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Uniformly hyperfinite algebras and locally compact transformation groups

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§0. J. Glimm [4] and E. G. Effros-F. Hahn [2] introduced and studied the C^*-algebras associated with locally compact transformation groups. In this paper, we shall represent uniformly hyperfinite algebras and the C^*-algebras of completely continuous operators as transformation group C^*-algebras.

Let $G^{(k)}$ $(k=1, 2, \cdots)$ be cyclic groups of finite order q_k, and consider the product groups $G_n=\prod_{k=1}^{n} G^{(k)}$ $(n=1, 2, \cdots)$. By left multiplications, $G^{(k)}$ and G_n are transformation groups on themselves. We denote by $Z^{(k)}$ and Z_n the groups $G^{(k)}$ and G_n which are considered as the underlying spaces of transformation groups. Then $(G^{(k)}, Z^{(k)})$ and (G_n, Z_n) are discrete finite transformation groups. Let $Z=\prod_{k=1}^{\infty} Z^{(k)}$ and let G be the restricted direct product $\prod_{k=1}^{\infty} G^{(k)}$ of $G^{(k)}$. Then (G, Z) is a locally compact transformation group such that G is a discrete countable group and Z is a compact Hausdorff space. Let $\mathfrak{A}(G^{(k)}, Z^{(k)})$ $(k=1, 2, \cdots)$, $\mathfrak{A}(G_n, Z_n)$ $(n=1, 2, \cdots)$, and $\mathfrak{A}(G, Z)$ be transformation group C^*-algebras associated with the transformation groups $(G^{(k)}, Z^{(k)})$ $(k=1, 2, \cdots)$, (G_n, Z_n) $(n=1, 2, \cdots)$, and (G, Z), respectively ([2], [4]). Then we shall show that $\mathfrak{A}(G, Z)$ is a uniformly hyperfinite algebra of type $\{p_n\}$, where $p_n=q_1q_2\cdots q_n$ $(n=1, 2, \cdots)$, and $\mathfrak{A}(G, Z)=C^*\text{-}\lim_n \mathfrak{A}(G_n, Z_n)=\otimes_k \mathfrak{A}(G^{(k)}, Z^{(k)})$ (Theorem 5). Conversely, let $\mathfrak{A}$ be a uniformly hyperfinite algebra of type $\{p_n\}$. Then we shall show that there exists a transformation group (G, Z) such that $\mathfrak{A}$ is *-isomorphic to $\mathfrak{A}(G, Z)$ where (G, Z) is as above (Corollary 6).

In §3, we shall show that if G is an infinite cyclic group, the C^*-algebra $\mathfrak{A}(G, G)$ associated with the transformation group (G, G) is *-isomorphic to the C^*-algebra $\mathfrak{LC}(H)$ of all completely continuous operators in a separable Hilbert space H (Theorem 9).

§ 1. Some lemmas.

Suppose that $Z^{(k)}$ $(k=1,2,\cdots)$ are compact Hausdorff spaces. Let $Z_n=\prod_{k=1}^{n} Z^{(k)}$ $(n=1,2,\cdots)$ and $Z=\prod_{k=1}^{\infty} Z^{(k)}$. If X is a compact Hausdorff space, $C(X)$ denotes the commutative C^*-algebra of all complex-valued continuous functions on X. Then it is easy to verify the following:

LEMMA 1. *There exists the C^*-inductive limit $C^*\text{-}\lim_n C(Z_n)$ of $C(Z_n)$ and $C(Z)$ is principally *-isomorphic to $C^*\text{-}\lim_n C(Z_n)$.*

Let $\mathfrak{A}(G, Z)$ be the transformation group C^*-algebra associated with a locally compact transformation group (G, Z) ([2], [4]). For each integer $n>0$, let M_n be the C^*-algebra of all complex $n\times n$-matrices. Then we have the following lemma:

LEMMA 2. *Suppose that G is a cyclic group of order n. Then, the transformation group C^*-algebra $\mathfrak{A}(G, G)$ is principally *-isomorphic to M_n.*

PROOF. It is clear that the C^*-algebra $\mathfrak{A}(G, G)$, associated with a discrete finite transformation group (G, G), is equal to the set of all complex-valued functions on $G\times G$.

Let s be the generator of G. For any $f\in\mathfrak{A}(G, G)$, let

$$\alpha_{pq}=f(s^{q-p}, s^{1-p}) \qquad (p, q=1, 2, \cdots, n).$$

Define

$$T_f=(\alpha_{pq})_{p,q=1,2,\cdots,n}.$$

Then it is easy to verify that T is a principal *-isomorphism of $\mathfrak{A}(G, G)$ onto M_n.

This completes the proof.

Suppose that $G^{(k)}$ $(k=1, 2, \cdots)$ are cyclic groups of finite orders q_k. Let $Z^{(k)}=G^{(k)}$, $G_n=\prod_{k=1}^{n} G^{(k)}$, and $Z_n=G_n$ as in the introduction. Then we have the following lemma:

LEMMA 3. *$\mathfrak{A}(G_n, Z_n)$ is principally *-isomorphic to the tensor product C^*-algebra $\bigotimes_{1\leqq k\leqq n}\mathfrak{A}(G^{(k)}, Z^{(k)})$ of $\mathfrak{A}(G^{(k)}, Z^{(k)})$.*

PROOF. Let σ_n be a linear mapping of $\bigotimes_{1\leqq k\leqq n}\mathfrak{A}(G^{(k)}, Z^{(k)})$ into $\mathfrak{A}(G_n, Z_n)$ such that

$$\sigma_n\Big(\bigotimes_{1\leqq k\leqq n} f^{(k)}\Big)(s^{(1)}, \cdots, s^{(n)}; \zeta^{(1)}, \cdots, \zeta^{(n)})=\prod_{k=1}^{n} f^{(k)}(s^{(k)}, \zeta^{(k)})$$

for any $\bigotimes_{1\leqq k\leqq n} f^{(k)}\in\bigotimes_{1\leqq k\leqq n}\mathfrak{A}(G^{(k)}, Z^{(k)})$. Then, it is easy to verify that σ_n is a principal *-isomorphism of $\bigotimes_{1\leqq k\leqq n}\mathfrak{A}(G^{(k)}, Z^{(k)})$ onto $\mathfrak{A}(G_n, Z_n)$.

This completes the proof.

By Lemmas 2, 3, we have

COROLLARY 4. *Suppose that* $G^{(k)}$, $Z^{(k)}$, G_n, *and* Z_n *are as in Lemma* 3. *Then* $\mathfrak{A}(G_n, Z_n)$ *is principally* $*$*-isomorphic to* M_{p_n}, *the* C^**-algebra of* $p_n \times p_n$*-matrices, where* $p_n = q_1 q_2 \cdots q_n$.

By Corollary 4, it follows that for each integer $n > 0$, there exists a $*$-isomorphism ρ_n of $\mathfrak{A}(G_n, Z_n)$ onto $\bigotimes_{1 \leqq k \leqq n} M_{q_k}$. For any integer $m, n: 0 < m < n$, define a $*$-isomorphism τ_{nm} of $\bigotimes_{1 \leqq k \leqq m} M_{q_k}$ into $\bigotimes_{1 \leqq k \leqq n} M_{q_k}$ by

$$\tau_{nm} = \rho_n \varphi_{nm} \rho_m^{-1},$$

where φ_{nm} is the canonical imbedding of $\mathfrak{A}(G_m, Z_m)$ into $\mathfrak{A}(G_n, Z_n)$ as in the proof of Theorem 5. Then we have

$$\tau_{nm}(\bigotimes_{1 \leqq k \leqq m} M_{q_k}) = (\bigotimes_{1 \leqq k \leqq m} M_{q_k}) \otimes (\bigotimes_{m+1 \leqq k \leqq n} I_{q_k}).$$

From this follows that the $*$-isomorphism τ_{nm} is a canonical imbedding of $\bigotimes_{1 \leqq k \leqq m} M_{q_k}$ into $\bigotimes_{1 \leqq k \leqq n} M_{q_k}$.

§ 2. Main theorem.

Let $\mathfrak{A}$ be a C^*-algebra with identity. $\mathfrak{A}$ is called *uniformly hyperfinite of type* $\{p_n\}$ if there exists a sequence of $p_n \times p_n$-matrix algebras M_{p_n} such that (i) $p_n \uparrow \infty$ as $n \to \infty$ and (ii) $\mathfrak{A} = C^*\text{-}\lim_n M_{p_n}$ (Glimm [3]).

Suppose that $G^{(k)}$, $Z^{(k)}$, G_n, Z_n, G, and Z are as in the introduction. Then we have the following theorem:

THEOREM 5. *The transformation group* C^**-algebra* $\mathfrak{A}(G, Z)$ *is the* C^**-inductive limit* $C^*\text{-}\lim_n \mathfrak{A}(G_n, Z_n)$ *of* $\mathfrak{A}(G_n, Z_n)$ *and is* $*$*-isomorphic to the infinite tensor product* C^**-algebra* $\otimes_k \mathfrak{A}(G^{(k)}, Z^{(k)})$ *of* $\mathfrak{A}(G^{(k)}, Z^{(k)})$. *Furthermore, these* C^**-algebras are uniformly hyperfinite algebras of type* $\{p_n\}$, *where* $p_n = q_1 q_2 \cdots q_n$ $(n = 1, 2, \cdots)$.

PROOF. Let $\mathfrak{A}_m = \mathfrak{A}(G_m, Z_m)$ $(m = 1, 2, \cdots)$. Let δ_s be the unit mass at s and $e^{(k)}$ the unit element in $G^{(k)}$. For each integers $m, n: 0 < m < n$, let φ_{nm} be the canonical imbedding of $\mathfrak{A}_m$ into $\mathfrak{A}_n$ defined by

$$\begin{aligned}\varphi_{nm}(f)(s^{(1)}, \cdots, s^{(n)}; \zeta^{(1)}, \cdots, \zeta^{(n)}) \\ = f(s^{(1)}, \cdots, s^{(m)}; \zeta^{(1)}, \cdots, \zeta^{(m)}) \prod_{k=m+1}^{n} \delta_{e^{(k)}}(s^{(k)}),\end{aligned}$$

where $f \in \mathfrak{A}_m$, $s^{(k)} \in G^{(k)}$, and $\zeta^{(k)} \in Z^{(k)}$ $(k = 1, 2, \cdots, n)$. Then it is obvious that φ_{nm} is a principal $*$-isomorphism and we have

$$\varphi_{nk} = \varphi_{nm} \varphi_{mk} \qquad (k < m < n).$$

Thus $\{\mathfrak{A}_m, \varphi_{nm}\}$ is an inductive system, hence there exists a C^*-inductive limit $C^*\text{-}\lim_n \mathfrak{A}_n$ of $\mathfrak{A}_n$.

We shall now show that $\mathfrak{A}(G, Z)=C^*\text{-}\lim_n \mathfrak{A}(G_n, Z_n)$. For each integer $m>0$, let φ_m be the canonical imbedding of $\mathfrak{A}_m$ into $C_0(G\times Z)\subseteqq \mathfrak{A}(G, Z)$ defined by

$$\varphi_m(f)(s^{(1)}, s^{(2)}, \cdots ; \zeta^{(1)}, \zeta^{(2)}, \cdots)$$
$$=f(s^{(1)}, \cdots, s^{(m)} ; \zeta^{(1)}, \cdots, \zeta^{(m)}) \prod_{k=m+1}^{\infty} \delta_{e^{(k)}}(s^{(k)}),$$

where $f\in\mathfrak{A}_m$, $s^{(k)}\in G^{(k)}$, and $\zeta^{(k)}\in Z^{(k)}$ $(k=1,2,\cdots)$. Then it is obvious that φ_m is a principal *-isomorphism and we have

$$\varphi_m=\varphi_n\varphi_{nm} \qquad (m<n),$$

hence it follows that $\varphi_m(\mathfrak{A}_m)\subseteqq\varphi_n(\mathfrak{A}_n)$. If we have the inclusion

$$(*) \qquad C_0(G\times Z)\subseteqq\overline{\bigcup_{m=1}^{\infty}\varphi_m(\mathfrak{A}_m)},$$

where the closure of $\bigcup_{m=1}^{\infty}\varphi_m(\mathfrak{A}_m)$ is in $\mathfrak{A}(G, Z)$, then we have

$$\mathfrak{A}(G, Z)=\overline{\bigcup_{m=1}^{\infty}\varphi_m(\mathfrak{A}_m)},$$

that is,

$$\mathfrak{A}(G, Z)=C^*\text{-}\lim_n \mathfrak{A}(G_n, Z_n).$$

Thus it suffices to show that the inclusion (*) holds. Since every element of $C_0(G\times Z)$ is of the form $\sum_{i=1}^{k}\delta_{s_i}\cdot h_i$, $s_i\in G$, $h_i\in C(Z)$, it suffices to show that for arbitrary $s\in G$, $h\in C(Z)$, $\delta_s\cdot h\in\overline{\bigcup_{m=1}^{\infty}\varphi_m(\mathfrak{A}_m)}$. For arbitrary $\varepsilon>0$, there exists a positive integer n and $g\in C(Z_n)$ such that

$$\|\psi_n(g)-h\|_\infty<\varepsilon, \qquad s^{(p)}=e^{(p)} \qquad (p\geqq n+1),$$

where ψ_n is the canonical imbedding of $C(Z_n)$ into $C(Z)$. Define a function $f\in\mathfrak{A}(G_n, Z_n)$ by

$$f(t^{(1)}, \cdots, t^{(n)} ; \zeta^{(1)}, \cdots, \zeta^{(n)})=g(\zeta^{(1)}, \cdots, \zeta^{(n)})\prod_{k=1}^{n}\delta_{s^{(k)}}(t^{(k)}).$$

Then we have $\varphi_n(f)\in\varphi_n(\mathfrak{A}_n)$ and

$$\|\varphi_n(f)-\delta_s\cdot h\|<\varepsilon,$$

that is, $\delta_s\cdot h\in\overline{\bigcup_{m=1}^{\infty}\varphi_m(\mathfrak{A}_m)}$. From this follows that

$$\mathfrak{A}(G, Z)=C^*\text{-}\lim_n \mathfrak{A}(G_n, Z_n).$$

By Corollary 4, $\mathfrak{A}(G, Z)$ is a uniformly hyperfinite algebra of type $\{p_n\}$. By Lemma 3, we have

$$\mathfrak{A}(G, Z)=\otimes_k \mathfrak{A}(G^{(k)}, Z^{(k)}).$$

This completes the proof.

Now, it is easy to verify a converse of Theorem 5, that is, every uniformly hyperfinite algebra $\mathfrak{A}$ is a transformation group C^*-algebra $\mathfrak{A}(G, Z)$.

COROLLARY 6. *Let $\mathfrak{A}$ be a uniformly hyperfinite algebra of type $\{p_n\}$. Then, there exists a locally compact transformation group (G, Z) such that $\mathfrak{A}$ is *-isomorphic to $\mathfrak{A}(G, Z)$, where (G, Z) is as in Theorem 5.*

Also, the following is obvious:

COROLLARY 7. *Let $\mathfrak{A}$, $(G^{(k)}, Z^{(k)})$ be as above. Then*

$$\mathfrak{A}=\otimes_k \mathfrak{A}(G^{(k)}, Z^{(k)})=\otimes_k M_{q_k}.$$

§ 3. The C^*-algebra of all completely continuous operators.

In § 1, Lemma 2, we showed that if G is a cyclic group of order n, the associated transformation group C^*-algebra $\mathfrak{A}(G, G)$ is *-isomorphic to M_n. We shall now show that if G is an infinite cyclic group, the associated transformation group C^*-algebra $\mathfrak{A}(G, G)$ is *-isomorphic to the C^*-algebra $\mathfrak{LC}(H)$ of all completely continuous operators in a separable Hilbert space H.

LEMMA 8 (Effros-Hahn [2]). *Let (G, Z) be a transformation group with G, Z locally compact, Hausdorff, and second countable. If G is discrete, amenable, and acts freely on Z, then the primitive ideal space $\mathrm{pr}\,\mathfrak{A}(G, Z)$ is homeomorphic to the quasi-orbit space $(Z/G)^\sim$.*

THEOREM 9. *Let G be an infinite cyclic group, and $Z=G$. Then, the associated transformation group C^*-algebra $\mathfrak{A}(G, Z)$ is *-isomorphic to $\mathfrak{LC}(H)$, the C^*-algebra of all completely continuous operators in a separable Hilbert space H.*

PROOF. We may assume that $G=Z=I$, where I is the integers. For $f\in C_0(I\times I)$, define an operator T_f of finite rank in $H=l^2(I)$ by

$$T_f e_m=\sum_{n=-\infty}^{+\infty} f(m-n, 1-n)e_n,$$

where $\{e_m : m\in I\}$ is the canonical orthonormal basis in H. It is clear that T is a linear mapping of $C_0(I\times I)$ into $\mathfrak{LC}(H)$. Furthermore, the mapping T is a *-homomorphism. In fact, we have

$$\begin{aligned}(T_{f*g}e_m|e_n)&=f*g(m-n, 1-n)\\ &=\sum_{p=-\infty}^{+\infty} f(p-n, 1-n)g(m-p, 1-p)\\ &=\sum_{p=-\infty}^{+\infty}(T_f e_p|e_n)(T_g e_m|e_p)\end{aligned}$$

$$=(T_f T_g e_m | e_n)$$

and

$$(T_{f^*} e_m | e_n) = f^*(m-n, 1-n)$$
$$= \overline{f(n-m, 1-m)}$$
$$= (T_f^* e_m | e_n) .$$

If $f \neq 0$, then there exist integers m, n such that $f(m-n, 1-n) \neq 0$. Hence we have

$$(T_f e_m | e_n) = f(m-n, 1-n) \neq 0 .$$

Thus the $*$-homomorphism T is one-to-one.

By definition of the norm ([2]), we have

$$\|T_f\| \leqq \|f\| ,$$

for all $f \in C_0(I \times I)$, so the $*$-isomorphism T is extended uniquely to a $*$-homomorphism of $\mathfrak{A}(I, I)$ into $\mathfrak{L}\mathfrak{C}(H)$. As the transformation group (I, I) is minimal, that is, the quasi-orbit space $(I/I)^{\sim}$ has only one-point, by Lemma 8, $\mathfrak{A}(I, I)$ is simple. Hence, the $*$-homomorphism T is a $*$-isomorphism of $\mathfrak{A}(I, I)$ into $\mathfrak{L}\mathfrak{C}(H)$. Since it is not difficult to verify that the representation T is irreducible, we have

$$\{T_f : f \in \mathfrak{A}(I, I)\} = \mathfrak{L}\mathfrak{C}(H) .$$

This completes the proof.

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