-1}{n-1}\right)^{1/n}-(11n-1)}+\frac{n}{\sqrt{n^2+4n-4}}$$
$$<\frac{1}{10}\sqrt{11\cdot 13\cdot\left(\frac{153}{13}\right)^{1/14}-153}+\frac{21}{\sqrt{521}}.$$

Since we have

$$\left(\frac{153}{13}\right)^{1/14} \fallingdotseq 1.19256, \qquad \frac{1}{10}\sqrt{11\cdot 13\cdot\left(\frac{153}{13}\right)^{1/14}-153} \fallingdotseq 0.41877,$$
$$\frac{21}{\sqrt{521}} \fallingdotseq 0.92003,$$

we get

$$\frac{1}{10}\sqrt{11\cdot 13\cdot\left(\frac{153}{13}\right)^{1/14}-153}+\frac{21}{\sqrt{521}}<1.33880<\sqrt{2}.$$

ii) When $n \geqq 21$,

$$\frac{1}{10}\sqrt{11(n-1)\left(\frac{11n-1}{n-1}\right)^{1/n}-(11n-1)}+\frac{n}{\sqrt{n^2+4n-4}}$$
$$<\frac{1}{10}\sqrt{220\cdot\left(\frac{23}{2}\right)^{1/21}-230}+1.$$

Since we have

$$\left(\frac{23}{2}\right)^{1/21} \fallingdotseq 1.12334, \qquad \frac{1}{10}\sqrt{220\cdot\left(\frac{23}{2}\right)^{1/21}-230} \fallingdotseq 0.41393,$$

we get

$$\frac{1}{10}\sqrt{220\cdot\left(\frac{23}{2}\right)^{1/21}-230}+1<1.41393<\sqrt{2}\,.$$

Thus we have proved the inequality

$$\sqrt{F(x)}+\sqrt{F(X(x))}<\sqrt{2} \qquad \text{for} \quad 0<x\leqq\sigma. \qquad \text{Q. E. D.}$$

Finally, by means of Lemma 1.2, (1.6), (1.8), Theorem 4.5 and Theorem 6.3, we obtain the following

MAIN THEOREM. *When* $n\geqq 3$, *the period function* T_n *given by* (1.1) *satisfies*

(i) $\pi<T_n(x_0)<\sqrt{2}\pi$ *for* $0<x_0<1$,

(ii) $\lim_{x_0\to 0} T_n(x_0)=\pi$ *and* $\lim_{x_0\to 1} T_n(x_0)=\sqrt{2}\pi$.

REMARK. In this paper, all numerical calculations have been done to sufficiently large number of decimal places and a seven figure table of logarithms has been used if necessary.

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