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Rings satisfying polynomial constraints

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§ 1. Introduction.

In a well-known paper [1] Herstein proved that if an associative ring R has the property that for each x in R there exists a polynomial $f_x(\lambda)$ (depending on x) with integer coefficients such that $x-x^2f_x(x)$ is in the center of R, then R is commutative. In this paper, we generalize Herstein's Theorem by essentially considering conditions on n elements $x_1, \cdots, x_n$ of R. We make extensive use of Herstein's methods throughout. A related problem has been recently investigated by the authors [5].

§ 2. Main results.

Throughout, R is an associate ring and $x_1, \cdots, x_n$ are elements of R. A *word* $w(x_1, \cdots, x_n)$ is simply a product in which each factor is x_i, for some $i=1, \cdots, n$. A *polynomial* $f(x_1, \cdots, x_n)$ is, then, an expression of the form $f(x_1, \cdots, x_n)=c_1w_1(x_1, \cdots, x_n)+\cdots+c_qw_q(x_1, \cdots, x_n)$, where the c_i are integers.

DEFINITION. Let n be a positive integer. An α_n-ring is an associative ring R with the property that for all $x_1, \cdots, x_n$ in R, there exists a polynomial $f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)$ (depending on $x_1, \cdots, x_n$) with integer coefficients such that: (a) degree of each x_i in every term of $f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)\geqq 2$, and (b) $x_1\cdots x_n-f_{x_1,\cdots,x_n}(x_1, \cdots, x_n)\in Z$, where Z denotes the center of R.

It is clear that subrings and homomorphic images of α_n-rings are again α_n-rings.

Our present object is to prove the following

THEOREM (Principal Theorem). *If R is an α_n-ring with center Z, then $R^n\subseteqq Z$ (and conversely).*

Since this theorem is true for $n=1$ (Herstein's Theorem), we shall assume that $n>1$ and

(2.0) FUNDAMENTAL INDUCTION HYPOTHESIS. The above theorem is true for α_{n-1}-rings.

In preparation for the proof of this theorem, we first establish the following lemmas.

LEMMA 2.0. *Let R be an α_n-ring, and let $x_1, \cdots, x_N \in R$. Then, for each positive integer m, and for each $N \geqq n$, there exists a polynomial $g_{x_1,\cdots,x_N}(x_1, \cdots, x_N)$ such that*

$$\text{(2.1)} \quad \begin{array}{l} \text{degree of } x_i \text{ in every term of } g_{x_1,\cdots,x_N}(x_1, \cdots, x_N) \geqq m, \text{ for each } i, \\ \text{and } x_1 \cdots x_N - g_{x_1,\cdots,x_N}(x_1, \cdots, x_N) \in Z, \end{array}$$

where Z is the center of R.

This lemma follows by induction. We omit the details.

LEMMA 2.1. *In an α_n-ring R, all the idempotents of R are in the center Z of R.*

PROOF. Let $e^2 = e \in R$, and let $x \in R$. Since R is an α_n-ring, there exists a polynomial $f = f_{e,e,\cdots,e,ex-exe}(e, e, \cdots, e, ex-exe)$ such that $e(ex-exe) - f \in Z$. Now, each word in this polynomial f involves e at least twice and involves $ex-exe$ at least twice (as a factor). Thus each word of f involves $(ex-exe)^2 = 0$, or involves $(ex-exe)e = 0$, and hence $f = 0$. Therefore, $e(ex-exe) \in Z$, that is, $ex-exe \in Z$. Hence, in particular, $e(ex-exe) = (ex-exe)e = 0$. Thus, $ex = exe$. A similar argument shows that $xe = exe$, and the lemma is proved.

LEMMA 2.2. *An α_n-ring R with an identity element is commutative.*

PROOF. Since R is an α_n-ring, there exists a polynomial $f = f_{x,1,1,\cdots,1}(x, 1, 1, \cdots, 1)$ such that $x \cdot 1 - f \in Z$, where f involves x at least twice (as a factor). Hence $f = x^2 p_x(x)$ for some polynomial $p_x(x)$, and thus $x - x^2 p_x(x) \in Z$. Therefore, by Herstein's Theorem [**1**], R is commutative.

LEMMA 2.3. *An α_n-ring R which is also semi-simple is commutative.*

PROOF. By Lemma 2.2, an α_n-complete matrix ring over a division ring is a field. Since a subring and a homomorphic image of an α_n-ring is again an α_n-ring, it follows, using the Jacobson density theorem [**3**; p. 33], that a primitive α_n-ring is commutative. Hence, a semi-simple α_n-ring is commutative [**3**; p. 14].

The annihilator, $A(S)$, of the ideal S is defined by

$$A(S) = \{x \in R \mid xS = (0) = Sx\} .$$

It is readily verified that $A(S)$ is an ideal in R.

LEMMA 2.4. *Let R be an α_n-ring with center Z such that R is subdirectly irreducible and not commutative. Let S be the minimal nonzero ideal in R, and let $A(S)$ be the annihilator of S. Then* (i) $S^2 = (0)$, (ii) $S \subseteqq Z$, *and* (iii) $R/A(S)$ *is commutative.*

PROOF. First, since R is subdirectly irreducible, the intersection of all nonzero ideals in R is a nonzero ideal S in R. Let J be the Jacobson radical of R. If $J = (0)$, then R is commutative (by Lemma 2.3), a contradiction. Hence $J \neq (0)$, and therefore $S \subseteqq J$. Let $s \in S$, $s \neq 0$. By Lemma 2.0, there exists a polynomial $p(s)$ (depending on s) with integer coefficients such that

(2.2) $$c=s^n-s^{2n+1}p(s)\in Z,\ c\in S\ (\text{since } s\in S),\ c\in J\ (\text{since } S\subseteqq J).$$

Now, since $c\in Z$, cS is an ideal in R and $cS\subseteqq S$. Hence $cS=S$ or $cS=(0)$. If $cS=S$, then $c^2S=S$, and hence there exists an $x\in S$ such that $c=c^2x$ (since $c\in S$). This implies that cx is idempotent (since $c\in Z$) and $cx\in S\subseteqq J$. Hence $cx=0$. Thus $c=0$, and hence $S=cS=(0)$, a contradiction. Therefore $cS\neq S$, and thus $cS=(0)$. Hence $cs=0$, and therefore by (2.2), $s^{n+1}-s^{2n+2}p(s)=0$. Thus $s^{n+1}p(s)$ is idempotent, and $s^{n+1}p(s)\in J$. Therefore $s^{n+1}p(s)=0$, and hence $s^{n+1}=s^{2n+2}p(s)=0$. Thus $s^{n+1}=0$ for all $s\in S$. Hence, S is locally nilpotent [**2**; p. 28]. We now assume that $S^2=S$ and get a contradiction. Let $s_1,\cdots,s_n\in S$. Then the subring, $\langle s_1,\cdots,s_n\rangle$, generated by $s_1,\cdots,s_n$ is nilpotent. Let r be the index of nilpotency of this subring. Now by Lemma 2.0, there exists a polynomial $f=f(s_1,\cdots,s_n)$ such that

$$s_1\cdots s_n-f(s_1,\cdots,s_n)\in Z;\ \text{degree of each } s_i \text{ in every term of } f\geqq r.$$

Hence $f=0$, and thus $s_1\cdots s_n\in Z$. Therefore, $S^n\subseteqq Z$. But $S=S^n$ (since $S^2=S$) and hence $S\subseteqq Z$. Since, moreover, $S^2=S\neq(0)$, there exists an $s\in S$ such that $sS\neq\{0\}$. Hence $sS=S$ (recall that $s\in Z$), and thus $S=S^{n+1}=(sS)^{n+1}=s^{n+1}S^{n+1}=(0)$, since $s^{n+1}=0$. Hence $S=(0)$, a contradiction. This contradiction shows that $S^2=(0)$.

To prove (ii), let $x=r_1\cdots r_{n-1}s$, where $r_1,\cdots,r_{n-1}\in R$. Since R is an α_n-ring, there exists a polynomial $f(r_1,\cdots,r_{n-1},s)$ where, in particular, the degree of s in every term of $f(r_1,\cdots,r_{n-1},s)\geqq 2$, and, moreover, $r_1\cdots r_{n-1}s-f(r_1,\cdots,r_{n-1},s)\in Z$. Since $f(r_1,\cdots,r_{n-1},s)\in S^2=(0)$, we get $r_1\cdots r_{n-1}s\in Z$. Hence $R^{n-1}S\subseteqq Z$. Similarly, $SR^{n-1}\subseteqq Z$. Moreover, since $RS\subseteqq S$, we have $RS=S$ or $RS=(0)$. Similarly, $SR=S$ or $SR=(0)$. Now, if $RS=S$, then $S=R^{n-1}S\subseteqq Z$ (as we have just shown). Similarly, if $SR=S$, then $S=SR^{n-1}\subseteqq Z$. The only case left is that in which $SR=RS=(0)$. But, again, $S\subseteqq Z$, and part (ii) is proved.

To prove part (iii), suppose $x,y\in R$, $s\in S$. Then, since $S\subseteqq Z$, we have $(xy)s=x(ys)=(ys)x=y(sx)=y(xs)=(yx)s$. Hence $(xy-yx)s=0$ for *all* $s\in S$, and thus $xy-yx\in A(S)$. Therefore $R/A(S)$ is commutative, and the lemma is proved.

LEMMA 2.5. *Let R be an α_n-ring, and let $x,y\in R$. Then $xy-yx$ is nilpotent.*

PROOF. The proof starts out as in [**3**; p. 221]. Thus suppose $z=xy-yx$, and suppose z is not nilpotent. Let M be the following nonvanishing m-system:

$$M=\{z^i\,|\,i \text{ is a positive integer}\}.$$

Since $0\notin M$, there exists, by Zorn's Lemma, an ideal P in R such that $M\cap P=\emptyset$, and where P is maximal with respect to the property of not intersecting

M. Moreover, it is easy to show that P is indeed a prime ideal in R [4; p. 65], and hence $\bar{R}=R/P$ is a prime ring. Now, since $z\in M$, $z\notin P$, and hence $xy-yx\notin P$. Therefore, R/P is not commutative. We claim that $\bar{R}$ is not subdirectly irreducible. For, suppose $\bar{R}$ is subdirectly irreducible. Since any homomorphic image of an α_n-ring is again an α_n-ring, it follows by Lemma 2.4, that the minimal nonzero ideal S of $\bar{R}$ has the following properties: $S^2=(0)$, $S\subseteq Z$ ($Z=$ center of $\bar{R}$). Now, let $s\in S$, $s\neq 0$. Since s is in the center of $\bar{R}$, we have $s\bar{R}s=s^2\bar{R}=(0)$, and hence $s=0$, since $\bar{R}$ is a prime ring. This contradiction shows that $\bar{R}$ is not subdirectly irreducible, and hence the intersection of all nonzero ideals in $\bar{R}$ is the zero ideal. Thus,

$$\bigcap_{B\supsetneqq P} B=P, \qquad \text{where } B \text{ is an ideal in } R. \tag{2.3}$$

Now, by the maximality of P, each ideal B above intersects M. Hence, for any such ideal B, we have $z^m\in B$ for some positive integer m. Next, consider the difference ring R/B. Letting $\bar{z}=z+B$, we get,

$$\bar{z}^m=0 \ (=\text{zero of } R/B). \tag{2.4}$$

Since R is an α_n-ring, R/B is an α_n-ring. Hence, by Lemma 2.0, we can find a polynomial $p(\bar{z})$ in which each term is of degree $\geqq m$ in $\bar{z}$ and such that $\bar{z}^n-p(\bar{z})\in Z(R/B)$, where $Z(R/B)=$ center of R/B. Since $p(\bar{z})=\bar{z}^m q(\bar{z})$ for some polynomial $q(\bar{z})$, it follows by (2.4) that $p(\bar{z})=0$, and hence $\bar{z}^n\in Z(R/B)$. Next, let $\bar{r}\in R/B$. By Lemma 2.0 again, there exists a polynomial $f=f(\bar{z}^n,\cdots,\bar{z}^n,\bar{r})$ with integer coefficients such that

$$\bar{z}^n\cdots\bar{z}^n\bar{r}-f(\bar{z}^n,\cdots,\bar{z}^n,\bar{r})\in Z(R/B)\,;\ \text{degree of } \bar{z}^n \text{ in each term of } f\geqq m. \tag{2.5}$$

Since $\bar{z}^n\in Z(R/B)$, we may collect together all the $\bar{z}^n$ factors in each word in the polynomial f in (2.5). Once this is done, it is easily seen by (2.4) and (2.5), that $f=0$ and hence $(\bar{z}^n)^{n-1}\bar{r}\in Z(R/B)$. Let $q=n(n-1)$. Again, since $\bar{z}^n\in Z(R/B)$, $\bar{z}^q\in Z(R/B)$. Hence, $\bar{z}^{q+1}=\bar{z}^q(\bar{x}\bar{y}-\bar{y}\bar{x})=(\bar{z}^q\bar{x})\bar{y}-\bar{z}^q\bar{y}\bar{x}=\bar{y}(\bar{z}^q\bar{x})-\bar{z}^q\bar{y}\bar{x}=\bar{y}(\bar{x}\bar{z}^q)-\bar{z}^q(\bar{y}\bar{x})=\bar{y}(\bar{x}\bar{z}^q)-(\bar{y}\bar{x})\bar{z}^q=0$. Thus $\bar{z}^{q+1}=0$, and hence $z^{q+1}\in B$ for all ideals $B\supsetneqq P$. Hence, by (2.3), $z^{q+1}\in P$, a contradiction, since $z^{q+1}\in M$ and $M\cap P=\emptyset$. This contradiction proves the lemma.

LEMMA 2.6. *Let R be an α_n-ring, and suppose $x\in R$. Suppose that there exists a positive integer k such that $x^kR^{n-1}\cup R^{n-1}x^k\subseteq Z$, where Z is the center of R. Then $xR^{n-1}\cup R^{n-1}x\subseteq Z$.*

PROOF. Let m be the smallest positive integer such that $x^mR^{n-1}\cup R^{n-1}x^m\subseteq Z$. We now assume $m>1$ and get a contradiction. Since $x^mR^{n-1}\cup R^{n-1}x^m\subseteq Z$, we have $Rx^mR^{n-1}\cup R^{n-1}x^mR\subseteq Z$. Now, let $y_1,\cdots,y_{n-1}\in R$. By Lemma 2.0, there exists a polynomial $g=g(x^{m-1}y_1,\cdots,x^{m-1}y_{n-1},x^{m-1})$ such that

$(x^{m-1}y_1)\cdots(x^{m-1}y_{n-1})x^{m-1}-g\in Z$; each argument in g occurs more than mn times in every term of g.

Then, as can be easily verified, each word in $g\in Rx^mR^{n-1}\subseteqq Z$. Hence, $(x^{m-1}R)^{n-1}x^{m-1}\subseteqq Z$. Therefore

$$\begin{aligned}R(x^{m-1}R)^{n+1}&=[R(x^{m-1}R)^{n-1}x^{m-1}]Rx^{m-1}R\\&=[(x^{m-1}R)^{n-1}x^{m-1}R]Rx^{m-1}R\subseteqq(x^{m-1}R)(x^{m-1}R)^{n-1}x^{m-1}R\\&=(x^{m-1}R)R(x^{m-1}R)^{n-1}x^{m-1}\\&\subseteqq(x^{m-1}R)(x^{m-1}R)(x^{m-1}R)^{n-2}x^{m-1}\\&\subseteqq(x^{m-1}R)(x^{m-1}R)^{n-2}x^{m-1}=(x^{m-1}R)^{n-1}x^{m-1}\subseteqq Z.\end{aligned}$$

Hence, $R(x^{m-1}R)^{n+1}\subseteqq Z$. Now, by Lemma 2.0, there exists a polynomial $h=h(x^{m-1}, y_1, \cdots, y_{n-1})$ such that

$$x^{m-1}y_1\cdots y_{n-1}-h\in Z;\ \text{degree of } x^{m-1} \text{ in every term of } h\geqq 2n+3.$$

Now, each word in $h\in R(x^{m-1}R)^{n+1}\subseteqq Z$, and hence $x^{m-1}y_1\cdots y_{n-1}\in Z$. Similarly, $y_1\cdots y_{n-1}x^{m-1}\in Z$. Thus, $x^{m-1}R^{n-1}\cup R^{n-1}x^{m-1}\subseteqq Z$, contradicting the minimality of m. This contradiction shows that $m=1$, and hence $xR^{n-1}\cup R^{n-1}x\subseteqq Z$. This proves the lemma.

LEMMA 2.7. *Let R be an α_n-ring with center Z, and let x be a nilpotent element in R. Then $xR^{n-1}\subseteqq Z$ and $R^{n-1}x\subseteqq Z$. Moreover the set of all nilpotent elements of $R^{2(n-1)}$ is contained in the center of $R^{2(n-1)}$, and hence form an ideal of $R^{2(n-1)}$.*

PROOF. Since x is nilpotent, $x^k=0$ for some positive integer k, and hence $x^kR^{n-1}\subseteqq Z$ and $R^{n-1}x^k\subseteqq Z$. Hence, by Lemma 2.6, $xR^{n-1}\cup R^{n-1}x\subseteqq Z$.

Next, suppose $r_1, \cdots, r_{2n-2}\in R$. Then, since $xR^{n-1}\cup R^{n-1}x\subseteqq Z$,

$$\begin{aligned}xr_1\cdots r_{2n-2}&=(r_n\cdots r_{2n-2})x(r_1\cdots r_{n-1})=(r_n\cdots r_{2n-2}x)(r_1\cdots r_{n-1})\\&=(r_1\cdots r_{n-1})(r_n\cdots r_{2n-2}x)=r_1\cdots r_{2n-2}x.\end{aligned}$$

Hence, the set of all nilpotent elements of R^{2n-2} is contained in the center of R^{2n-2}, and thus form an ideal of R^{2n-2}.

Now, an easy combination of Lemmas 2.5 and 2.7 yields the following

COROLLARY 2.8. *Let R be an α_n-ring. Then the commutator ideal of R^{2n-2} is contained in its center.*

LEMMA 2.9. *Let R be an α_n-ring which is subdirectly irreducible and not commutative, and let S be the minimal nonzero ideal in R. If, further, the commutator ideal of R is contained in the center Z of R, then $A(S)R^{n-1}\subseteqq Z$ and $R^{n-1}A(S)\subseteqq Z$, where $A(S)$ is the annihilator of S.*

PROOF. Let $x\in A(S)$. By Lemma 2.0, there exist integers α_i, β_i, m, p

such that

$$x^n - \sum_{i=2n}^{m} \alpha_i x^i \in Z, \tag{2.6}$$

$$x^{n+1} - \sum_{i=2n+2}^{p} \beta_i x^i \in Z. \tag{2.7}$$

Let $[x, y] = xy - yx$. We claim that $x^n[x, y] = 0$ for all y in R. For, suppose that $x^n[x, y] \neq 0$ for some y in R. Then $x^{n-1}[x, y] \neq 0$. Now, by (2.6), we get

$$[x^n, y] = \sum_{i=2n}^{m} \alpha_i [x^i, y]. \tag{2.8}$$

Moreover, our hypothesis implies that $[x, y]$ commutes with x. Using this fact, an easy induction shows that [**3**; p. 221]

$$[x^k, y] = k x^{k-1}[x, y], \qquad k \text{ any positive integer}. \tag{2.9}$$

Combining (2.8) and (2.9), we get

$$n x^{n-1}[x, y] = \sum_{i=2n}^{m} \alpha_i i x^{i-1}[x, y] = \left(\sum_{i=2n}^{m} \alpha_i i x^{i-n}\right) x^{n-1}[x, y]. \tag{2.10}$$

A similar argument, now applied to (2.7), yields

$$(n+1) x^n[x, y] = \left(\sum_{i=2n+2}^{p} \beta_i i x^{i-n-1}\right) x^n[x, y]. \tag{2.11}$$

Now, let $s \in S$, $s \neq 0$. By Lemma 2.4, $S \subseteq Z$. Moreover, since $x^n[x, y] \neq 0$ and $x^{n-1}[x, y] \neq 0$ and S is the minimal nonzero ideal in S, we get

$$s \in (x^{n-1}[x, y]) \cap (x^n[x, y]). \tag{2.12}$$

Furthermore, since $x^{n-1}[x, y]$ and $x^n[x, y]$ are both in the commutator ideal of R, we have, by hypothesis, that

$$x^{n-1}[x, y] \in Z \quad \text{and} \quad x^n[x, y] \in Z. \tag{2.13}$$

Now, an easy combination of (2.10), (2.11), (2.12), and (2.13), together with the hypothesis that $x \in A(S)$, yields

$$ns = \left(\sum_{i=2n}^{m} \alpha_i i x^{i-n}\right) s = 0, \tag{2.14}$$

and

$$(n+1)s = \left(\sum_{i=2n+2}^{p} \beta_i i x^{i-n-1}\right) s = 0. \tag{2.15}$$

Hence $s = (n+1)s - ns = 0$, a contradiction. This contradiction shows that $x^n[x, y] = 0$ for all y in R. Combining this with (2.9), we get

$$[x^k, y] = k x^{k-n-1}(x^n[x, y]) = 0 \qquad \text{for all } k \geqq n+1,$$

and hence

(2.16) $\qquad x^k \in Z \quad$ for all integers $k \geqq n+1$, and all $x \in A(S)$.

Combining (2.16) and (2.6), we get $x^n \in Z$, and hence

(2.17) $\qquad x^k \in Z \quad$ for all integers $k \geqq n$, and all $x \in A(S)$.

Now, suppose $x, y \in A(S)$. By Lemma 2.0, there exists a polynomial $f = f(y_1, x^{n+1}, \cdots, x^{n+1})$ such that

(2.18) $\qquad y\underbrace{x^{n+1} \cdots x^{n+1}}_{(n-1)} - f(y, x^{n+1}, \cdots, x^{n+1}) \in Z$; degree of each argument in every term in $f \geqq n+1$.

Since, by (2.17), $x^{n+1} \in Z$, we can find integers α_i such that f has the form

(2.19) $$f(y, x^{n+1}, \cdots, x^{n+1}) = \sum_i \alpha_i (x^{n+1})^{s_i} y^{l_i}; \qquad l_i \geqq n+1, \text{ each } i.$$

Therefore, by (2.17) and (2.19), we get $f(y, x^{n+1}, \cdots, x^{n+1}) \in Z$, and hence by (2.18),

(2.20) $$y(x^{n+1})^{n-1} = (x^{n+1})^{n-1} y \in Z.$$

Hence, $x^{(n+1)(n-1)+1}R^{n-1} = x^{(n+1)(n-1)}(xR^{n-1}) \subseteqq x^{(n+1)(n-1)}A(S) \subseteqq Z$. Combining this with (2.16), we get $x^k R^{n-1} \cup R^{n-1} x^k \subseteqq Z$ (where $k = (n+1)(n-1)+1$). Hence, by Lemma 2.6, we have $xR^{n-1} \cup R^{n-1}x \subseteqq Z$ for *all* $x \in A(S)$, and the lemma follows.

COROLLARY 2.10. *Under all the hypotheses of Lemma* 2.9, *if* $A(S) = R$, *then* $R^n \subseteqq Z$.

LEMMA 2.11. *Let* R *be a ring satisfying all the hypotheses of Lemma* 2.9. *If, further,* $A(S) \neq R$, *then* $sR = S$ *for all* $s \in S$, $s \neq 0$.

PROOF. The proof is as in [**1**]. Thus, suppose $s \in S$, $s \neq 0$. By Lemma 2.4, $S \subseteqq Z$, and hence sR is an ideal in R. Since $sR \subseteqq S$, we must have $sR = S$ or $sR = (0)$. If $sR = (0)$, then $A = \{x \mid x \in S,\ xR = (0)\}$ is a nonzero ideal in R, and hence $S \subseteqq A$. This implies that $SR = (0)$. Since $S \subseteqq Z$, we also have $RS = (0)$, which contradicts the hypothesis $A(S) \neq R$. Hence $sR \neq (0)$ and thus $sR = S$. This proves the lemma.

LEMMA 2.12. *Under all the hypotheses of Lemma* 2.11, *we have that* $R/A(S)$ *is a commutative ring with identity. Indeed, there exists an element* $e \in Z$ *such that* $e + A(S)$ *is the identity element of* $R/A(S)$.

PROOF. First, by Lemma 2.4, $R/A(S)$ is commutative and $S \subseteqq Z$. Now, since $A(S) \neq R$, there exists an element $x \in R$, $x \notin A(S)$. Let $s \in S$, $s \neq 0$. Suppose that $sx = 0$. We shall show that this leads to a contradiction. Now, $(Rs)x = R(sx) = (0)$. But, by Lemma 2.11 and the fact that $s \in Z$, we get $Rs = sR = S$, and hence $xS = Sx = (Rs)x = (0)$. Thus $x \in A(S)$, a contradiction. Hence $sx \neq 0$, and thus by Lemma 2.11, $R(sx) = (sx)R = S$. Therefore, for some $y \in R$, $s = ysx = syx$, since $s \in Z$. Let $e = yx$. Then, for all $r \in R$, $s(re - r) = 0$. Thus $Rs(re - r) = (0)$, and hence (by Lemma 2.11 again) $S(re - r) = (0)$. Thus $re - r$

$\in A(S)$. Similarly, $s(er-r)=0$, and hence $Rs(er-r)=(0)$, which implies $S(er-r)=(0)$. Thus $er-r\in A(S)$. Hence $e+A(S)$ is the identity of $R/A(S)$. Moreover, $e^2-e\in A(S)$, and hence, by Lemma 2.9, $e^{n+1}-e^n=(e^2-e)e^{n-1}\in Z$. Now, if $e\notin Z$, then there exists an element y in R such that $[e,y]=ey-ye\neq 0$. Since $[e^{n+1}-e^n, y]=0$, we have $[e^{n+1}, y]=[e^n, y]$. Hence, by (2.9), $(n+1)e^n[e,y]=ne^{n-1}[e,y]$. Therefore, $((n+1)e^n-ne^{n-1})[e,y]=0$. Now, let $s\in S$, $s\neq 0$. Since $([e,y])\neq(0)$, we must have $s\in([e,y])$. But, by hypothesis, $[e,y]\in Z$. These facts, together with the equation $(n+1)e^n[e,y]-ne^{n-1}[e,y]=0$, show that $((n+1)e^n-ne^{n-1})s=0$. Hence $(n+1)e^n-ne^{n-1}\in A(S)$, and thus $e\in A(S)$ (since $e+A(S)$ is the identity of $R/A(S)$). This implies that $R=A(S)$, a contradiction. Thus the assumption that $e\notin Z$ led to a contradiction. Hence $e\in Z$, and the lemma is proved.

LEMMA 2.13. *In the notation, and under all the hypotheses, of Lemma* 2.12, *we have that the ring* $(eR)^{n-1}\subseteq Z(eR)$.

PROOF. Since R is an α_n-ring, we have that for all $r_1,\cdots,r_n$ in R, there exists a polynomial $f=f(e,r_1,\cdots,r_{n-1})$ such that

(2.21) $$er_1\cdots r_{n-1}-f(e,r_1,\cdots,r_{n-1})\in Z;\quad \text{degree of each argument in every term of } f\geqq 2.$$

Moreover, by Lemma 2.12,

(2.22) $$e\in Z \quad\text{and}\quad e+A(S) \text{ is the identity of } R/A(S).$$

Now, let $w_i=w_i(e,r_1,\cdots,r_{n-1})$ be a typical word in f. Then, since $e\in Z$,

(2.23) $$w_i=w_i(e,r_1,\cdots,r_{n-1})=e^{k_i}w_i'(r_1,\cdots,r_{n-1})=e^{k_i}w_i';\quad k_i\geqq 2.$$

Let

(2.24) $$l_i=\text{degree of } r_1 \text{ in } w_i'+\cdots+\text{degree of } r_{n-1} \text{ in } w_i'.$$

By (2.22), $e^{k_i}-e^{l_i}\in A(S)$, and hence by Lemma 2.9, we have

(2.25) $$(e^{k_i}-e^{l_i})w_i'(r_1,\cdots,r_{n-1})\in Z.$$

Moreover, since $e\in Z$, we have by (2.24), $w_i'(er_1,\cdots,er_{n-1})=e^{l_i}w_i'(r_1,\cdots,r_{n-1})$. Combining this with (2.23) and (2.25), we get

(2.26) $$w_i(e,r_1,\cdots,r_{n-1})-w_i'(er_1,\cdots,er_{n-1})\in Z.$$

Let

(2.27) $$\begin{aligned} f(e,r_1,\cdots,r_{n-1})&=\sum_i c_iw_i(e,r_1,\cdots,r_{n-1})\\ g(er_1,\cdots,er_{n-1})&=\sum_i c_iw_i'(er_1,\cdots,er_{n-1})\end{aligned}\quad(\text{the } c_i \text{ integers}).$$

Then, by (2.26), $f(e,r_1,\cdots,r_{n-1})-g(er_1,\cdots,er_{n-1})\in Z$, and hence by (2.21), we get

(2.28) $$er_1 \cdots r_{n-1} - g(er_1, \cdots, er_{n-1}) \in Z\,.$$

Now, by (2.22), $e^{n-1}-e \in A(S)$, and hence by Lemma 2.9,

(2.29) $$(e^{n-1}-e)r_1 \cdots r_{n-1} \in Z\,.$$

By (2.22), $e^{n-1}r_1 \cdots r_{n-1} = (er_1) \cdots (er_{n-1})$. Combining this with (2.29) and (2.28), we get

(2.30) $$(er_1) \cdots (er_{n-1}) - g(er_1, \cdots, er_{n-1}) \in Z\,.$$

Moreover, by (2.23) and (2.21), each word $w_i'(r_1, \cdots, r_{n-1})$ involves every r_j at least twice, and hence the degree of each er_j in every term of $g(er_1, \cdots, er_{n-1}) \geqq 2$. This, together with (2.30), now shows that eR is an α_{n-1}-ring. Hence, by (2.0), $(eR)^{n-1} \subseteqq Z(eR)$, and the lemma is proved.

LEMMA 2.14. *Suppose R, Z, S, $A(S)$, e are as in Lemmas 2.11 and 2.12, and suppose that all the hypotheses of Lemma 2.11 hold. Then $R^n \subseteqq Z$.*

PROOF. Let $r_1, \cdots, r_n \in R$. By Lemma 2.12, $e^n r_1 - r_1 \in A(S)$ and $e \in Z$. Hence, by Lemma 2.9, $e^n r_1 \cdots r_n - r_1 \cdots r_n \in Z$. Let $y \in R$. By Lemma 2.13,

$$\begin{aligned}[r_1 \cdots r_n, y] &= [e^n r_1 \cdots r_n, y] = e^n r_1 \cdots r_n y - y e^n r_1 \cdots r_n \\ &= [(er_1 r_2)(er_3) \cdots (er_n)](ey) - (ey)[(er_1 r_2)(er_3) \cdots (er_n)] \\ &= 0\,.\end{aligned}$$

Thus, $[r_1 \cdots r_n, y] = 0$, and the lemma is proved.

Now, an easy combination of Corollary 2.10, Lemma 2.14, and Birkhoff's Theorem that every ring is isomorphic to a subdirect sum of subdirectly irreducible rings [**3**; p. 219], yields

COROLLARY 2.15. *Let R be an α_n-ring such that the commutator ideal in R is contained in the center Z of R. Then $R^n \subseteqq Z$.*

We are now in a position to prove the Principal Theorem.

PROOF OF THE PRINCIPAL THEOREM: By Corollary 2.8 and Corollary 2.15, we have

(2.31) $$R^{(2n-2)n} \text{ is a commutative ring}\,.$$

Now, suppose $x, y \in R^{(2n-2)n}$, and suppose $r \in R$. Then $yr \in R^{(2n-2)n}$, $rx \in R^{(2n-2)n}$, and hence using (2.31), we get

$$(xy)r = x(yr) = (yr)x = y(rx) = (rx)y = r(xy)\,.$$

Thus xy is in the center $Z(R)$ of R. Therefore

(2.32) $$(R^{(2n-2)n})^2 \subseteqq Z(R)\,.$$

Now, let $y_1, \cdots, y_n \in R$. Then, by Lemma 2.0, we can find a polynomial $f = f_{y_1, \cdots, y_n}(y_1, \cdots, y_n)$ such that

(2.33) $$y_1 \cdots y_n - f_{y_1,\cdots,y_n}(y_1, \cdots, y_n) \in Z; \quad \text{degree of } y_1 \text{ in each term of } f \geqq 2(2n-2)n.$$

Since $f \in R^{2(2n-2)n} \subseteqq Z$ (by (2.32)), we have $f \in Z$. Combining this with (2.33), we obtain $y_1 \cdots y_n \in Z$, and hence $R^n \subseteqq Z$. The converse, of course, is trivial. This proves the theorem.

Finally, we remark that in [5], the authors have given examples which show that the hypotheses regarding the degrees (in the definition of an α_n-ring) are indeed essential for the validity of our principal theorem.

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