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On the asymptotic behavior of resolvent kernels for elliptic operators

By Michihiro NAGASE

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§1. Introduction.

Let $A=A(X, D_x)$ be a formally self-adjoint elliptic differential operator of order m which is defined on a bounded domain Ω in the n-dimensional real space R^n and $\tilde{A}$ be a self-adjoint realization of A in $L^2(\Omega)$ with domain in $H_m(\Omega)$, which is bounded from below. The asymptotic distribution of eigenvalues $\{\lambda_j\}$ of $\tilde{A}$ has been studied in many papers.

S. Agmon showed the asymptotic formula of the form

$$N(t)=\sum_{\lambda j<t} 1=c\, t^{n/m}+O(t^{(n-\sigma)/m}) \quad \cdots\cdots(1)$$

where σ is an arbitrary number less than 1 if the principal part of A has constant coefficients and less than 1/2 if the principal part has variable coefficients (see [3]).

The estimate (1) was shown by using the asymptotic estimate for spectral functions $e(t; x, x)$ of $\tilde{A}$, which was derived from the asymptotic estimate of resolvent kernels for the operator defined on R^n (see [3]).

The purpose of the present paper is to give another simple proof of the asymptotic formula (see Theorem in section 4) of resolvent kernels for the operator of order $m>n$ which is defined on R^n.

For the proof we shall use the parametrix of an elliptic differential operator, which was used by L. Hörmander [5] to obtain the asymptotic behavior of spectral functions and its Riesz mean.

Using the method of the present paper we can obtain similar results to those of [7] for the case of semi-elliptic operators.

§2. Notation and lemmas.

We use following notation;

$$D_x=(D_{x1}, \cdots, D_{xn}), \quad D_{xj}=-i\partial_{xj}=-i(\partial/\partial xj), \qquad j=1, \cdots, n,$$

$$\alpha=(\alpha_1, \cdots, \alpha_n),\ \alpha_j;\ \text{non negative integer for } j=1, \cdots, n,$$

$$|\alpha|=\alpha_1+\cdots+\alpha_n\,, \qquad D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}\,,$$

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n} \text{ and } |x|=(x_1^2+\cdots+x_n^2)^{1/2} \qquad \text{for } x\in R^n\,.$$

We consider the differential operator

$$A(X, D_x)=\sum_{j=0}^{m}\mathcal{A}_j(X, D_x)=\sum_{j=0}^{m}\sum_{|\alpha|=j}\mathcal{A}_\alpha(x)D_x^\alpha$$

of order $m>n$ with coefficients of class $\mathcal{B}^\infty(R_x^n)$, where $\mathcal{B}^\infty(R_x^n)=\{f(x)\in C^\infty(R_x^n);\ |D_x^\alpha f(x)|\leqq C_\alpha$ for any $\alpha\}$.

In what follows we assume that the principal part $\mathcal{A}_m(x, \xi)$ of $A(X, D_x)$ is real valued for $\xi\in R^n$ and uniformly elliptic, that is, there is a constant C_0 such that

$$\mathcal{A}_m(x, \xi)=\sum_{|\alpha|=m}\mathcal{A}_\alpha(x)\xi^\alpha\geqq C_0|\xi|^m \qquad (C_0>0)\,. \tag{2.1}$$

For a complex number λ we denote the distance from λ to positive real axis $(0, \infty)$ by $d(\lambda)$, that is,

$$d(\lambda)=\begin{cases}|\mathcal{I}m\lambda| & \text{for } \mathrm{Re}\,\lambda\geqq 0\\ |\lambda| & \text{for } \mathrm{Re}\,\lambda<0\,.\end{cases} \tag{2.2}$$

LEMMA 2.1. *Let δ be a positive number less than* 1. *Under assumptions that $d(\lambda)\geqq C_0|\lambda|^{1-\delta}$ and $|\lambda|\geqq C_1$ $(C_0, C_1>0)$, there exists a positive constant C_2 such that*

$$|\mathcal{A}_m(x, \xi)-\lambda|\geqq C_2(|\xi|^m+|\lambda|)^{1-\delta}\,. \tag{2.3}$$

PROOF. We put $\lambda=\mu+i\nu$. When $\mu\leqq 0$, we have

$$\begin{aligned}|\mathcal{A}_m(x, \xi)-\lambda| &\geqq (1/2)(\mathcal{A}_m(x, \xi)-\mu+|\nu|)\\ &\geqq (1/2)(C_0|\xi|^m-\mu+|\nu|)\\ &\geqq (1/2)(C_0|\xi|^m+|\lambda|)\\ &\geqq C(|\xi|^m+|\lambda|)^{1-\delta}\,.\end{aligned}$$

When $0<\mu<(1/2)C_0|\xi|^m$, we have

$$\begin{aligned}|\mathcal{A}_m(x, \xi)-\lambda| &\geqq (1/2)(\mathcal{A}_m(x, \xi)-\mu+|\nu|)\\ &\geqq (1/2)(C_0|\xi|^m-\mu+|\nu|)\\ &\geqq (1/2)((1/2)C_0|\xi|^m+|\nu|)\\ &\geqq (1/2)((1/4)C_0|\xi|^m+(1/2)\mu+|\nu|)\\ &\geqq C_2'(|\xi|^m+|\lambda|)\\ &\geqq C_2''(|\xi|^m+|\lambda|)^{1-\delta}\,.\end{aligned}$$

When $\mu \geqq (1/2)C_0|\xi|^m$, we have $|\nu| = d(\lambda) \geqq C_0|\lambda|^{1-\delta}$, so

$$|\mathcal{A}_m(x, \xi)-\lambda| \geqq |\nu| \geqq (1/2)C_0|\lambda|^{1-\delta}+(1/2)|\nu|$$
$$\geqq (1/2)C_0\mu^{1-\delta}+(1/2)C_0|\lambda|^{1-\delta}$$
$$\geqq C_2'''(|\xi|^m+|\lambda|)^{1-\delta}. \qquad \text{Q.E.D.}$$

For any function $p(x, \xi) \in \mathcal{S}^s_{1,0}$, where $\mathcal{S}^s_{1,0} = \{p(x, \xi) \in C^\infty(R^n_x \times R^n_\xi)$; $|D^\beta_x \partial^\alpha_\xi p(x, \xi)| \leqq C_{\alpha,\beta}(1+|\xi|)^{s-|\alpha|}$ for any $\alpha, \beta\}$, we define the operator $p(X, D_x)$ by

$$p(X, D_x)u(x) = \int_{R^n} e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi) d\!\!\!{}^-\xi \qquad \text{for } u \in \mathcal{S}$$

where $d\!\!\!{}^-\xi = (2\pi)^{-n}d\xi$, $\hat{u}(\xi) = \int_{R^n} e^{-ix\cdot\xi}u(x)dx$ and $\mathcal{S} = \{u(x) \in C^\infty(R^n_x)$; $\lim_{|x|\to\infty} |x|^k |D^\alpha_x u(x)| = 0$ for any real number k and multi-integer $\alpha\}$.

The operator is called a pseudo-differential operator with its symbol $p(x, \xi)$, and we denote $\sigma(p(X, D_x)) = p(x, \xi)$.

For any symbol $p(x, \xi)$ we denote $p^{(\alpha)}_{(\beta)}(x, \xi) = D^\beta_x \partial^\alpha_\xi p(x, \xi)$, $p^{(\alpha)}(x, \xi) = \partial^\alpha_\xi p(x, \xi)$ and $p_{(\beta)}(x, \xi) = D^\beta_x p(x, \xi)$ (see [8]). In the followings we denote $\int_{R^n}$ by $\int$.

§ 3. Parametrix of the operator $A(X, D_x)-\lambda$.

For the brevity of notations we use

$$(3.1) \qquad \begin{cases} p_m(\lambda, x, \xi) = \mathcal{A}_m(x, \xi)-\lambda \\ p_j(\lambda, x, \xi) = \mathcal{A}_j(x, \xi), \qquad j = 0, 1, \cdots, m-1. \end{cases}$$

We determine the series $\{q_k(\lambda, x, \xi)\}_{k=0}^\infty$ of symbols as follows.

Using the expansion formula for symbols of pseudo-differential operators (see [6], [8]), we obtain formally that

$$\sigma(\{A(X, D_x)-\lambda\} \cdot \sum_{k=0}^\infty q_k(\lambda, X, D_x))$$
$$\sim \sigma(\sum_{j=0}^m \sum_{k=0}^\infty p_{m-j}(\lambda, X, D_x) \cdot q_k(\lambda, X, D_x))$$
$$\sim \sum_{j=0}^m \sum_{k=0}^\infty \sum_\alpha \frac{1}{\alpha!} p^{(\alpha)}_{m-j}(\lambda, x, \xi) q_{k(\alpha)}(\lambda, x, \xi)$$
$$\sim \sum_{l=0}^\infty \sum_{\substack{k+j+|\alpha|=l \\ 0\leqq j\leqq m}} \frac{1}{\alpha!} p^{(\alpha)}_{m-j}(\lambda, x, \xi) q_{k(\alpha)}(\lambda, x, \xi).$$

So we define the symbols $q_k(\lambda, x, \xi)$, $k = 0, 1, \cdots$, as follows;

$$(3.2) \qquad p_m(\lambda, x, \xi) q_0(\lambda, x, \xi) = 1,$$

(3.3) $$p_m(\lambda, x, \xi)q_l(\lambda, x, \xi)+\sum_{\substack{k+j+|\alpha|=l\\ k\neq l}}\frac{1}{\alpha!}p_{m-j}^{(\alpha)}(\lambda, x, \xi)q_{k(\alpha)}(\lambda, x, \xi)=0, \quad \text{for } l=1, 2, \cdots.$$

Then we have following propositions.

PROPOSITION 3.1. *The symbols* $\{q_k(\lambda, x, \xi)\}_{k=0}^{\infty}$ *have following properties;*

(3.4) $$q_k(t^m\lambda, x, t\xi)=t^{-m-k}q_k(\lambda, x, \xi) \quad \textit{for any } t>0,$$

(3.5) $$q_0(\lambda, x, \xi)=\frac{1}{p_m(\lambda, x, \xi)}=\frac{1}{\mathcal{A}_m(x, \xi)-\lambda},$$

(3.6) $$q_k(\lambda, x, \xi)=\sum_{j=1}^{2k}\frac{p_{k,j}(x, \xi)}{p_m(\lambda, x, \xi)^{j+1}}, \quad k=1, 2, \cdots$$

where $p_{k,j}(x, \xi)$ *are homogeneous polynomials in* ξ *of degree* $mj-k$ *with coefficients of class* $\mathcal{B}^\infty(R_x^n)$ *for* $mj-k\geqq 0$ *and* $p_{k,j}(x, \xi)=0$ *for* $mj-k<0$. *In particular if* $p_m(\lambda, x, \xi)$ *is independent of* x, *that is, the principal part of* $A(X, D_x)$ *has constant coefficients, then*

(3.7) $$q_k(\lambda, x, \xi)=\sum_{j=1}^{k}\frac{p_{k,j}(x, \xi)}{p_m(\lambda, x, \xi)^{j+1}}, \quad k=1, 2, \cdots$$

where $p_{k,j}(x, \xi)$ *have the same property as the case of variable coefficients.*

PROOF. The equality (3.5) is clear from (3.2), and (3.4) follows from (3.5) and (3.6). So we show the equality (3.6) by induction in k.

We note that the equality (3.3) can be rewritten in the form

(3.3)′ $$q_l(\lambda, x, \xi)p_m(\lambda, x, \xi)+\sum_{\substack{k+j+|\alpha|=l\\ k\neq l}}\frac{1}{\alpha!}\mathcal{A}_{m-j}^{(\alpha)}(x, \xi)q_{k(\alpha)}(\lambda, x, \xi)=0.$$

For $l=1$ we have

(3.3)″ $$q_1(\lambda, x, \xi)p_m(\lambda, x, \xi)+\mathcal{A}_{m-1}(x, \xi)q_0(\lambda, x, \xi) +\sum_{|\alpha|=1}\mathcal{A}_m^{(\alpha)}(x, \xi)q_{0(\alpha)}(\lambda, x, \xi)=0.$$

When $|\alpha|+|\beta|\neq 0$, we have by induction in $|\alpha|+|\beta|$

(3.8) $$q_{0(\beta)}^{(\alpha)}(\lambda, x, \xi)=\sum_{j=1}^{|\alpha|+|\beta|}\frac{p_{0,j,\beta}^{\alpha}(x, \xi)}{p_m(\lambda, x, \xi)^{j+1}}$$

where $p_{0,j,\beta}^{\alpha}(x, \xi)$ are homogeneous polynomials in ξ of order $mj-|\alpha|$ $(\geqq 0)$ with coefficients of class $\mathcal{B}^\infty(R_x^n)$.

The equality (3.6) for $k=1$ follows from (3.3)″ and (3.8).

We assume that (3.6) is true for any $k\leqq l$. Then we have by induction in $|\alpha|+|\beta|$, for $1\leqq k\leqq l$

(3.9) $$q_{k(\beta)}^{(\alpha)}(\lambda, x, \xi)=\sum_{j=2}^{2k+|\alpha|+|\beta|}\frac{p_{k,j,\beta}^{\alpha}(x, \xi)}{p_m(\lambda, x, \xi)^{j+1}}$$

where $p_{k,j,\beta}^{\alpha}(x, \xi)$ are homogeneous polynomials in ξ of degree $mj-k-|\alpha|$ with coefficients of class $\mathcal{B}^{\infty}(R_x^n)$.

From (3.8), (3.9) and (3.3)$'$, we have

$$q_{l+1}(\lambda, x, \xi)p_m(\lambda, x, \xi)$$

$$=-\Big\{\sum_{\substack{k+j+|\alpha|=l+1,\\ k\neq l+1\\ k\neq 0}}\sum_{i=2}^{2k+|\alpha|}\frac{1}{\alpha!}\mathcal{A}_{m-j}^{(\alpha)}(x, \xi)\frac{p_{k,i,\alpha}(x, \xi)}{p_m(\lambda, x, \xi)^{i+1}}$$

$$+\sum_{\substack{j+|\alpha|=l+1\\ |\alpha|\neq 0}}\sum_{i=0}^{|\alpha|}\frac{1}{\alpha!}\mathcal{A}_{m-j}^{(\alpha)}(x, \xi)\frac{p_{0,i,\alpha}(x, \xi)}{p_m(\lambda, x, \xi)^{i+1}}$$

$$+\mathcal{A}_{m-(l+1)}(x, \xi)q_0(\lambda, x, \xi)\Big\}.$$

In the above equality the second and third terms are equal to zero if $m\leqq l+1$. Thus we obtain (3.6) for $l+1$.

When $p_m(\lambda, x, \xi)$ is independent of x, we have in place of (3.8) and (3.9),

$$q_{0(\beta)}^{(\alpha)}(\lambda, x, \xi)=\sum_{j=1}^{|\alpha|}\frac{p_{0,j,\beta}^{\alpha}(x, \xi)}{p_m(\lambda, x, \xi)^{j+1}} \tag{3.8$'$}$$

for $|\alpha|\neq 0$, and if $|\beta|\neq 0$ then the right hand side of (3.8)$'$ is equal to zero,

$$q_{k(\beta)}^{(\alpha)}(\lambda, x, \xi)=\sum_{j=1}^{k+|\alpha|}\frac{p_{k,j,\beta}^{\alpha}(x, \xi)}{p_m(\lambda, x, \xi)^{j+1}}, \qquad k=1, 2, \cdots \tag{3.9$'$}$$

where $p_{k,j,\beta}^{\alpha}(x, \xi)$ are the same as in (3.8) and (3.9).

Using (3.8)$'$ and (3.9)$'$ we get (3.7) for the case of constant coefficients.

Q. E. D.

PROPOSITION 3.2. *When $d(\lambda)\geqq C_0|\lambda|^{1-\delta}$ and $|\lambda|\geqq C_1$, we have*

$$|q_{k(\beta)}^{(\alpha)}(\lambda, x, \xi)|\leqq C_{k,\alpha,\beta}(|\xi|^m+|\lambda|)^{\delta(2k+|\alpha|+|\beta|)-(1/m)(k+|\alpha|)}. \tag{3.10}$$

In particular if $p_m(\lambda, x, \xi)$ is independent of x, we have

$$|q_{k(\beta)}^{(\alpha)}(\lambda, x, \xi)|\leqq C_{k,\alpha,\beta}(|\xi|^m+|\lambda|)^{\delta(k+|\alpha|)-(1/m)(k+|\alpha|)}. \tag{3.11}$$

PROOF. By Lemma 2.1 and (3.9) we have

$$|q_{k(\beta)}^{(\alpha)}(\lambda, x, \xi)|\leqq\sum_{j=2}^{2k+|\alpha|+|\beta|}\frac{|p_{k,j,\beta}^{\alpha}(x, \xi)|}{|p_m(\lambda, x, \xi)^{j+1}|}$$

$$\leqq C'_{k,\alpha,\beta}\sum_{\substack{j=2\\ mj-k-|\alpha|\geqq 0}}^{2k+|\alpha|+|\beta|}|\xi|^{mj-k-|\alpha|}(|\xi|^m+|\lambda|)^{-(1-\delta)(j+1)}$$

$$\leqq C''_{k,\alpha,\beta}\sum_{j=2}^{2k+|\alpha|+|\beta|}(|\xi|^m+|\lambda|)^{\delta j-(1/m)(k+|\alpha|)}$$

$$\leqq C_{k,\alpha,\beta}(|\xi|^m+|\lambda|)^{\delta(2k+|\alpha|+|\beta|)-(1/m)(k+|\alpha|)}.$$

When $p_m(\lambda, x, \xi)$ has constant coefficients, using Lemma 2.1 and (3.9)′ we obtain,

$$|q_{k(\beta)}^{(\alpha)}(\lambda, x, \xi)| \leqq C'_{k,\alpha,\beta} \sum_{j=2}^{k+|\alpha|} (|\xi|^m + |\lambda|)^{\delta j-(1/m)(k+|\alpha|)}$$
$$\leqq C_{k,\alpha,\beta}(|\xi|^m + |\lambda|)^{\delta(k+|\alpha|)-(1/m)(k+|\alpha|)}.$$

Q. E. D.

REMARK. Considering λ as a parameter, we have $q_k(\lambda, x, \xi) \in S_{1,0}^{-m-k}$ for $d(\lambda) \neq 0$, that is,

$$|q_{k(\beta)}^{(\alpha)}(\lambda, x, \xi)| \leqq C_{k,\alpha,\beta,\lambda}(|\xi|^2+1)^{-(1/2)(m+k+|\alpha|)}.$$

For any natural number N we put

$$Q_N(\lambda, x, \xi) = \sum_{k=0}^{N} q_k(\lambda, x, \xi). \tag{3.12}$$

PROPOSITION 3.3. *It holds that*

$$\{A(X, D_x)-\lambda\} \cdot Q_N(\lambda, X, D_x)u(x) = u(x) + R_N(\lambda, X, D_x)u(x) \tag{3.13}$$

for any $u \in \mathcal{S}$ *where*

$$R_N(\lambda, x, \xi) = \sum_{\substack{k+j+|\alpha| \geqq N+1 \\ k \leqq N}} \frac{1}{\alpha!} p_{m-j}^{(\alpha)}(\lambda, x, \xi) q_{k_{(\alpha)}}(\lambda, x, \xi). \tag{3.14}$$

PROOF. Using the expansion formula for symbols and equalities (3.2) and (3.3) we have

$$\sigma(\{A(X, D_x)-\lambda\} \cdot Q_N(\lambda, X, D_x))$$
$$= \sum_{j=0}^{m} \cdot \sum_{k=0}^{N} \sigma(p_{m-j}(\lambda, X, D_x) \cdot q_k(\lambda, X, D_x))$$
$$= \sum_{l=0}^{N} \sum_{j+k+|\alpha|=l} \frac{1}{\alpha!} p_{m-j}^{(\alpha)}(\lambda, x, \xi) q_{k_{(\alpha)}}(\lambda, x, \xi)$$
$$+ \sum_{\substack{j+k+|\alpha| \geqq N+1 \\ k \leqq N}} \frac{1}{\alpha!} p_{m-j}^{(\alpha)}(\lambda, x, \xi) q_{k_{(\alpha)}}(\lambda, x, \xi)$$
$$= 1 + R_N(\lambda, x, \xi).$$

Thus, we obtain the proposition. Q. E. D.

PROPOSITION 3.4. *When* $d(\lambda) \geqq C_0|\lambda|^{1-\delta}$ *and* $|\lambda| > C_1$, *it holds that*

$$|R_{N(\beta)}^{(\alpha)}(\lambda, x, \xi)| \leqq C_{N,\alpha,\beta}(|\xi|^m + |\lambda|)^{\theta_1} \tag{3.15}$$

where $\theta_1 = \delta(2N+|\alpha|+|\beta|+m) - (1/m)(N+|\alpha|) + 2$.

In particular if $p_m(\lambda, x, \xi)$ *has constant coefficients, we have*

$$|R_{N(\beta)}^{(\alpha)}(\lambda, x, \xi)| \leqq C_{N,\alpha,\beta}(|\xi|^m + |\lambda|)^{\theta_2} \tag{3.16}$$

where $\theta_2 = \delta(N+|\alpha|) - (1/m)(N+|\alpha|) + 2$.

PROOF. By the definition of $R_N(\lambda, x, \xi)$, we have

$$R_{N(\beta)}^{(\alpha)}(\lambda, x, \xi)=\sum_{\substack{k+j+|\gamma|\geqq N+1\\ k\leqq N\\ |\alpha_1+\gamma|+j\leqq m}}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}}\frac{1}{\gamma!}C_{\alpha_1,\alpha_2,\beta_1,\beta_2}$$

$$p_{m-j}{}_{(\beta_1)}^{(\gamma+\alpha_1)}(\lambda, x, \xi)q_{k}{}_{(\gamma+\beta_2)}^{(\alpha_2)}(\lambda, x, \xi).$$

Hence, using Proposition 3.2 we obtain

$$\begin{aligned}|R_{N}{}_{(\beta)}^{(\alpha)}(\lambda, x, \xi)| &\leqq C'_{N,\alpha,\beta}\sum|p_{m-j}{}_{(\beta_1)}^{(\gamma+\alpha_1)}(\lambda, x, \xi)|\times|q_{k}{}_{(\gamma+\beta_2)}^{(\alpha_2)}(\lambda, x, \xi)|\\ &\leqq C''_{N,\alpha,\beta}\sum(|\xi|^m+|\lambda|)^{(1/m)(m-j-|\gamma+\alpha_1|)+\delta(2k+|\alpha_2|+|\gamma+\beta_2|)-(1/m)(k+|\alpha_2|)}\\ &\leqq C'''_{N,\alpha,\beta}\sum(|\xi|^m+|\lambda|)^{\delta(2k+|\alpha_2|+|\gamma+\beta_2|)-(1/m)(k+|\alpha|)+(1/m)(m-j-|\gamma|)}\end{aligned}$$

where summations can be taken for $k+j+|\gamma|\geqq N+1$, $k\leqq N$, $|\alpha_1|+|\gamma|+j\leqq m$, $\alpha_1+\alpha_2=\alpha$ and $\beta_1+\beta_2=\beta$.

So we get

$$\begin{aligned}|R_{N}{}_{(\beta)}^{(\alpha)}(\lambda, x, \xi)| &\leqq C\sum_{N-m\leqq k\leqq N}(|\xi|^m+|\lambda|)^{\theta_1-1}\\ &\leqq C_{N,\alpha,\beta}(|\xi|^m+|\lambda|)^{\theta_1}.\end{aligned}$$

The inequality (3.16) can be shown by the same way using (3.11). Q.E.D.

§ 4. Asymptotic behavior of resolvent kernels.

Since the estimate

$$\|u\|_m\leqq C(\|A(X, D_x)u\|_0+\|u\|_0) \qquad \text{for } u\in\mathcal{S} \tag{4.1}$$

holds where $\|u\|_s$ is the Sobolev norm, that is,

$$\|u\|_s^2=\int(1+|\xi|^2)^s|\hat{u}(\xi)|^2d\xi,$$

considering the operator $A=A(X, D_x)$ in $L^2(R_x^n)$ with domain $C_0(R_x^n)$, A can be extended uniquely to a closed operator $\tilde{A}$ with its domain $H_m(R^n)=\{u\in S';\ \|u\|_m<\infty\}$.

Since the principal part $\mathcal{A}_m(x, \xi)$ of A is real valued, A can be written in the form $A(X, D_x)=A_0(X, D_x)+B(X, D_x)$ where $A_0(X, D_x)$ is formally self-adjoint and $B(X, D_x)$ is an operator of order $m-1$. So using the same method as in S. Agmon [1], we can prove that the resolvent set of A contains the set $\Lambda=\{d(\lambda)\geqq C_0|\lambda|^{1-1/m}, |\lambda|\geqq C_1\}$ for some C_0, $C_1>0$.

Furthermore we have the same assertion for the formal adjoint A^* of A and $\tilde{A}^*=(\tilde{A})^*$. By the closed graph theorem it holds that for $\lambda\in\Lambda$, $(\tilde{A}-\lambda)^{-1}$ and $(\tilde{A}^*-\lambda)^{-1}$ are bounded operators from $L^2(R^n)$ to $H_m(R^n)$.

Since $m>n$, by the kernel theorem, which was shown by S. Agmon [2], we get that the resolvent operator $(\tilde{A}-\lambda)^{-1}$ is an integral operator with continuous and bounded kernel $R_\lambda(x, y)$, that is,

$$R_\lambda u(x)=(\tilde{A}-\lambda)^{-1}u(x)=\int R_\lambda(x, y)u(y)dy \qquad \text{for } u\in L^2(R_x^n), \tag{4.2}$$

and

$$|R_\lambda(x, y)|\leqq C\frac{|\lambda|^{(n/m)}}{d(\lambda)} \qquad \text{for } \lambda\in\Lambda. \tag{4.3}$$

LEMMA 4.1. *For any* $u\in\mathcal{S}$ *we have*

$$R_\lambda u-Q_N(\lambda, X, D_x)u=-R_\lambda\cdot R_N(\lambda, X, D_x)u. \tag{4.4}$$

PROOF. By definitions we have

$$(\tilde{A}-\lambda)\cdot R_\lambda u=R_\lambda\cdot(A-\lambda)u=u \qquad \text{for any } u\in\mathcal{S}.$$

So by Proposition 3.3 we obtain

$$\begin{aligned} R_\lambda u-Q_N(\lambda, X, D_x)u &= R_\lambda u-R_\lambda\cdot(A-\lambda)\cdot Q_N(\lambda, X, D_x)u \\ &= R_\lambda u-R_\lambda\{u+R_N(\lambda, X, D_x)u\} \\ &= -R_\lambda\cdot R_N(\lambda, X, D_x)u. \end{aligned}$$

Q. E. D.

From the equality it can be written in the form;

$$R_\lambda(x, y)=\sum_{j=0}^{N} q'_j(\lambda, x, x-y)-\int R_\lambda(x, z)R'_N(\lambda, z, z-y)dz \tag{4.5}$$

where,

$$q'_j(\lambda, x, z)=\int e^{iz\cdot\xi}q_j(\lambda, x, \xi)d\xi, \tag{4.6}$$

$$R'_N(\lambda, x, z)=\int e^{iz\cdot\xi}R_N(\lambda, x, \xi)d\xi. \tag{4.7}$$

In fact, the integrability of right hand side of (4.6) and (4.7) is true because $q_j(\lambda, x, \xi)\in\mathcal{S}_{1,0}^{-m-j}$ and $R_N(\lambda, x, \xi)\in\mathcal{S}_{1,0}^{-m-N-1}$ and $m>n$. By (4.4) we have

$$\begin{aligned} \int R_\lambda(x, y)u(y)dy &= \sum_{j=0}^{N}\int e^{ix\cdot\xi}q_j(\lambda, x, \xi)\hat{u}(\xi)d\xi \\ &\quad -\int R_\lambda(x, z)\int e^{iz\cdot\xi}R_N(\lambda, z, \xi)\hat{u}(\xi)d\xi dz \\ &= \sum_{j=0}^{N}\int e^{ix\cdot\xi}q_j(\lambda, x, \xi)\int e^{-iy\cdot\xi}u(y)dyd\xi \\ &\quad -\int R_\lambda(x, z)\int e^{iz\cdot\xi}R_N(\lambda, z, \xi)\int e^{-iy\cdot\xi}u(y)dyd\xi dz. \end{aligned}$$

Since $u\in\mathcal{S}$, changing the order of integrals we have

$$\int R_\lambda(x, y)u(y)dy = \sum_{j=0}^{N} \iint e^{i(x-y)\cdot\xi} q_j(\lambda, x, \xi)d\xi u(y)dy$$

$$-\int R_\lambda(x, z) \iint e^{i(z-y)\cdot\xi} R_N(\lambda, z, \xi)d\xi u(y)dydz$$

$$= \sum_{j=0}^{N} \int q'_j(\lambda, x, x-y)u(y)dy$$

$$-\int R_\lambda(x, z) \int R'_N(\lambda, z, z-y)u(y)dydz .$$

So we obtain (4.5) if $R_\lambda(x, z)R'_N(\lambda, z, z-y)$ is integrable in z. We shall refer to the integrability of $R_\lambda(x, z)R'_N(\lambda, z, z-y)$ in the proof of the following theorem.

THEOREM. (i) *When* $d(\lambda) \geqq C_0|\lambda|^{1-1/(2m)+\varepsilon}$ *and* $C_1 \leqq |\lambda|$ $(0<\varepsilon<m/2)$, *we have*

$$|(-\lambda)^{1-n/m}R_\lambda(x, x) - \sum_{j=0}^{N} C_j(x)(-\lambda)^{-j/m}| \leqq C_{N,\varepsilon}|\lambda|^{-(N+1)/m} \tag{4.8}$$

for any N.

(ii) *If* $p_m(\lambda, x, \xi)$ *is independent of* x, *we have*

$$|(-\lambda)^{1-n/m}R_\lambda(x, x) - \sum_{j=0}^{N} C_j(x)(-\lambda)^{-j/m}| \leqq C_{N,\varepsilon}|\lambda|^{-(N+1)/m} \tag{4.9}$$

for $d(\lambda) \geqq C_0|\lambda|^{1-1/m+\varepsilon}$, $|\lambda| \geqq C_1$ *and* $0<\varepsilon<1/m$.

In these inequalities $C_j(x)$ are functions in $\mathcal{B}^\infty(R^n_x)$ and in particular,

$$C_0(x) = \int \frac{1}{p_m(-1, x, \xi)} d\xi = \int \frac{1}{\mathcal{A}_m(x, \xi)+1} d\xi . \tag{4.10}$$

PROOF. By (4.6) we have

$$q'_j(\lambda, x, 0) = \int q_j(\lambda, x, \xi)d\xi .$$

Hence, by the Proposition 3.1 we get

$$q'_j(\lambda, x, 0) = (-\lambda)^{n/m-1-j/m} \int q_j(-1, x, \xi)d\xi$$

$$= C_j(x)(-\lambda)^{n/m-1-j/m} .$$

For any multi-integer α we get

$$z^\alpha \cdot R'_N(\lambda, x, z) = z^\alpha \int e^{iz\cdot\xi} R_N(\lambda, x, \xi)d\xi$$

$$= (-i)^{|\alpha|} \int e^{iz\cdot\xi} R_N^{(\alpha)}(\lambda, x, \xi)d\xi .$$

Using (3.15) for $\delta = 1/2m - \varepsilon$, we obtain

$$|z^{\alpha}\cdot R'_N(\lambda, x, z)|\leqq C_\alpha\int(|\xi|^m+|\lambda|)^{(1/2m-\varepsilon)(2N+m+|\alpha|)-(1/m)(|\alpha|+N)+2}d\xi$$

$$\leqq C'_\alpha\int(|\xi|^m+|\lambda|)^{-|\alpha|/2m-(2N+m+|\alpha|)\varepsilon+3}d\xi .$$

Thus, taking a large N such that $2Nm\varepsilon-3>n$, we have

$$|z^{\alpha}\cdot R'_N(\lambda, x, z)|\leqq C'_{\alpha,N}|\lambda|^{-(2N+m+|\alpha|)\varepsilon-|\alpha|/2m+n/m+3} .$$

Hence, we obtain

$$|R'_N(\lambda, x, z)|\leqq C'_{N,k}(1+|z|)^{-k}|\lambda|^{-(2N+m)\varepsilon+4}$$

for any k and sufficiently large N.

Using (4.3) and taking $k>N$, we have

$$\left|\int R_\lambda(x, z)\cdot R'_N(\lambda, z, z-y)dz\right|\leqq C\frac{|\lambda|^{n/m}}{d(\lambda)}|\lambda|^{-(2N+m)\varepsilon+4}\int(1+|z-y|)^{-k}dz$$

$$\leqq C\frac{1}{d(\lambda)}|\lambda|^{-(2N+m)\varepsilon+5} .$$

Thus, by (4.5) we have

$$R_\lambda(x, x)=\sum_{j=0}^{N}q'_j(\lambda, x, 0)+\int R_\lambda(x, z)R'_N(\lambda, z, z-y)dz$$

$$=\sum_{j=0}^{N}C_j(x)(-\lambda)^{n/m-1-j/m}+O\Big(\frac{1}{d(\lambda)}|\lambda|^{-(2N+m)+5}\Big)$$

for $|\lambda|\geqq C_1$ and $d(\lambda)\geqq C_0|\lambda|^{1-1/2m+\varepsilon}$.

Hence, we have

$$R_\lambda(x, x)-\sum_{j=0}^{N}C_j(x)(-\lambda)^{n/m-j/m-1}$$

$$=\sum_{j=N+1}^{N+M}C_j(x)(-\lambda)^{n/m-1-j/m}+O\Big(\frac{1}{d(\lambda)}|\lambda|^{-(2N+2M)\varepsilon-m\varepsilon+5}\Big) .$$

Now taking M sufficiently large we get the estimate (4.8).

The estimate (4.9) can be shown by the same way by using estimates for the case of constant coefficients. Q. E. D.

When $A(X, D_x)$ is formally self-adjoint and defined in a bounded open domain Ω, we consider a self-adjoint realization $\tilde{A}$ with domain $D(\tilde{A})\subset H_m(\Omega)$, which is bounded from below.

S. Agmon [3] obtained the asymptotic estimate for spectral functions and the asymptotic distribution (1) of eigenvalues by using Theorem 3.5.

When the operator $A(X, D_x)$ has the form $A(X, D_x)=A_0(X, D_x)+B(X, D_x)$ where $A_0(X, D_x)$ is formally self-adjoint and $B(X, D_x)$ is of order $m-1$, S. Agmon [1] showed that

(i) the resolvent set of $\tilde{A}=\tilde{A}_0+\tilde{B}$ contains the set

$$\Lambda = \{\lambda ; d(\lambda) \geqq C_0 |\lambda|^{1-1/m}, \ |\lambda| > C_1\} \qquad \text{for some } C_0, C_1 > 0$$

where $\tilde{A}_0$ is a self-adjoint realization of A_0 with domain $D(\tilde{A}_0) \subset H_m(\Omega)$, and $\tilde{B}$ is an operator with domain $D(\tilde{B}) \subset H_{m-1}(\Omega)$ and satisfies

$$\|\tilde{B}u\|_{0,\Omega} \leqq C\|u\|_{m-1,\Omega} \qquad \text{for } u \in D(\tilde{B}),$$

(ii) A has discrete eigenvalues $\{\lambda_j\}_{j=1}$,

(iii)

$$N(t) = \sum_{\mathrm{Re}\,\lambda_j < t} 1 = C_0 t^{n/m} + o(t^{n/m}).$$

Using the Theorem and the method in S. Agmon [1] and [3], we have

$$N(t) = \sum_{\mathrm{Re}\,\lambda_j < t} 1 = C_0 t^{n/m} + O(t^{(n-\sigma)/m})$$

where σ is the same as in section 1 (see Maruo [9]).

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Michihiro NAGASE
Department of Mathematics
Yamaguchi University
Yoshida, Yamaguchi-shi
Japan