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An addition formula for Kodaira dimensions of analytic fibre bundles whose fibres are Moišezon manifolds

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§ 0. Introduction.

Let K_M be the canonical line bundle of a compact complex manifold M. If $\dim H^0(M, \mathcal{O}(K_M^{\otimes m}))=N+1\geqq 2$ we have a meromorphic mapping $\Phi_{mK}: M\to \boldsymbol{P}^N$ of M into $\boldsymbol{P}^N$. When m is a positive integer the meromorphic mapping Φ_{mK} is called pluricanonical mapping. In this case the *Kodaira dimension* $\kappa(M)$ of M is, by definition

$$\kappa(M)=\max_{m\in L} \dim \Phi_{mK}(M)\ ,$$

where $L=\{m\in \boldsymbol{N} \mid \dim H^0(M, \mathcal{O}(K_M^{\otimes m}))\geqq 2\}$. When $H^0(M, \mathcal{O}(K_M^{\otimes m}))=0$ for all positive integers, we define the Kodaira dimension $\kappa(M)$ of M to be $-\infty$. When $\dim H^0(M, \mathcal{O}(K_M^{\otimes m}))\leqq 1$ for all positive integers m and there exists a positive integer m_0 such that $\dim H^0(M, \mathcal{O}(K_M^{\otimes m_0}))=1$, we define $\kappa(M)=0$. As for the fundamental properties of Kodaira dimension, see [**3**].

By a Moišezon manifold V we mean an n-dimensional compact complex manifold that has n algebraically independent meromorphic functions.

The main purpose of the present paper is to prove the following

MAIN THEOREM. *Let $\pi: M\to S$ be a fibre bundle over a compact complex manifold S whose fibre and structure group are a Moišezon manifold V and the group* Aut(V) *of analytic automorphisms of V respectively. Then we have an equality*

$$\kappa(M)=\kappa(V)+\kappa(S)\ .$$

To prove Main Theorem we need to analyze the action of Aut(V) on the vector space $H^0(V, \mathcal{O}(K_V^{\otimes m}))$. More generally the group Bim(V) of all bimeromorphic mappings of V acts on $H^0(V, \mathcal{O}(K_V^{\otimes m}))$ for any positive integer m. Hence we have a representation $\rho_m: \mathrm{Bim}(V)\to GL(H^0(V, \mathcal{O}(K_V^{\otimes m})))$. We call this representation pluricanonical representation. A group G is called periodic if each element g of G is of finite order. In § 1 we shall prove the following

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THEOREM 1. *Let V be a Moišezon manifold.*

1) $\rho_m(\mathrm{Bim}(V))$ *is a periodic subgroup of* $GL(H^0(V, \mathcal{O}(K_V^{\otimes m})))$ *for every positive integer* m.

2) *The representation* ρ_m *is equivalent to a unitary representation.*

3) *When* $\Phi_{mK}(V)$ *is not a ruled variety,* $\rho_m(\mathrm{Bim}(V))$, *hence a fortiori* $\rho_m(\mathrm{Aut}(V))$ *is a finite group.*

In § 2 we shall prove Main Theorem.

Main Theorem was first conjectured by S. Iitaka. He proved the theorem when the fibre V is an abelian variety. He also gave counter examples of the above two theorems, when we only assume that the manifold V is a compact complex manifold. (See Remark 1, Remark 4 below.)

§ 1. Pluricanonical representations and Proof of Theorem 1.

Let K_V be the canonical line bundle of an n-dimensional compact complex manifold V. For any positive integer m we can consider an element φ of $H^0(V, \mathcal{O}(K_V^{\otimes m}))$ as a holomorphic m-tuple differential n-form. That is, in a coordinate neighborhood $\mathcal{U}$ of V with a system of local coordinates $(z_1, \cdots, z_n)$, φ is expressed in the form

$$\varphi = f(z_1, \cdots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^m$$

where $f(z_1, \cdots, z_n)$ is holomorphic in $\mathcal{U}$.

Let $g: W \to V$ be a generically surjective meromorphic mapping of a compact complex manifold W into a compact complex manifold V of the same dimension n. Then for any element φ of $H^0(V, \mathcal{O}(K_V^{\otimes m}))$ we can define the pull back $g^*(\varphi)$ of φ as an m-tuple n-form. Since the point set where g is not holomorphic is of at least codimension 2, by Hartog's theorem, $g^*(\varphi)$ is a holomorphic m-tuple n-form on W and defines an element of $H^0(W, \mathcal{O}(K_W^{\otimes m}))$. Moreover, we can define for a meromorphic mapping g the homomorphism of the free parts of the cohomology groups

$$g_k^*: \ H^k(V, \boldsymbol{Z})_0 \longrightarrow H^K(W, \boldsymbol{Z})_0$$

as follows; since g is defined by an analytic subvariety (graph of g) of $W \times V$, we take a nonsingular model W^* of it with canonical projections f and h,

$$\begin{array}{ccc} & W^* & \\ {}^{f}\swarrow & & \searrow^{h} \\ W & \xrightarrow{\quad g \quad} & V \end{array}$$

and consider a homomorphism $f_*^{2n-k}: H_{2n-k}(W^*) \to H_{2n-k}(W)$. We define $f_k^*: H^k(W^*)_0 \to H^k(W)_0$ to be the dual of the image by f_*^{2n-k} of Poincaré dual and also define $g_k^* = h_k^* \cdot f_k^*$. It is easy to check that the definition of g_k^* is independent of the choice of W^*. By this homomorphism we obtain the homomorphism

$$g_k^*: \ H^k(V, \boldsymbol{C}) \longrightarrow H^k(W, \boldsymbol{C}).$$

We can regard $H^0(V, \mathcal{O}(K_V))$ and $H^0(W, \mathcal{O}(K_W))$ as subspaces of $H^n(V, \boldsymbol{C})$ and $H^n(W, \boldsymbol{C})$, respectively. Then for any element φ of $H^0(V, \mathcal{O}(K_V))$ we have

$$g^*(\varphi) = g_n^*(\varphi).$$

PROPOSITION 1. *Let g be a bimeromorphic mapping of an n-dimensional compact complex manifold V. If we have*

$$g^*(\varphi) = \alpha\varphi, \qquad \alpha \in \boldsymbol{C},$$

for some non zero element φ of $H^0(V, \mathcal{O}(K_V^{\otimes m}))$, then α is an algebraic integer. Moreover the degree $[\boldsymbol{Q}(\alpha): \boldsymbol{Q}]$ of the algebraic extension $\boldsymbol{Q}(\alpha)$ over $\boldsymbol{Q}$ is bounded above by the constant $N(\varphi)$, which depends on φ but does not depend on the bimeromorphic mapping g.

PROOF. Case 1. $m=1$. φ is a holomorphic n-form. Since we have

$$g^*(\varphi) = g_n^*(\varphi),$$

α is an eigenvalue of the automorphism g_n^* of $H^n(V, \boldsymbol{Z})_0$. Hence α is an algebraic integer. The degree of the minimal equation of α with coefficients in $\boldsymbol{Z}$ is bounded above by the n-th Betti number $b_n(V)$ of V.

Case 2. $m \geqq 2$. Let $\{\mathcal{V}_i\}_{i \in I}$ be a sufficiently fine finite open covering of V, where $\mathcal{V}_i$ is a coordinate neighborhood of V with a system of local coordinates $(z_i^1, \cdots, z_i^n)$. In terms of these local coordinates φ is expressed in the form

$$\varphi_i(z_i^1, \cdots, z_i^n)(dz_i^1 \wedge \cdots \wedge dz_i^n)^m,$$

where $\varphi_i(z_i^1, \cdots, z_i^n)$ is holomorphic in $\mathcal{V}_i$. Let $\boldsymbol{K}$ be a complex manifold which is a total space of the canonical line bundle K_V. The complex manifold $\boldsymbol{K}$ is covered by coordinate neighborhoods $\mathcal{U}_i$ with a system of coordinates $(z_i^1, \cdots, z_i^n, w_i)$. $\mathcal{U}_i$ is complex analytically isomorphic to $\mathcal{V}_i \times \boldsymbol{C}$. We shall define a subvariety V' of $\boldsymbol{K}$ by equations

$$(w_i)^m = \varphi_i(z_i^1, \cdots, z_i^n),$$

for any $i \in I$. It is easy to see that a holomorphic n-form $w_i dz_i^1 \wedge \cdots \wedge dz_i^n$ on $\mathcal{U}_i$ defines a global holomorphic n-form Ψ on $\boldsymbol{K}$.

Moreover a bimeromorphic mapping g induces a bimeromorphic mapping $g_{\boldsymbol{K}}$ of $\boldsymbol{K}$. In fact, if $g(\mathcal{V}_i) \subset \mathcal{V}_j$, then $g_{\boldsymbol{K}}|_{\mathcal{U}_i}: \mathcal{U}_i \rightarrow \mathcal{U}_j$ is expressed by the above local coordinates in the form

$$(z_i^1, \cdots, z_i^n, w_i) \longrightarrow \left(g^1(z_i), \cdots, g^n(z_i), \left(\det \frac{\partial(g^1(z_i), \cdots, g^n(z_i))}{\partial(z_i^1, \cdots, z_i^n)}\right)^{-1} w_i\right).$$

Let m_β be an analytic automorphism of $\boldsymbol{K}$ defined by

$$m_\beta:\quad (z_i^1, \cdots, z_i^n, w_i) \longrightarrow (z_i^1, \cdots, z_i^n, \beta w_i),$$

for each $i \in I$, where β is one of the m-th root of α.

Since $g^*(\varphi) = \alpha\varphi$, the bimeromorphic mapping $m_\beta \circ g_{\boldsymbol{K}}$ induces a bimeromorphic mapping of V' onto V'.

By a suitable sequence of monoidal transformations of the manifold $\boldsymbol{K}$ with non-singular centers, we can obtain a manifold $\tilde{\boldsymbol{K}}$ and the strict transform W of V', which is a non-singular model of the variety V' ([**2**]). Then the bimeromorphic mapping $m_\beta \circ g_{\boldsymbol{K}}$ of $\boldsymbol{K}$ can be extended to the bimeromorphic mapping $\tilde{h}$ of $\tilde{\boldsymbol{K}}$ which induces a bimeromorphic mapping h of W.

Let $f_1: W \to V'$ be a surjective holomorphic mapping, which is induced from the inverse mapping of the above monoidal transformations of $\boldsymbol{K}$. Let $f_2: V' \to V$ be a finite surjective holomorphic map defined by

$$f_2:\quad (z_i^1, \cdots, z_i^n, w_i) \longrightarrow (z_i^1, \cdots, z_i^n).$$

We set $f = f_2 \circ f_1$.

The holomorphic n-form Ψ can be lifted to a holomorphic n-form $\tilde{\Psi}$ on $\tilde{\boldsymbol{K}}$, which induces a holomorphic n-form ω on W. From the arguments above it is easy to see that

$$\omega^{\otimes m} = f^*(\varphi).$$

Moreover since $(m_\beta \circ g_{\boldsymbol{K}})^*(\Psi) = \beta\Psi$, it follows

$$h^*(\omega) = \beta\omega \quad \text{and} \quad \beta^m = \alpha.$$

Hence by Case 1, β is an algebraic integer and $[\boldsymbol{Q}(\beta): \boldsymbol{Q}] \leqq b_n(W)$. This implies α is an algebraic integer and $[\boldsymbol{Q}(\alpha): \boldsymbol{Q}] \leqq b_n(W)$. Since $b_n(W)$ depends only on φ and does not depend on g, we complete the proof.

PROPOSITION 2. *Let V, g, φ and α be the same as those of Proposition* 1. *Then we have $|\alpha| = 1$. Moreover when V is a Moišezon manifold α is a root of unity.*

PROOF. We use the same notations as above. By $(\varphi \wedge \bar{\varphi})^{1/m}$ we denote a differential $2n$-form on V defined over $\mathcal{V}_i$ in the form

$$(\sqrt{-1})^{-n^2} |\varphi_i(z_i^1, \cdots, z_i^n)|^{2/m} dz_i^1 \wedge \cdots \wedge dz_i^n \wedge d\bar{z}_i^1 \wedge \cdots \wedge d\bar{z}_i^n.$$

We set

$$\|\varphi\| = \left(\int_V (\varphi \wedge \bar{\varphi})^{1/m}\right)^{1/2}.$$

Then we have

$$0 < \|\varphi\|^2 = \int_V (\varphi \wedge \bar{\varphi})^{1/m} = \int_V (g^*\varphi \wedge \overline{g^*\varphi})^{1/m} = \|g^*\varphi\|^2 = |\alpha|^{2/m} \|\varphi\|^2.$$

Hence we have $|\alpha| = 1$.

Next we shall prove the latter half of the Proposition. By a theorem of Moišezon, for any Moišezon manifold there exists a non-singular projective model of it. Hence we may assume V to be projective. We fix an imbedding of V into $\boldsymbol{P}^N$ for some N and set $I(V)$ the defining ideal of V. For an automorphism σ of the complex number field and a homogeneous polynomial $f(z)=f(z_0, \cdots, z_N)$, we define $f^\sigma(z)=(f(z_0^{\sigma^{-1}}, \cdots, z_N^{\sigma^{-1}}))^\sigma$ and also define $I(V)^\sigma=\{f^\sigma; f\in I(V)\}$. Another projective manifold V^σ is defined by the ideal $I(V)^\sigma$. Then a meromorphic mapping g^σ of V^σ is defined to be $g^\sigma(z)=(g(z^{\sigma^{-1}}))^\sigma$ symbolically. Similarly for an element φ of $H^0(V, \mathcal{O}(K_V^{\otimes m}))$ we define an element φ^σ of $H^0(V^\sigma, \mathcal{O}(K_{V^\sigma}^{\otimes m}))$. Then it follows $(g^\sigma)^*\varphi^\sigma=(g^*\varphi)^\sigma$. In fact

$$(g^\sigma)^*\varphi^\sigma(z)=\varphi^\sigma(g^\sigma(z))=(\varphi((g^\sigma(z))^{\sigma^{-1}}))^\sigma=((\varphi\circ g)(z^{\sigma^{-1}}))^\sigma=(g^*\varphi)^\sigma(z)\,.$$

Hence if $g^*\varphi=\alpha\varphi$ then we have $(g^\sigma)^*\varphi^\sigma=\alpha^\sigma\varphi^\sigma$. The above argument implies $|\alpha^\sigma|=1$. Hence α is a root of unity. q. e. d.

REMARK 1. *Proposition does not hold for a compact complex manifold in general.* In fact, S. Iitaka made the following example. Let a, b, c be three roots of the equation

$$x^3+3x+1=0\,.$$

We assume a to be real. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ be six roots of the equation

$$z^6+3z^2+1=0\,,$$

such that

$$\alpha_1^2=\alpha_2^2=a\,, \qquad \beta_1^2=\beta_2^2=b\,, \qquad \gamma_1^2=\gamma_2^2=c\,.$$

We set

$$\Omega=\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 & \alpha_1^5 \\ 1 & \beta_1 & \beta_1^2 & \beta_1^3 & \beta_1^4 & \beta_1^5 \\ 1 & \beta_2 & \beta_2^2 & \beta_2^3 & \beta_2^4 & \beta_2^5 \end{pmatrix}.$$

Then there exists a three-dimensional complex torus T with a period matrix Ω. Left multiplication of the matrix $\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}$ defines a holomorphic automorphism g of the complex torus T. Then we have

$$g^*(dz_1\wedge dz_2\wedge dz_3)=\alpha dz_1\wedge dz_2\wedge dz_3$$

where $\alpha=\alpha_1\beta_1\beta_2=-\alpha_1 b$.

On the other hand the Galois group of $L=\boldsymbol{Q}(a, b, c)$ over $\boldsymbol{Q}$ is a symmetric group S_3.

Hence there exists an automorphism σ of L such that $\alpha^\sigma=\beta_1\gamma_1\gamma_2=-\beta_1 c$. Since

$$|a|>1\,, \qquad |c|=1/\sqrt{|a|}<1\,, \qquad |\beta_1|=\sqrt{|c|}$$

we have

$$|\alpha^{\sigma}|=\sqrt{|c|}^{3}<1.$$

Hence α is not a root of unity.

PROPOSITION 3. *Let V be a compact complex manifold and let ρ_m: Bim (V) $\rightarrow GL(H^0(V, \mathcal{O}(K_V^{\otimes m})))$ be a pluricanonical representation. Then for any element g of* Bim (V), *$\rho_m(g)$ is semi-simple.*

PROOF. If $\rho_m(g)$ is not semi-simple there exist two linearly independent elements φ_1, φ_2 of $H^0(V, \mathcal{O}(K_V^{\otimes m}))$ such that

$$g^*\varphi_1=\alpha\varphi_1+\varphi_2,$$

$$g^*\varphi_2=\alpha\varphi_2, \qquad |\alpha|=1,$$

where α is an algebraic integer by Proposition 1. Then

$$(g^l)^*\varphi_1=\alpha^l\varphi_1+l\alpha^{l-1}\varphi_2.$$

Since g^l is a bimeromorphic mapping of V we have

$$\|(g^l)^*\varphi_1\|=\|\varphi_1\|.$$

On the other hand we have

$$\|(g^l)^*\varphi_1\|^2=(\sqrt{-1})^{-n^2}\int|\alpha^l\varphi_{1,i}+l\alpha^{l-1}\varphi_{2,i}|^{2/m}dz_i^1\wedge\cdots\wedge dz_i^n\wedge d\bar{z}_i^1\wedge\cdots\wedge d\bar{z}_i^n$$

$$=(\sqrt{-1})^{-n^2}l^{2/m}\int\left|\frac{\varphi_{1,i}}{l}+\frac{\varphi_{2,i}}{\alpha}\right|^{2/m}dz_i^1\wedge\cdots\wedge dz_i^n\wedge d\bar{z}_i^1\wedge\cdots\wedge d\bar{z}_i^n.$$

It is easy to see that there exists a positive number A such that

$$(\sqrt{-1})^{-n^2}\int\left|\frac{\varphi_{1,i}}{l}+\frac{\varphi_{2,i}}{\alpha}\right|^{2/m}dz_i^1\wedge\cdots\wedge dz_i^n\wedge d\bar{z}_i^1\wedge\cdots\wedge d\bar{z}_i^n\geqq A$$

for any sufficiently large positive integer l. Hence

$$\lim_{l\to\infty}\|(g^l)^*\varphi_1\|^2=+\infty.$$

This contradicts the fact $\|\varphi_1\|=\|(g^l)^*\varphi_1\|$ for all l. q. e. d.

PROOF OF THEOREM 1. 1) By Proposition 2 every eigenvalue of $\rho_m(g)$ is a root of unity for any element g of Bim (V). By Proposition 3 $\rho_m(g)$ is diagonalizable. Hence $\rho_m(g)$ is of finite order.

2) Schur proved that 1) implies 2) ([**1**] § 36.11).

3) Let $\varphi_0, \varphi_1, \cdots, \varphi_N$ be a basis of $H^0(V, \mathcal{O}(K_V^{\otimes m}))$. We can assume the pluricanonical mapping Φ_{mK} is defined by this basis.

Let S be $\Phi_{mK}(V)$ and Lin (S) a subgroup of Aut (S) consisting of the elements induced by the projective transformations of the ambient space $\boldsymbol{P}(H^0(V, \mathcal{O}(K_V^{\otimes m})))$ which leave S invariant. The group Lin (S) is obviously an algebraic group. Since S is not a ruled variety by the assumption, $\mathrm{Lin}(S)$ is discrete ([4]). Hence $\mathrm{Lin}(S)$ is a finite group. On the other hand $\rho_m(\mathrm{Bim}(V)) \subset \mathrm{Lin}(S)$. Hence $\rho_m(\mathrm{Bim}(V))$ is a finite group. q.e.d.

REMARK 2. It is interesting to know *whether the third part of Theorem* 1 *is valid for all Moišezon manifolds*. When V is an elliptic surface of general type (i.e. $\kappa(V)=1$), even if $\Phi_{mK}(V)=\boldsymbol{P}^1$, we can easily show that $\rho_m(\mathrm{Bim}(V))$ is a finite group.

REMARK 3. When V is a compact complex manifold *the pluricanonical representation* $\rho_m: \mathrm{Aut}(V) \to GL(H^0(V, \mathcal{O}(K_V^{\otimes m})))$ *maps the connected component* $\mathrm{Aut}^0(V)$ *of* $\mathrm{Aut}(V)$ *onto the identity matrix*. This is an immediate consequence of Proposition 3.

§ 2. Proof of Main Theorem.

Let $\{\mathcal{U}_i\}_{i\in I}$ be a finite covering of S by small open subsets $\mathcal{U}_i$ with systems of local coordinates $(u_i^1, \cdots, u_i^l)$. By $\mathrm{Aut}(V)$ we denote the sheaf of germs of holomorphic sections of $\mathrm{Aut}(V)$. The complex fibre bundle $\pi: M \to S$ is determined by a 1-cocycle $\{F_{ij}\}$ where $F_{ij} \in H^0(\mathcal{U}_i \cap \mathcal{U}_j, \mathrm{Aut}(V))$. Let $\{\mathcal{V}_j\}_{j\in J}$ be a sufficiently fine finite open covering of V where $\mathcal{V}_i$ is a coordinate neighborhood with a system of local coordinates $(z_i^1, \cdots, z_i^n)$. The fibre bundle M is covered by a finite open covering $\{\mathcal{M}_{ij}\}$ such that an open set $\mathcal{M}_{ij}$ is analytically isomorphic to $\mathcal{U}_i \times \mathcal{V}_j$. The transition functions $\{K_{(i,j)(k,l)}(M)\}$ of the canonical line bundle K_M of M is given by

$$1)\quad K_{(i,j)(k,l)}(M) = \left(\det \frac{\partial(u_i^1, \cdots, u_i^l, F_{ik}^1(u_k, z_l), \cdots, F_{ik}^n(u_k, z_l))}{\partial(u_k^1, \cdots, u_k^l, z_l^1, \cdots, z_l^n)}\right)^{-1}$$

$$= \det\left(\frac{\partial(u_i^1, \cdots, u_i^l)}{\partial(u_k^1, \cdots, u_k^l)}\right)^{-1} \cdot \det\left(\frac{\partial(F_{ik}^1(u_k, z_l), \cdots, F_{ik}^n(u_k, z_l))}{\partial(z_l^1, \cdots, z_l^n)}\right)^{-1}.$$

Hence we have

$$K_M = \pi^*(K_S) \otimes L\,,$$

where L is a line bundle determined by transition functions

$$\left\{\det\left(\frac{\partial(F_{ik}^1(u_k, z_l), \cdots, F_{ik}^n(u_k, z_l))}{\partial(z_l^1, \cdots, z_l^n)}\right)^{-1}\right\}.$$

If we restrict the line bundle L to the fibre $M_s = \pi^{-1}(s)$, $s \in S$, then $L|_{\pi^{-1}(s)}$ is nothing but the canonical line bundle K_{M_s}.

By $\pi_*(K_M^{\otimes m})$ and $\pi_*(L^{\otimes m})$ we denote the vector bundles associated to the locally free sheaves $\pi_*(\mathcal{O}(K_M^{\otimes m}))$ and $\pi_*(\mathcal{O}(L^{\otimes m}))$ respectively. We have

$$\pi_*(K_M^{\otimes m}) = K_S^{\otimes m} \otimes \pi_*(L^{\otimes m})\,.$$

On $\mathcal{U}_i$, $\pi_*(K_M^{\otimes m})|_{\mathcal{U}_i}$ is analytically isomorphic to a trivial vector bundle

$\mathcal{U}_i \times H^0(V, \mathcal{O}(K_V^{\otimes m}))$.

From 1) we can easily show that transition functions $\{G_{ij}\}$ of this vector bundle are given by

$$G_{ij} = \rho_m(F_{ij}) \cdot K_{ij}(S),$$

where ρ_m is the pluricanonical representation of $\mathrm{Aut}(V)$ into $GL(H^0(V, \mathcal{O}(K_V^{\otimes m})))$ and $\{K_{ij}(S)\}$ are transition functions of the canonical line bundle K_S of S.

Let A_m be a subgroup of $\rho_m(\mathrm{Aut}(V))$ generated by $\rho_m(F_{ij})$ for all $(i, j) \in I \times J$. By Theorem 1 and Remark 3, A_m is a finitely generated periodic subgroup. Hence by the theorem of Schur A_m is a finite group ([1] § 36.2). Then there exists a finite unramified covering manifold $f: \tilde{S} \to S$ such that the induced vector bundle $f^*(\pi_*(K_M^{\otimes m}))$ is analytically isomorphic to $K_{\tilde{S}}^{\otimes m} \otimes H^0(V, \mathcal{O}(mK_V))$. From this we infer that $\kappa(M) = -\infty$ follows from $\kappa(S) = -\infty$ or $\kappa(V) = -\infty$.

In the other case, we let $\tilde{f}: (\tilde{M}, \tilde{\pi}, \tilde{S}) \to (M, \pi, S)$ be a lift of $\pi: M \to S$ over $\tilde{S}$. Note that

$$\begin{aligned} H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}^{\otimes m})) &= H^0(\tilde{S}, \tilde{\pi}_* \mathcal{O}(K_{\tilde{M}}^{\otimes m})) \\ &= H^0(\tilde{S}, \tilde{\pi}_* \tilde{f}^* \mathcal{O}(K_M^{\otimes m})) \\ &= H^0(\tilde{S}, f^* \pi_* \mathcal{O}(K_M^{\otimes m})). \end{aligned}$$

Then combining this with

$$H^0(\tilde{S}, f^* \pi_* \mathcal{O}(K_M^{\otimes m})) = H^0(\tilde{S}, \mathcal{O}(K_{\tilde{S}}^{\otimes m})) \otimes H^0(V, \mathcal{O}(K_V^{\otimes m}))$$

we have

$$H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}^{\otimes m})) = H^0(\tilde{S}, \mathcal{O}(K_{\tilde{S}}^{\otimes m})) \otimes H^0(V, \mathcal{O}(K_V^{\otimes m})).$$

From this it follows $\kappa(\tilde{M}) = \kappa(\tilde{S}) + \kappa(V)$. Recalling $\kappa(\tilde{M}) = \kappa(M)$ and $\kappa(\tilde{S}) = \kappa(S)$, we obtain $\kappa(M) = \kappa(S) + \kappa(V)$ as required. q. e. d.

REMARK 4. Let T be a three-dimensional complex torus constructed in Remark 1. Let E be an elliptic curve with fundamental periods $\{1, \omega\}$. Let G be a free abelian group of analytic automorphisms of $\boldsymbol{C} \times T$ generated by two automorphisms

$$f_1: \quad (z, q) \longmapsto (z+1, q),$$
$$f_2: \quad (z, q) \longmapsto (z+\omega, g(q)),$$

where $g: T \to T$ is an analytic automorphism of the complex torus T constructed in Remark 1. Then G acts on $\boldsymbol{C} \times T$ properly discontinuously and its action has no fixed points. The quotient manifold $M = \boldsymbol{C} \times T/G$ is a fibre bundle over E whose fibre and structure group are T and $\mathrm{Aut}(T)$ respectively. By the result of Remark 1 we infer readily that $\kappa(M) = -\infty$.

Hence *Main Theorem does not hold in general without the assumption that V is a Moišezon manifold.*

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