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On exceptional values of meromorphic functions of divergence class

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§ 1. Introduction.

In this paper, following the methods used in [5] and [6], we investigate the value distribution of meromorphic functions of divergence class or of infinite order in the plane $|z|<\infty$. Let $f(z)$ be a meromorphic function having divergence class of order ρ, $0<\rho<\infty$, in $|z|<\infty$; that is, $\int^{\infty}T(t, f)/t^{1+\rho}dt=\infty$. Then, it is known that there are at most two G-exceptional values which satisfy $\int^{\infty}N(t, a)/t^{1+\rho}dt=O(1)$. Further, these values are not always exceptional in the sense of Nevanlinna ([7]) and conversely there is a meromorphic function $g(z)$ of divergence class such that $\delta(0, g)=1$ and the value 0 is not G-exceptional (Example 2, § 4). These examples show that these two notions of exceptionality of values are independent of each other in a sense. Then, how many values are there for $f(z)$ which satisfy $\delta(a, f)=1$ or are G-exceptional? We start from this question, discuss some relations among Borel exceptional values, G-exceptional values and Nevanlinna exceptional values, and introduce a new notion of exceptionality of values for meromorphic functions of divergence class (§ 2).

Any meromorphic function $h(z)$ of infinite order in $|z|<\infty$ is of divergence class in a sense, because for any large number λ, $\int^{\infty}T(t, h)/t^{1+\lambda}dt=\infty$. Analogizing with the case of finite order, we introduce notions of exceptionality of values for meromorphic functions of infinite order and give some relations with the Nevanlinna deficient values (§ 3).

Some examples are given in § 4.

We will use the symbols of the Nevanlinna theory:

$$T(r, f),\quad m(r, a),\quad N(r, a),\quad \delta(a, f),\quad S(r, f)\quad \text{etc.}$$

freely ([2], [4]).

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§ 2. Meromorphic functions of divergence class.

In this section, we discuss meromorphic functions of divergence class of finite order. Let $f(z)$ be a meromorphic function having divergence class of order ρ $(0<\rho<\infty)$ in $|z|<\infty$:

$$\int^{\infty}\frac{T(t, f)}{t^{1+\rho}}dt=\infty .$$

It is said ([7]) that a value w is G-exceptional for $f(z)$ when

$$\int^{\infty}\frac{N(t, w)}{t^{1+\rho}}dt=O(1) .$$

It is known that the number $N(G)$ of the elements of the set G of G-exceptional values for $f(z)$ is at most two ([7]). In [6], we considered the defect relation in relation to the Borel exceptional values. Here, we consider the defect relation of meromorphic functions of divergence class in relation to the G-exceptional values. As a Borel exceptional value is G-exceptional and the converse is not always true (see Example 1 in § 4), the discussions done here are wider than those in [6].

DEFINITION 1. For $r>1$, $0\leqq\alpha\leqq\rho$ and any value a

$$T_{\alpha}(r, f)=\int_{1}^{r}\frac{T(t, f)}{t^{1+\alpha}}dt , \qquad N_{\alpha}(r, a)=\int_{1}^{r}\frac{N(t, a)}{t^{1+\alpha}}dt .$$

LEMMA 1.

$$\lim_{r\to\infty}T_{\rho}(r, f)=\infty .$$

Because $f(z)$ has divergence class and $T(t, f)$ is non-negative.

DEFINITION 2. For any value a

$$\delta_{\alpha}(a, f)=1-\limsup_{r\to\infty}\frac{N_{\alpha}(r, a)}{T_{\alpha}(r, f)} , \qquad \Delta_{\alpha}(a, f)=1-\liminf_{r\to\infty}\frac{N_{\alpha}(r, a)}{T_{\alpha}(r, f)} .$$

As in Proposition 5 ([5]), we have

LEMMA 2. *For any* $0\leqq\alpha\leqq\rho$,

$$0\leqq\delta(a, f)\leqq\delta_{\alpha}(a, f)\leqq\delta_{\rho}(a, f)\leqq\Delta_{\rho}(a, f)\leqq\Delta_{\alpha}(a, f)\leqq\Delta(a, f)\leqq 1 .$$

LEMMA 3. *If w is G-exceptional for $f(z)$, then* $\delta_{\rho}(w, f)=1$.

Because $\lim_{r\to\infty}T_{\rho}(r, f)=\infty$ by Lemma 1 and $N_{\rho}(r, w)=O(1)$ by definition.

PROPOSITION 1. *Let $a_1, \cdots, a_q$ $(q\geqq 3)$ be q different values. Then, for all* $r>1$

$$(q-2)T_{\rho}(r, f)<\sum_{i=1}^{q}N_{\rho}(r, a_i)+O(1) .$$

Because

$$\int_1^r \frac{\log t}{t^{1+\rho}} dt = O(1),$$

we obtain this proposition from the second fundamental theorem of Nevanlinna directly.

From this, we have the so-called defect relation:

COROLLARY 1.

$$\sum_a \delta_\rho(a, f) \leqq 2.$$

Applying Lemma 3, we have

COROLLARY 2. $N(G) \leqq 2$.

To consider relations between the order and the number of G-exceptional values, as in [5] we give the following.

DEFINITION 3. For $0 \leqq \alpha \leqq \rho$,

$$K_\alpha(f) = \limsup_{r\to\infty} \frac{N_\alpha(r, 0) + N_\alpha(r, f)}{T_\alpha(r, f)}.$$

As in Proposition 6 ([5]), we have

PROPOSITION 2. *For any* $0 \leqq \alpha \leqq \rho$,

$$K_\rho(f) \leqq K_\alpha(f) \leqq K(f),$$

where $K(f) = \limsup_{r\to\infty} (N(r, 0) + N(r, f))/T(r, f)$.

PROPOSITION 3. *If* ρ *is not integer,*

$$K_\rho = \inf_f K_\rho(f) \geqq \begin{cases} 1-\rho & \text{for } 0<\rho<1 \\ (q+1-\rho)(\rho-q)/\rho c(q) & \text{for } [\rho] = q \geqq 1, \end{cases}$$

where f *ranges over all meromorphic functions having divergence class of order* ρ *and* $c(q) = 2(q+1)(\log(q+1)+2)$.

We can prove this as in the case of Theorem 3 in [5].

COROLLARY 3. *If* $K_\rho(f) = 0$ *(therefore if* $N(G) = 2$*), then* ρ *is integer.*

REMARK. Even if $K_\rho(f) = 0$, $f(z)$ is not always of regular growth as Theorem 5 in [7] shows.

As cited in §1, the G-exceptionality and the Nevanlinna exceptionality are independent of each other in a sense. But considering $\delta_\rho(a, f)$, we can control the two notions into a single relation. That is, we can prove the following

THEOREM 1. *For any meromorphic function* $f(z)$ *having divergence class of order* ρ $(0<\rho<\infty)$ *in* $|z|<\infty$, *the following inequality holds:*

$$\sum_{a \notin G} \delta_\rho(a, f) \leqq 2 - N(G).$$

PROOF. 1) The case when $G = \emptyset$. See Corollary 1. 2) The case when $G \neq \emptyset$. Let $G = \{a_i\}_{i=1}^n$ $(n=1$ or $2)$ and $q>2$, then from Proposition 1

$$(q-2)T_\rho(r, f) < \sum_{a_i \notin G} N_\rho(r, a_i) + O(1).$$

Using Lemma 1 and by definition of $\delta_\rho(a, f)$, we have

$$\sum_{a_i \notin G} \delta_\rho(a_i, f) \leqq 2-n = 2-N(G).$$

This inequality reduces to this theorem as usual.

COROLLARY 4.

$$\sum_{a \notin G} \delta(a, f) \leqq 2-N(G).$$

COROLLARY 5. *The number of values which are G-exceptional for $f(z)$ or at which $f(z)$ has the maximal Nevanlinna deficiency is at most two.*

LEMMA 4. *$f'(z)$ has also divergence class of order ρ* ([**1**]).

Using this lemma, we can prove the following proposition as in Theorem 1 ([**5**]).

PROPOSITION 3.

$$\sum_{a \neq \infty} \delta_\rho(a, f) \leqq \liminf_{r \to \infty} \frac{T_\rho(r, f')}{T_\rho(r, f)} \leqq \limsup_{r \to \infty} \frac{T_\rho(r, f')}{T_\rho(r, f)} \leqq 2-\delta_\rho(\infty, f).$$

As in Theorem 2 ([**5**]), we have

PROPOSITION 4.

$$\sum_{a \neq \infty} \delta_\rho(a, f) \leqq (2-\delta_\rho(\infty, f))\delta_\rho(0, f').$$

THEOREM 2. *Let $f(z)$ be any meromorphic function having divergence class of order $\rho (0 < \rho < \infty)$ in $|z| < \infty$. If ρ is not integer and $N(G) > 0$, then $N(G) = 1$ and*

$$\sum_{a \notin G} \delta_\rho(a, f) \leqq 1-K_\rho.$$

PROOF. If $N(G) = 2$, we may assume that $G = \{0, \infty\}$ by using a linear transformation if necessary. Then, by Lemma 3

$$\delta_\rho(0, f) = \delta_\rho(\infty, f) = 1.$$

This implies $K_\rho(f) = 0$, so that from Corollary 3, ρ is integer. This is a contradiction. $N(G)$ must be equal to 1.

Thus, we may suppose $G = \{\infty\}$ as above. Then,

$$\delta_\rho(\infty, f') = 1.$$

As $T_\rho(r, f') \nearrow \infty$ (Lemma 4) and as $N(r, f') \leqq 2N(r, f)$,

$$N_\rho(r, f') \leqq 2N_\rho(r, f) = O(1).$$

Suppose that

(1) $$\sum_{a \neq \infty} \delta_\rho(a, f) > 1-K_\rho.$$

Then, from Proposition 4

$$\sum_{a \neq \infty} \delta_\rho(a, f) \leqq \delta_\rho(0, f')$$

as $\delta_\rho(\infty, f) = 1$, so that

$$1 - K_\rho < \delta_\rho(0, f') .$$

Adding both sides $\delta_\rho(\infty, f') = 1$, we have

$$2 - K_\rho < \delta_\rho(0, f') + \delta_\rho(\infty, f') . \tag{2}$$

As

$$K_\rho(f') \leqq 2 - \delta_\rho(0, f') - \delta_\rho(\infty, f') ,$$

we have by (2)

$$K_\rho(f') < K_\rho ,$$

which is a contradiction, because $f'(z)$ has divergence class of order ρ, and so $K_\rho \leqq K_\rho(f')$ by definition. This shows that the inequality (1) is false. We have the result.

Next, introducing a new notion of exceptionality of values for meromorphic functions of divergence class, we will give precise forms for some results given above. Let $f(z)$ have divergence class of order $\rho (0 < \rho < \infty)$ as above.

DEFINITION 4. For $r > 1$ and $0 \leqq \lambda \leqq \rho$,

$$T_{0,\lambda}(r, f) = \int_1^r \frac{T_0(t, f)}{t^{1+\lambda}} dt .$$

LEMMA 5. *$\lim_{r \to \infty} T_{0,\rho}(r, f) = \infty$ if and only if $\lim_{r \to \infty} T_\rho(r, f) = \infty$.*

PROOF. By a simple computation, we obtain

$$T_\rho(r, f) = \frac{T_0(r, f)}{r^\rho} + \rho \int_1^r \frac{T_0(t, f)}{t^{1+\rho}} dt . \tag{3}$$

It is trivial that $T_\rho(r, f)$ and $T_{0,\rho}(r, f)$ increase as r increases. If $\lim_{r \to \infty} T_{0,\rho}(r, f) = \infty$, then $T_\rho(r, f)$ tends to infinity because

$$T_\rho(r, f) \geqq \rho T_{0,\rho}(r, f)$$

by (3). Conversely, if $T_{0,\rho}(r, f) = O(1)$, we see that $T_0(r, f)/r^\rho$ tends to 0 for $r \to \infty$, so that $T_\rho(r, f) = O(1)$.

DEFINITION 5. For $r \geqq r_0$ and $0 \leqq \lambda \leqq \rho$

$$T_\lambda^*(r, f) = \int_{r_0}^r \frac{T_0(t, f)}{t^{1+\lambda} T_{0,\lambda}(t, f)} dt$$

where r_0 is a positive number such that $T_0(r_0, f) > 0$.

PROPOSITION 5. *$f(z)$ is of divergence class if and only if*

$$\lim_{r \to \infty} T_\rho^*(r, f) = \infty .$$

PROOF. As

$$T_\rho^*(r, f) = \log T_{0,\rho}(r, f) - O(1),$$

we have the result by Lemma 5.

DEFINITION 6. For any value a,

1) $$N_\lambda^*(r, a) = \int_{r_0}^{r} \frac{N_0(t, a)}{t^{1+\lambda} T_{0,\lambda}(t, f)} dt \quad \text{for } r \geqq r_0 \text{ and } 0 \leqq \lambda \leqq \rho;$$

2) $$\delta_\lambda^*(a, f) = 1 - \limsup_{r \to \infty} \frac{N_\lambda^*(r, a)}{T_\lambda^*(r, f)}.$$

DEFINITION 7. A value a is said to be G_1-exceptional for $f(z)$ when $N_\rho^*(r, a) = O(1)$ $(r \to \infty)$.

REMARK. "G-exceptional" is "G_1-exceptional" trivially. But the converse is not always true as Example 3 (§ 4) shows.

Let $N(G_1)$ be the number of the elements of the set G_1 of G_1-exceptional values for $f(z)$. Then, how many G_1-exceptional values are there? For this question, we have

THEOREM 3. *For any meromorphic function $f(z)$ of divergence class of order ρ $(0 < \rho < \infty)$, it holds*

$$\sum_{a \notin G_1} \delta_\rho^*(a, f) \leqq 2 - N(G_1).$$

PROOF. From the second fundamental theorem of Nevanlinna, we have easily

$$(q-2) T_0(r, f) < \sum_{i=1}^{q} N_0(r, a) + O((\log r)^2)$$

for all $r > 1$ and q different values a_i $(i = 1, \cdots, q; q \geqq 3)$.

Dividing this by $r^{1+\rho} T_{0,\rho}(r, f)$ and integrating from r_0 to r, it reduces to

(4) $$(q-2) T_\rho^*(r, f) < \sum_{i=1}^{q} N_\rho^*(r, a_i) + O(1).$$

From this inequality, as $\lim_{r \to \infty} T_\rho^*(r, f) = \infty$, we have the following defect relation as usual:

$$\sum_a \delta_\rho^*(a, f) \leqq 2.$$

When $G_1 \neq \emptyset$, if $a \in G_1$, then $\delta_\rho^*(a, f) = 1$. This implies $N(G_1) \leqq 2$ from the above inequality. Let $G_1 = \{a_i\}_{i=1}^{n}$ $(n = 1$ or $2)$. For $q \geqq 3$, from (4) and Proposition 5

$$\sum_{a_i \notin G_1} \delta_\rho^*(a_i, f) \leqq 2 - n = 2 - N(G_1).$$

This reduces to the desired result as usual.

COROLLARY 6.

$$\sum_{a \in G_1} \delta(a, f) \leqq 2 - N(G_1).$$

Because, as in Lemma 2, $\delta(a, f) \leqq \delta_\rho^*(a, f)$.

COROLLARY 7. $N(G_1) \leqq 2$.

REMARK. When $N(G)=2$, ρ is integer (Corollary 3); but even when $N(G_1)=2$, it is still open whether ρ is integer or not.

§ 3. Meromorphic functions of infinite order.

In this section, introducing some new exceptionalities of values which include the Borel exceptionality, we generalize some results given in § 2 to the case of infinite order.

Let $f(z)$ be a meromorphic function of infinite order in $|z|<\infty$ and

$$T_0(r, f)=\int_1^r \frac{T(t, f)}{t} dt, \qquad N_0(r, a)=\int_1^r \frac{N(t, a)}{t} dt$$

$$m_0(r, f)=\int_1^r \frac{m(t, f)}{t} dt, \qquad S_0(r, f)=\int_1^r \frac{S(t, f)}{t} dt$$

(see [**5**]).

First, using these notations we give a modified second fundamental theorem of Nevanlinna without exceptional intervals.

PROPOFITION 6. *Let $a_1, \cdots, a_q$ $(q \geqq 3)$ be q different values. Then, for all $r>1$*

$$(q-2)T_0(r, f) < \sum_{i=1}^{q} N_0(r, a_i) + S_0(r, f) \tag{5}$$

where

$$S_0(r, f) = O((\log r)^2) + O\Big(\int_1^r \frac{\log^+ T(t, f)}{t} dt\Big) \tag{6}$$

for all $r>1$ and

$$\lim_{r\to\infty} \frac{S_0(r, f)}{T_0(r, f)} = 0. \tag{7}$$

PROOF. We have the inequality (5) directly from the second fundamental theorem of Nevanlinna, and so we have only to prove (6) and (7). As is known (see [**4**])

$$S(r, f) < 8\log^+ R + 6\log^+ \frac{1}{R-r} + 8\log^+ T(R, f) + O(1) \tag{8}$$

for $1<r<R$. As in the proof of Lemma 2 ([**4**], p. 62), putting

$$R = r + \frac{r'-r}{r'}$$

for $r' \geqq r > 1$ and dividing both sides of (8) by r, we integrate them from

$r=r_0$ to $r=r'$ $(1<r_0<r')$. We estimate each term of the righthand side of (8) as follows.

i) $$\int_{r_0}^{r'}\frac{\log^+ R}{r}dr=\int_{r_0}^{r'}\frac{\log(r+(r'-r)/r')}{r}dr\leqq\int_{r_0}^{r'}\frac{\log(r+1)}{r}dr$$
$$\leqq\frac{(\log r')^2}{2}+\log r'.$$

ii) $$\int_{r_0}^{r'}\frac{\log^+1/(R-r)}{r}dr=\int_{r_0}^{r'}\frac{\log r'/(r'-r)}{r}dr$$
$$=\int_{u_0}^{\infty}\frac{\log u}{(u-1)u}du\qquad\binom{u=r'/(r'-r)}{u_0=r'/(r'-r_0)}$$
$$=\int_{u_0}^{e}\frac{\log u}{(u-1)u}du+\int_{e}^{\infty}\frac{\log u}{(u-1)u}du\quad(r'>er_0/(e-1))$$
$$<\int_{u_0}^{e}\frac{\log u}{u-1}du+O(1)\qquad(\text{as } u_0>1)$$
$$<\log\frac{e-1}{u_0-1}+O(1)<\log r'+O(1).$$

iii) $$\int_{r_0}^{r'}\frac{\log^+T(R,f)}{r}dr=\int_{t_0}^{t'}\frac{\log^+T(t,f)}{t}\left(\frac{t}{t-1}\right)dt$$
$$<\frac{r_0}{r_0-1}\int_{r_0}^{r'}\frac{\log^+T(t,f)}{t}dt$$

where $t=r+(r'-r)/r'$, $t_0=r_0+(r'-r_0)/r'$ and $t'=r'$. Using i), ii) and iii), we obtain (6) easily.

Next, as $\lim_{r\to\infty}T_0(r,f)/(\log r)^2=\infty$ ([6], Proposition 1) and $\lim_{r\to\infty}\log^+T(r,f)/T(r,f)=0$, we can prove (7) easily from (6).

We wish to generalize some results obtained in § 2 to the case of infinite order. For this purpose, this proposition 6 plays a fundamental role. First of all, it is necessary to define a divergence class for meromorphic functions of infinite order. Analogizing with the case of finite order (cf. Prop. 5), it is natural to say that $f(z)$ has divergence class when, for all $0\leqq\lambda<\infty$

$$\int^{\infty}\frac{T_0(t,f)}{t^{1+\lambda}T_{0,\lambda}(t,f)}dt=\infty.$$

From this point of view, we give some definitions.

DEFINITION 8. For $r>r_0$ and any λ $(0\leqq\lambda<\infty)$

$$T_\lambda^*(r,f)=\int_{r_0}^{r}T_0(t,f)/t^{1+\lambda}T_{0,\lambda}(t,f)dt,$$

$$N_\lambda^*(r,a)=\int_{r_0}^{r}N_0(t,a)/t^{1+\lambda}T_{0,\lambda}(t,f)dt$$

where $T_{0,\lambda}(r, f)=\int_1^r T_0(t, f)/t^{1+\lambda}dt$ and r_0 is a positive number such that $T_0(r_0, f)>0$.

PROPOSITION 7. *For any meromorphic function $f(z)$ of infinite order in $|z|<\infty$,*

i) $T_\lambda^*(r, f)\nearrow\infty$ *for* $r\nearrow\infty$ *and* $\limsup_{r\to\infty} T_\lambda^*(r, f)/\log r=\infty$ *for any* λ $(0\leqq\lambda<\infty)$;

ii) *for* $\alpha\leqq\beta$, $T_\alpha^*(r, f)\geqq T_\beta^*(r, f)$.

PROOF. By definition,

$$T_\lambda^*(r, f)=\log T_{0,\lambda}(r, f)-O(1)$$

and $T_{0,\lambda}(r, f)$ increases to infinity monotonously when r tends to infinity for any λ, positive or zero, as in Proposition 1 ([**5**]), so that we have the first assertion of i). To prove the second, we note first that, as in Proposition 1 ([**5**])

$$\limsup_{r\to\infty}\frac{\log T_0(r, f)}{\log r}=\infty .$$

Using this, as in Proposition 1 ([**5**]), we have for any λ $(\geqq 0)$

$$\limsup_{r\to\infty}\frac{\log T_{0,\lambda}(r, f)}{\log r}=\infty .$$

This shows the second assertion of i).

Integration by parts gives us easily

$$T_{0,\beta}(r, f)=\varepsilon\int_1^r\frac{T_{0,\alpha}(t, f)}{t^{1+\varepsilon}}dt+\frac{T_{0,\alpha}(r, f)}{r^\varepsilon}$$

where $\varepsilon=\beta-\alpha$. From this we obtain

$$r^\beta T_{0,\beta}(r, f)\geqq r^\alpha T_{0,\alpha}(r, f),$$

so that we have ii) by definition.

It is our aim to define a divergence class for meromorphic functions of infinite order. But, these properties force us to assert that any meromorphic function of infinite order is of divergence class in a sense. Therefore, contrary to the case of finite order, the following discussion is valid for all meromorphic functions of infinite order.

Form now on, let $f(z)$ be any meromorphic function of infinite order in $|z|<\infty$.

DEFINITION 9. 1) A value w is said to be

i) G_1-exceptional for $f(z)$ when for some λ, $N_\lambda^*(r, w)=O(1)$;

ii) P_1-exceptional for $f(z)$ when for some λ, $N_\lambda^*(r, w)=O(\log r)$.

2) For any value a

$$\delta_\lambda^*(a, f)=1-\limsup_{r\to\infty}\frac{N_\lambda^*(r, a)}{T_\lambda^*(r, f)}, \qquad \Delta_\lambda^*(a, f)=1-\liminf_{r\to\infty}\frac{N_\lambda^*(r, a)}{T_\lambda^*(r, f)}.$$

REMARK. i) Trivially, "Borel-exceptional" is "G_1-exceptional" and "G_1-exceptional" is "P_1-exceptional"; where a value w is Borel exceptional when $N(r, w)$ is of finite order.

ii) There is an example of meromorphic function which has a G_1-exceptional value not being Borel exceptional (§ 4, Example 4).

Let $N(G_1)$ (resp. $N(P_1)$) be the number of the elements of the set G_1 (resp. P_1) of G_1 (resp. P_1)-exceptional values for $f(z)$.

THEOREM 4. *Let $f(z)$ be any meromorphic function of infinite order in $|z|<\infty$. Then, there is a λ_0 such that for all $\lambda\geqq\lambda_0$*

$$\sum_{a\notin P_1}\delta_\lambda^*(a, f)\leqq 2-N(P_1).$$

PROOF. Dividing the inequality (5) of Proposition 6 by $r^{1+\lambda}T_{0,\lambda}(r, f)$ and integrating from r_0 to r, we have for $r\geqq r_0$

$$(q-2)T_\lambda^*(r, f)<\sum_{i=1}^{q}N_\lambda^*(r, a_i)+S_\lambda^*(r, f) \tag{9}$$

where

$$S_\lambda^*(r, f)=o(T_\lambda^*(r, f)) \qquad (r\to\infty). \tag{10}$$

The relation (10) is derived from (7) easily.

1) The case $P_1=\emptyset$. In this case, we can prove this theorem easily from (9) by using (10) and Definition 9-2).

2) The case $P_1\neq\emptyset$. Let $P_1=\{a_i\}_{i=1}^{N(P_1)}$ and n be any finite number such that $1\leqq n\leqq N(P_1)$. Then, there is a λ_0 such that for any $\lambda\geqq\lambda_0$

$$N_\lambda^*(r, a_i)=O(\log r) \qquad (r\to\infty;\ i=1, \cdots, n).$$

Using this, we have from (9) for $q>\max(2, n)$ and $r\geqq r_0$

$$(q-2)T_\lambda^*(r, f)<O(\log r)+\sum_{i=n+1}^{q}N_\lambda^*(r, a_i)+S_\lambda^*(r, f)$$

and so by (10) and Proposition 7-i),

$$\sum_{i=n+1}^{q}\delta_\lambda^*(a_i, f)\leqq 2-n.$$

This implies $N(P_1)\leqq 2$ and

$$\sum_{a\notin P_1}\delta_\lambda^*(a, f)\leqq 2-N(P_1).$$

COROLLARY 8. $$N(G_1)\leqq N(P_1)\leqq 2.$$

COROLLARY 9. $$\sum_{a\notin P_1}\delta(a, f)\leqq 2-N(P_1).$$

Becanse $\delta(a, f) \leqq \delta_{\lambda}^{*}(a, f)$ for any λ.

REMARK. This is an extension of Corollary 1 ([6]) for the case of infinite order.

§ 4. Examples.

In this section, we give several examples which explain relations among exceptional values used in § 2 and § 3. First we note that Valiron [7] gave an example with G-exceptional values which are not Nevanlinna exceptional values.

EXAMPLE 1. *Example of a G-exceptional value which is not Borel exceptional* (cf. § 2).

Let

$$f(z)=\prod_{n=2}^{\infty}\left(1+\frac{z}{n(\log n)^2}\right).$$

Then,

$$T(r, f) \sim \frac{r}{\log r}, \qquad n(r, 1/f) \sim \frac{r}{(\log r)^2}$$

(see [2], p. 29). (Here, $p(r) \sim q(r)$ means $\lim_{r\to\infty} p(r)/q(r)=1$.) Therefore, $\rho=1$ and $f(z)$ has divergence class of order 1 as

$$T_1(r) \sim \log\log r.$$

The order of $N(r, 1/f)$, which is equal to that of $n(r, 1/f)$, is one.

On the other hand, in this case

$$\int_1^{\infty} \frac{N(t, 0)}{t^2} dt = \int_1^{\infty} \frac{n(t, 0)}{t^2} dt = O(1)$$

because $\int_2^{\infty} 1/r(\log r)^2 dr < \infty$.

That is, the value 0 is G-exceptional but not Borel exceptional for $f(z)$.

EXAMPLE 2. *Example of $f(z)$ such that $\delta(0, f)=1$ but the value 0 is not G-exceptional* (cf. § 1).

For any positive integer q, let

$$f(z)=\prod_{n=2}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{q}\left(\frac{z}{a_n}\right)^q\right)$$

with

$$a_n=-[n(\log n)^{\alpha}]^{1/\rho},$$

where $\rho=q$ and $0<\alpha<1$. Then,

$$T(r, f) \sim \frac{r^{\rho}(\log r)^{1-\alpha}}{\pi\rho^{\alpha+1}(1-\alpha)}, \qquad N(r, 0) \sim \frac{r^{\rho}(\log r)^{-\alpha}}{\rho^{\alpha+1}} \tag{11}$$

(see [4], p. 18-19).

Since from (11)

$$T_\rho(r, f) \sim (\log r)^{2-\alpha}/(2-\alpha)(1-\alpha)\pi\rho^{\alpha+1},$$

$f(z)$ has divergence class of order ρ.

As

$$N_\rho(r, 0) \sim (\log r)^{1-\alpha}/(1-\alpha)\rho^{\alpha+1},$$

the value 0 is not G-exceptional, but from (11)

$$\delta(0, f) = 1.$$

EXAMPLE 3. *Example of a G_1-exceptional value which is not G-exceptional* (cf. Remark of Definition 7).

Let $f(z)$ be the same as in Example 2. Then the value 0 is not G-exceptional by Example 2. We show that it is G_1-exceptional.

Put

$$A(r) = r^\rho(\log r)^{1-\alpha}, \qquad B(r) = r^\rho(\log r)^{-\alpha}.$$

Then

$$A_0(r) = \int_1^r A(t)/t dt \geqq r^\rho/\rho - O(1)$$

and

$$A_{0,\rho}(r) = \int_1^r A_0(t)/t^{1+\rho} dt \geqq (\log r)/\rho - O(1).$$

Further

$$B_0(r) = \int_{e^2}^r B(t)/t dt \leqq 2r^\rho(\log r)^{-\alpha}/\rho$$

and

$$B_\rho^*(r) = \int_{e^2}^r \frac{B_0(t)}{t^{1+\rho}A_{0,\rho}(t)} dt \leqq K\int_{e^2}^r \frac{1}{t(\log t)^{1+\alpha}} dt = O(1)$$

where K is a positive constant. Applying these estimates, we have from (11) that the value 0 is G_1-exceptional.

EXAMPLE 4. *Example of a G_1-exceptional value which is not Borel exceptional* (case of infinite order) (cf. Remark of Definition 9).

Let

$$f(z) = (\exp e^{z/p} - 1) \exp e^z$$

for $1 < p < \infty$. Put

$$g_1(z) = \exp e^{z/p} - 1 \quad \text{and} \quad g_2(z) = \exp e^z.$$

Then

$$T(r, f) \sim e^r/(2\pi^3 r)^{1/2} \qquad (r \to \infty)$$

because

$$T(r, g_2) \sim e^r/(2\pi^3 r)^{1/2} \quad \text{and} \quad T(r, g_1) = o(T(r, g_2)) \qquad (r \to \infty)$$

(see [2], p. 7 and p. 19-20). Therefore, $\rho=\infty$.

On the other hand, 0 and ∞ are Picard exceptional, and so Borel exceptional for $\exp e^{z/p}$. As there are at most two Borel exceptional values for meromorphic functions of infinite order (see [3] or [6]), the value 1 is not Borel exceptional for $\exp e^{z/p}$. This shows that the order of $N(r, 1/g_1)$ is infinite because that of $g_1(z)$ is infinite. Therefore, the value 0 is not Borel exceptional for $f(z)$. As, for a positive constant K and $r\geqq r_0(K)$

$$N(r, 1/f)=N(r, 1/g_1)\leqq T(r, g_1)+O(1)\leqq Ke^{r/p},$$

it holds that

$$N_0(r, 0)\leqq Ke^{r/p}\log r$$

for $r\geqq r_0$. Further, for a positive constant K_1 and $r\geqq r_1(K_1)$

$$T_0(r, f)\geqq K_1e^r/r^{3/2} \quad \text{and so} \quad r^{1+\lambda}T_{0,\lambda}(r, f)\geqq K_1e^r/r^{1/2},$$

so that for any non-negative λ

$$N_\lambda^*(r, 0)\leqq\frac{K}{K_1}\int_1^r\frac{t^{1/2}e^{t/p}\log t}{e^t}dt+O(1)$$
$$=O(1)$$

as $p>1$. This shows that for $f(z)$ the value 0 is G_1-exceptional but not being Borel exceptional.

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