J. Math. Soc. Japan
Vol. 25, No. 1, 1973

Scattering theory for differential operators, I, operator theory

By S. T. KURODA

(Received Dec. 21, 1971)
(Revised Aug. 31, 1972)

§ 1. Introduction.

The present paper is intended to be the first of a series of papers aimed at dealing with a spectral and scattering theory for some partial differential operators by application of the so-called abstract stationary method. We take the attitude of studying problems in operator-theoretical terms as far as possible and then handling differential operators by applying the obtained results.

Some problems considered in the mathematical theory of scattering are: i) to investigate the structure of the absolutely continuous spectrum of a perturbed operator; ii) to prove the existence and the completeness of wave operators; iii) to establish the discreteness, as defined in § 5, of the singular spectrum; and iv) to construct eigenfunction expansions. Among many works concerning these problems we only mention a work of Ikebe in 1960 ([**5**]) and a group of more recent works on the abstract stationary method[1] ([**11**], [**9**], [**14**]). In [**5**] Ikebe treated the Schrödinger operator $-\Delta+q(x)$ by the integral equation method under the main assumption that $q(x)=O(|x|^{-\delta})$, $\delta>2$, as $|x|\to\infty$. With the aid of a theorem of Kato [**7**] concerning the growth property of the solution of $-\Delta u+qu=\lambda u$, he solved i) - iv) with a sharper result that there is no singular continuous spectrum except for non-positive eigenvalues. (The method was later applied to exterior problems in [**17**], [**6**], etc.) On the other hand, it was shown in [**11**] etc. that problems i) and ii) (and iv) partly) can be handled by the abstract stationary method. In particular, it was shown by Kato [**9**] that problems i) and ii) for Schrödinger operators can be solved for $\delta>1$ and that the sharper result as Ikebe's holds for $\delta>5/4$. Some more results were announced in [**14**].

Recently, S. Agmon investigated the spectral problem of differential operators by a new method based on a weighted elliptic estimate and an-

1) For an overall exposition of the scattering theory with an extensive list of literatures, the reader is referred to Kato [**10**].

nounced that problems i), ii) and iv) can be solved for self-adjoint elliptic operators of an arbitrary order (i) and iii) in [1] and iv) in a lecture at the Mathematics Research Institute, Oberwolfach, 1971). The main assumption is that all the coefficients approach to constants with the order $O(|x|^{-\delta})$, $\delta>1$, $|x|\rightarrow\infty$. Stimulated by Agmon's work, the present author tried to study problem iii) in the framework of the operator-theoretical approach.

One purpose of the present paper is to present the results thus obtained with a detailed proof. In this respect the main part is § 5 and the main theorem is Theorem 5.21. Another purpose of the paper is to give a reformulation of the abstract stationary method with special regards to the following points: a) the attention is restricted to the so-called smooth perturbation; and b) the discussion is based on a new form of assumptions (especially Assumption 3.3) which makes the applicability of the results broader. It is our intention to give a relatively quick proof in the generality broad enough for many applications. There are also some new elements in the method of proof. § 3 and § 4 will be devoted for the latter purpose. They also provide materials necessary in § 5.

Applications to self-adjoint operators will be given in a subsequent paper ([**15**]), where a relatively easy check of the assumptions introduced in this paper will be performed. In this way we will obtain the result equivalent to Agmon's as far as problems i) and iii) are concerned. We shall leave problem iv) to a later investigation. However, it may be mentioned that the perturbed spectral representation $F_{\pm}$ constructed in §§ 3 and 4 (cf. Theorem 3.11) takes over, at least partly, the role of eigenfunction expansions[2)]. For instance, problem ii) can be solved with the aid of $F_{\pm}$ (cf. Theorems 3.12 and 3.13).

In Agmon's work (and in some of others) the principle of limiting absorption or something similar plays a decisive role. In our approach this principle is somewhat hidden behind the existence of the boundary values of the resolvents or some related operator valued functions. To explain how the usual form of the principle is derived, we shall make a short comment in § 6.1.

In § 6.2 a remark will be made on the scattering matrix.

Finally, we mention that another abstract stationary theory for scattering has been developed by Birman and others and applied to various partial differential operators (see [**2**] and references quoted there). They are mostly concerned with problems i) and ii) and use conditions involving the trace class to ensure the existence of boundary values. It seems that Birman's

2) In the respect that $F_{\pm}$ are constructed first our approach is somewhat related to works of Rejto [**16**], Howland [**4**], and Goldstein [**3**].

theory can be applied to a wider type of differential operators with, however, more stringent conditions on the decay of perturbing coefficients.

§ 2. Preliminaries.

2.1. Notations. The following notations will be used throughout the present paper.

$\mathfrak{H}, \mathfrak{K}, \mathfrak{C}, \cdots$ denote Hilbert spaces. If required for clarity, we will distinguish norms and inner products in various spaces by the subscript like $\|u\|_{\mathfrak{K}}$, $(u, v)_{\mathfrak{C}}$. When no distinction is necessary, we simply write $\|u\|$, (u, v), etc. for various spaces.

All operators appearing in this paper are assumed to be linear and densely defined unless otherwise stated. The domain and the range of an operator A are denoted by $\mathfrak{D}(A)$ and $\mathfrak{R}(A)$, respectively. $\mathfrak{N}(A)=\{u\in\mathfrak{D}(A)\,|\,Au=0\}$ is the null space of A. The resolvent set of A is denoted by $\rho(A)$ and the resolvent by $R(\zeta\,;A)=(A-\zeta)^{-1}$. The closure of a closable operator A is denoted by $[A]^a$. For simplicity of the exposition we agree that a statement containing $[A]^a$ includes implicitly the assertion that A is closable.

$B(\mathfrak{H}, \mathfrak{K})$ stands for the Banach space of all bounded linear operators on $\mathfrak{H}$ to $\mathfrak{K}$ and $B_\infty(\mathfrak{H}, \mathfrak{K})$ the subspace of $B(\mathfrak{H}, \mathfrak{K})$ consisting of all compact operators in $B(\mathfrak{H}, \mathfrak{K})$.

Let I be a Borel subset of $R^1=(-\infty, \infty)$. For a Hilbert space $\mathfrak{C}$ the Hilbert space of all strongly measurable $\mathfrak{C}$-valued functions with square integrable norm is denoted by $L^2(I\,;\mathfrak{C})$. Here, the measure in I is understood to be the Lebesgue measure.

Let I be an open set in R^1 and let $I'\subset I$ be a Borel set. For brevity we agree to write $I'\Subset I$ when the closure I'^a of I' is a *compact* subset of I.

2.2. Factorization scheme. We will always deal with two self-adjoint operators H_1 and H_2. Among a few ways of expressing H_2 as a perturbation of H_1 the so-called factorization scheme due to Kato [8] is rather inclusive and convenient. It will be formulated below in a form suitable for our purpose.

Let H_1 and H_2 be self-adjoint operators in a Hilbert space $\mathfrak{H}$. For brevity we put $R_j(\zeta)=R(\zeta\,;H_j)$, $\zeta\in\rho(H_j)$. We consider the situation formally expressed as $H_2=H_1+A^*CB=H_1+B^*C^*A$.

ASSUMPTION 2.1. A and B are closed operators from $\mathfrak{H}$ to another Hilbert space $\mathfrak{K}$. Furthermore, $C\in B(\mathfrak{K})$.

ASSUMPTION 2.2. $\mathfrak{D}(A)\supset\mathfrak{D}(H_1)$, $\mathfrak{D}(B)\supset\mathfrak{D}(H_1)$.

From these assumptions it follows that

$$AR_1(\zeta)\in B(\mathfrak{H}, \mathfrak{K})\,, \qquad BR_1(\zeta)\in B(\mathfrak{H}, \mathfrak{K})\,,$$

so that $BR_1(\zeta)A^*$ is densely defined. One easily sees that

$$[BR_1(\zeta)A^*]^a \subset B[R_1(\zeta)A^*]^a = B(AR_1(\bar{\zeta}))^*, \tag{2.1}$$

all the members being closed operators. (When one member of the above relation is a bounded operator, the inclusion is replaced by the equality.)

ASSUMPTION 2.3. $[BR_1(\zeta)A^*]^a \in B(\mathfrak{K})$ for one or equivalently all $\zeta \in \rho(H_1)$[3].

We introduce the following notations:

$$Q_1(\zeta) = C[BR_1(\zeta)A^*]^a \in B(\mathfrak{K}), \qquad \zeta \in \rho(H_1); \tag{2.2}$$

$$G_1(\zeta) = 1 + Q_1(\zeta) \in B(\mathfrak{K}). \tag{2.3}$$

ASSUMPTION 2.4. For every $\zeta \in \rho(H_1) \cap \rho(H_2)$ the inverse $G_1(\zeta)^{-1}$ of $G_1(\zeta)$ exists and belongs to $B(\mathfrak{K})$. Furthermore

$$R_2(\zeta) = R_1(\zeta) - [R_1(\zeta)A^*]^a G_1(\zeta)^{-1} CBR_1(\zeta) \tag{2.4}$$

holds for every $\zeta \in \rho(H_1) \cap \rho(H_2)$.

DEFINITION 2.5. When H_1, H_2, A, B, and C satisfy Assumptions 2.1-2.4, we write

$$H_2 \sim H_1 + A^*CB \sim H_1 + B^*C^*A. \tag{2.5}$$

The following two propositions are proved along the line given in Kato [8].

PROPOSITION 2.6. *Let $\zeta \in \rho(H_1)$. Then, $G_1(\zeta)^{-1}$ exists as an operator in $B(\mathfrak{K})$ if and only if $\zeta \in \rho(H_2)$.*

PROPOSITION 2.7. *Suppose that $H_2 \sim H_1 + A^*CB \sim H_1 + B^*C^*A$ and put*

$$Q_2(\zeta) = C[BR_2(\zeta)A^*]^a \in B(\mathfrak{K}), \qquad \zeta \in \rho(H_2), \tag{2.6}$$

$$G_2(\zeta) = 1 - Q_2(\zeta) \in B(\mathfrak{K}). \tag{2.7}$$

Then, we have

$$G_2(\zeta) = G_1(\zeta)^{-1}, \tag{2.8}$$

$$CBR_2(\zeta) = G_1(\zeta)^{-1} CBR_1(\zeta), \tag{2.9}$$

$$[R_2(\zeta)A^*]^a = [R_1(\zeta)A^*]^a G_1(\zeta)^{-1} \tag{2.10}$$

for $\zeta \in \rho(H_1) \cap \rho(H_2)$. Furthermore

$$H_1 \sim H_2 - A^*CB \sim H_2 - B^*C^*A.$$

2.3. Perturbation given by quadratic forms. The scheme given in § 2.2 is realized if H_1 and H_2 are linked through sesqui-linear forms associated with them. We now discuss such a situation. It is useful not only in appli-

3) According to the agreement made in § 2.1, this assumption includes the requirement that $BR_1(\zeta)A^*$ is closable.

cations but in some part of the abstract approach.

Let $H_j=\int_{-\infty}^{\infty} E_j(d\lambda)$, $j=1, 2$, where E_j is the spectral measure associated with H_j, and put

$$|H_j|^\alpha=\int_{-\infty}^{\infty}|\lambda|^\alpha E_j(d\lambda), \qquad \operatorname{sgn} H_j=\int_{-\infty}^{\infty}\operatorname{sgn}\lambda E_j(d\lambda). \tag{2.11}$$

DEFINITION 2.8. For any θ, $0\leqq\theta\leqq 1/2$, and $j=1, 2$ the sesqui-linear form $h_j^{(\theta)}$ on $\mathfrak{D}(|H_j|^\theta)\times\mathfrak{D}(|H_j|^{1-\theta})$ is defined by

$$h_j^{(\theta)}[u, v]=(\operatorname{sgn} H_j\cdot|H_j|^\theta u,\ |H_j|^{1-\theta}v), \tag{2.12}$$
$$u\in\mathfrak{D}(|H_j|^\theta),\ v\in\mathfrak{D}(|H_j|^{1-\theta}).$$

THEOREM 2.9. *Suppose that there exists* θ, $0\leqq\theta\leqq 1/2$, *such that*

$$\mathfrak{D}(|H_1|^{1-\theta})=\mathfrak{D}(|H_2|^{1-\theta})\equiv\mathfrak{D}_{1-\theta}. \tag{2.13}$$

Put $\mathfrak{D}_\theta=\mathfrak{D}(|H_1|^\theta)=\mathfrak{D}(|H_2|^\theta)$[4]. *Let* $\mathfrak{K}$, A, B, *and* C *be as in Assumption* 2.1 *and assume that*

$$\mathfrak{D}(A)\supset\mathfrak{D}_{1-\theta}, \qquad \mathfrak{D}(B)\supset\mathfrak{D}_\theta, \tag{2.14}$$

$$h_2^{(\theta)}[u, v]=h_1^{(\theta)}[u, v]+(CBu, Av), \qquad u\in\mathfrak{D}_\theta,\ v\in\mathfrak{D}_{1-\theta}. \tag{2.15}$$

Then, $H_2\sim H_1+A^*CB\sim H_1+B^*C^*A$ *in the sense of Definition* 2.5.

EXAMPLE 2.10. If $\theta=1/2$ and H_j are bounded from below, then $h_j^{(1/2)}$ is precisely the quadratic form associated with H_j in the sense of Friedrichs. If $\theta=0$, then we have $\mathfrak{D}(A)\supset\mathfrak{D}(H_1)$, $B\in B(\mathfrak{H}, \mathfrak{K})$, $\mathfrak{D}(H_2)=\mathfrak{D}(H_1)$, and $H_2=H_1+B^*C^*A$. Only these two values of θ may be of practical interest. But no additional complication will be introduced by considering other values of θ simultaneously.

PROOF OF THEOREM 2.9. Assumption 2.2 is obviously satisfied. Put $|R_1(\zeta)|=\int|\lambda-\zeta|^{-1}E_1(d\lambda)$ and $W(\zeta)=\int\exp(-i\arg(\lambda-\zeta))E_1(d\lambda)$. Then

$$\begin{aligned} BR_1(\zeta)A^* &= B|R_1(\zeta)|^\theta W(\zeta)|R_1(\zeta)|^{1-\theta}A^* \\ &\subset B|R_1(\zeta)|^\theta W(\zeta)(A|R_1(\zeta)|^{1-\theta})^*\in B(\mathfrak{K}), \end{aligned} \tag{2.16}$$

whence follows Assumption 2.3. Note that $[BR_2(\zeta)A^*]^a\in B(\mathfrak{K})$ is obtained similarly.

Assumption 2.4 will be verified by the following succession of lemmas.

LEMMA 2.11. *Let* $u=[R_1(\zeta)A^*]^a w$, $w\in\mathfrak{K}$. *Then*, $u\in\mathfrak{D}_\theta$ *and*

$$(h_1^{(\theta)}-\zeta)[u, v]=(w, Av) \qquad \text{for any } v\in\mathfrak{D}_{1-\theta}. \tag{2.17}$$

PROOF. Factorizing $R_1(\zeta)$ as in (2.16), we see readily that $u\in\mathfrak{D}_\theta$. If

4) The second equality follows from (2.13) by means of interpolation.

$v\in\mathfrak{D}(H_1)$, $(h_1^{(\theta)}-\zeta)[u, v]=(u, (H_1-\bar{\zeta})v)=(w, Av)$ so that (2.17) holds. For a general $v\in\mathfrak{D}_{1-\theta}$, (2.17) is proved by the limit procedure. In fact, it suffices to write (2.17) for $v_n=E_1([-n, n])v\in\mathfrak{D}(H_1)$ and note that $Av_n=A|R_1(\zeta)|^{1-\theta}\cdot|H_1-\zeta|^{1-\theta}v_n\to Av$.

LEMMA 2.12. *$G_1(\zeta)$ is one-to-one for every $\zeta\in\rho(H_1)\cap\rho(H_2)$.*

PROOF. Suppose $G_1(\zeta)w=w+Q_1(\zeta)w=0$, $w\in\mathfrak{K}$, and put $u=[R_1(\zeta)A^*]^a w$. From Lemma 2.11 and the relation $w=-Q_1(\zeta)w=-CBu$ it follows that $(h_1^{(\theta)}-\zeta)[u, v]=-(CBu, Av)$ and hence $(h_2^{(\theta)}-\zeta)[u, v]=0$ for all $v\in\mathfrak{D}_{1-\theta}$. This implies that $u\in\mathfrak{D}(H_2)$ and $(H_2-\zeta)u=0$. Since $\zeta\in\rho(H_2)$, we must have $u=0$ and hence $w=-CBu=0$. q. e. d.

LEMMA 2.13. *For every $\zeta\in\rho(H_1)\cap\rho(H_2)$ we have*

$$R_1(\zeta)=R_2(\zeta)+[R_1(\zeta)A^*]^a CBR_2(\zeta)\,, \tag{2.18}$$

$$CBR_1(\zeta)=G_1(\zeta)CBR_2(\zeta)\,, \tag{2.19}$$

$$G_1(\zeta)(1-C[BR_2(\zeta)A^*]^a)=1\,. \tag{2.20}$$

PROOF. Let $u, v\in\mathfrak{H}$ and put $u'=R_2(\zeta)u$. Then,

$$\begin{aligned}(R_1(\zeta)u, v)&=((H_2-\zeta)u', R_1(\bar{\zeta})v)\\&=(h_1^{(\theta)}-\zeta)[u', R_1(\bar{\zeta})v]+(CBu', AR_1(\bar{\zeta})v)\\&=(u', v)+([R_1(\zeta)A^*]^a CBu', v)\,,\end{aligned} \tag{2.21}$$

from which (2.18) follows. By inserting $v=B^*w$, $w\in\mathfrak{D}(B^*)$, into (2.21) and noting that w ranges over a dense subset of $\mathfrak{K}$, we see immediately that $BR_1(\zeta)=BR_2(\zeta)+[BR_1(\zeta)A^*]^a CBR_2(\zeta)$. Multiplying by C from the left and recalling (2.3), we get (2.19). Finally, we multiply (2.19) by A^* from the right. Then (2.20) follows after a simple manipulation. q. e. d.

Completion of the proof of Theorem 2.9. The existence of $G_1(\zeta)^{-1}\in B(\mathfrak{K})$ follows from Lemma 2.12 and (2.20). Then (2.19) implies $CBR_2(\zeta)=G_1(\zeta)^{-1}CBR_1(\zeta)$. (2.4) is proved by inserting this into (2.18). q. e. d.

§ 3. Perturbation of spectral representations and construction of wave operators.

3.1. Assumptions. In order to formulate main theorems, we will introduce several assumptions. We first write down all the assumptions and make some comments afterwards (cf. Remark 3.6, Proposition 3.7).

Let $H_j=\int\lambda E_j(d\lambda)$, $j=1, 2$, be self-adjoint in $\mathfrak{H}$ and let $\mathfrak{K}$, A, B and C satisfy Assumption 2.1. Let I be a non-empty open set in R^1. Since we want to discuss the problem in a form localized with respect to the spectral parameter, our attention will be restricted to the spectral properties of the operators $E_j(I)H_j$. The set I will be fixed throughout the entire discussion.

ASSUMPTION 3.1. $H_2 \sim H_1 + A^*CB \sim H_1 + B^*C^*A$ in the sense of Definition 2.5.

ASSUMPTION 3.2. There exist a Hilbert space $\mathfrak{C}$ and a unitary operator F from $E_1(I)\mathfrak{H}$ onto $L^2(I;\mathfrak{C})$ such that for every Borel set $I' \subset I$ one has $FE_1(I')F^{-1} = \chi_{I'}\cdot$, where $\chi_{I'}\cdot$ stands for the operator of multiplication by the characteristic function $\chi_{I'}$ of I' (i.e. $(\chi_{I'}\cdot u)(\lambda) = \chi_{I'}(\lambda)u(\lambda)$ a.e. in I).

ASSUMPTION 3.3. There exist $B(\mathfrak{K}, \mathfrak{C})$-valued functions $T(\lambda; A)$ and $T(\lambda; B)$, $\lambda \in I$, on I such that: i) $T(\cdot\,; A)$ and $T(\cdot\,; B)$ are locally Hölder continuous in I with respect to the operator norm; and ii) there exist dense subsets $\mathfrak{D} \subset \mathfrak{D}(A^*)$ and $\mathfrak{D}' \subset \mathfrak{D}(B^*)$ such that for any $u \in \mathfrak{D}$ and $v \in \mathfrak{D}'$ one has

$$T(\lambda; A)u = (FE_1(I)A^*u)(\lambda) \qquad \text{a.e. in } I, \tag{3.1}$$

$$T(\lambda; B)v = (FE_1(I)B^*v)(\lambda) \qquad \text{a.e. in } I. \tag{3.2}$$

ASSUMPTION 3.4. For one or equivalently all $\zeta \in \rho(H_1)$

$$\text{either } BR_1(\zeta) \in B_\infty(\mathfrak{K}) \quad \text{or} \quad AR_1(\zeta) \in B_\infty(\mathfrak{K}). \tag{3.3}$$

ASSUMPTION 3.5. The subspace generated by $\{E_j(I')A^*u \mid u \in \mathfrak{D}(A^*),\ I'$ is a Borel subset of $I\}$ is dense in $E_j(I)\mathfrak{H}$, $j = 1, 2$.

REMARK 3.6. Since we are concerned mostly with applications to differential operators, we assumed from the outset that H_1 has a nice spectral representation in I (Assumption 3.2). Assumption 3.2 implies in particular that H_1 is absolutely continuous in I. If $\mathfrak{H}$ is separable, it is equivalent to assuming that H_1 is absolutely continuous in I with constant multiplicity. In a separable case $\mathfrak{C}$ and F are determined uniquely, up to the unitary equivalence for $\mathfrak{C}$ and up to a decomposable unitary operator in $L^2(I;\mathfrak{C})$ for F. However, we need to single out one particular $\mathfrak{C}$ and F to state Assumption 3.3.

Assumptions 3.3 and 3.4 are our main assumptions. Assumption 3.3, a prototype of which was introduced in [13], plays an essential role in application and allows the coefficients of perturbing differential operator to decay as slow as $O(|x|^{-(1+\varepsilon)})$, $\varepsilon > 0$. In Assumption 3.4, a relative compactness of the perturbation, the appearance of the only one factor A or B of the perturbation allows the perturbing differential operator to have the same order as the unperturbed one. Assumption 3.5 was introduced to exclude the subspace where the perturbation has no effects.

PROPOSITION 3.7. *Let Assumptions* 2.2, 3.2 *and* 3.3 *be satisfied. If $I' \Subset I$, then $[E_1(I')A^*]^a \in B(\mathfrak{K}, \mathfrak{H})$ and*

$$\chi_{I'}(\lambda)T(\lambda; A)u = (F[E_1(I')A^*]^a u)(\lambda), \qquad a.e., \tag{3.4}$$

for any $u \in \mathfrak{K}$. Similar statement holds for B, too. Furthermore, (3.1) *and* (3.2) *hold for any* $u \in \mathfrak{D}(A^*)$ *and* $v \in \mathfrak{D}(B^*)$.

COROLLARY 3.8. $T(\lambda; A)$ *and* $T(\lambda; B)$ *are unique and are independent of the choice of* $\mathfrak{D}$ *and* $\mathfrak{D}'$.

PROOF OF PROPOSITION 3.7. $[E_1(I')A^*]^a \in B(\mathfrak{K}, \mathfrak{H})$ follows from Assumption 2.2 at once. Next let $u \in \mathfrak{K}$ and let $u_n \in \mathfrak{D}$, $u_n \to u$. (3.1) and Assumption 3.2 show that

$$\chi_{I'}(\lambda)T(\lambda; A)u_n = (FE_1(I')A^*u_n)(\lambda) = (F[E_1(I')A^*]^a u_n)(\lambda), \qquad \text{a. e.}$$

(3.4) follows from this by letting $n \to \infty$ (along with a suitable subsequence, if necessary). Other assertions are checked easily. q. e. d.

3.2. Theorems. Several familiar conclusions in the scattering theory can be derived from the assumptions made in § 3.1. These are summarized in the following theorems, of which the first two are of preliminary nature. The following notations will be used[5]:

$$\Pi^{\pm} = \{\zeta \mid \mathrm{Im}\, \zeta \gtrless 0\}, \qquad \Pi_I^{\pm} = \Pi^{\pm} \cup I.$$

THEOREM 3.9. *The* $B(\mathfrak{K})$*-valued function* $G_1 : \Pi^{\pm} \to B(\mathfrak{K})$ *can be extended uniquely to a locally Hölder continuous* $B(\mathfrak{K})$*-valued function* $G_{1\pm} : \Pi_I^{\pm} \to B(\mathfrak{K})$. *In particular, for every* $\lambda \in I$ *the boundary value*

$$\lim_{\varepsilon \downarrow 0} G_1(\lambda \pm i\varepsilon) = G_{1\pm}(\lambda)$$

exists with respect to the operator norm.

THEOREM 3.10. *Let*

$$e_{\pm} = \{\lambda \in I \mid G_{1\pm}(\lambda) \text{ is not one-to-one}\}. \tag{3.5}$$

Then, $e_{\pm}$ *is a closed set with the (one-dimensional) Lebesgue measure* 0. *If* $\lambda \in I - e_{\pm}$, *then* $G_{1\pm}(\lambda)^{-1} \in B(\mathfrak{K})$. *The* $B(\mathfrak{K})$*-valued function* $G_2 : \Pi^{\pm} \to B(\mathfrak{K})$ *can be extended uniquely to a locally Hölder continuous* $B(\mathfrak{K})$*-valued function* $G_{2\pm}$: $\Pi_{I-e_{\pm}}^{\pm} \to B(\mathfrak{K})$. $G_{2\pm}$ *is the inverse of* $G_{1\pm}$. *In particular, for every* $\lambda \in I - e_{\pm}$ *the boundary value*

$$\lim_{\varepsilon \downarrow 0} G_2(\lambda \pm i\varepsilon) = G_{2\pm}(\lambda)$$

exists with respect to the operator norm and it satisfies $G_{2\pm}(\lambda) = G_{1\pm}(\lambda)^{-1}$.

In the sequel when we want to refer to the extension of Q_j, $j = 1, 2$, or other operator valued functions, we will use the notation similar to $G_{1\pm}$ without any comment. Thus, for instance $G_{1\pm}(\lambda) = 1 + Q_{1\pm}(\lambda)$. Usually in the literature the notation $G_{1\pm}$ is used only for the boundary values $G_{1\pm}(\lambda)$ (more often denoted by $G_1(\lambda \pm i0)$). In this paper, however, $G_{1\pm}(\zeta)$ will be used also

5) Here and in what follows, whenever the double sign $\pm$, $\mp$ appear in an assertion or a formula, we tacitly agree that two statements or formulae are asserted to hold, the one for the upper signs taken throughout and the other for the lower signs.

for non-real ζ. Thus $G_{1\pm}(\zeta)=G_1(\zeta)$ if $\operatorname{Im}\zeta\neq 0$.

Let us now put

$$e=e_+\cup e_- \tag{3.6}$$

and state our main theorems.

THEOREM 3.11. *Let Assumptions* 3.1-3.5 *be satisfied and let* $G_{2\pm}$ *and* e *be as above. Then, there exists a uniquely determined unitary operator* $F_\pm$ *from* $E_2(I-e)\mathfrak{H}$ *onto* $L^2(I;\mathfrak{C})$ *such that for every Borel set* $I'\subset I-e$ *and every* $u\in\mathfrak{D}(A^*)$ *one has*

$$(F_\pm E_2(I')A^*u)(\lambda)=\chi_{I'}(\lambda)T(\lambda;A)G_{2\pm}(\lambda)u \qquad a.e.\ in\ I. \tag{3.7}$$

Furthermore, $F_\pm$ *satisfies* $F_\pm E_2(I')F_\pm^{-1}=\chi_{I'}\cdot$ *for every Borel set* $I'\subset I-e$.

THEOREM 3.12. *Let* $W_\pm=W_\pm(H_2,H_1;I)=F_\pm^*F$. *Then,* $W_\pm$ *is a unitary operator from* $E_1(I)\mathfrak{H}$ *onto* $E_2(I-e)\mathfrak{H}$ *and satisfies the intertwining relation* $H_2W_\pm=W_\pm H_1$ *on* $E_1(I)\mathfrak{H}$. *The operator* $S=S(H_2,H_1;I)=W_+^*W_-$ *is a unitary operator in* $E_1(I)\mathfrak{H}$ *which commutes with* H_1.

THEOREM 3.13. *Let Assumptions* 3.1-3.5 *be satisfied and let* $W_\pm=W_\pm(H_2,H_1;I)$ *be as constructed in Theorem* 3.12. *Let* ϕ *be a real valued Borel measurable function on* I *such that*

$$\int_0^\infty\left|\int_I f(\lambda)e^{-it\phi(\lambda)-i\xi\lambda}d\lambda\right|^2 d\xi\longrightarrow 0\,,\qquad t\to\infty\,, \tag{3.8}$$

for any $f\in L^2(I)$. *Then, for any* $u\in E_1(I)\mathfrak{H}$ *the limit in the next formula exists and*

$$\lim_{t\to\pm\infty}e^{it\phi(H_2)}e^{-it\phi(H_1)}u=W_\pm u\,,\qquad u\in E_1(I)\mathfrak{H}\,. \tag{3.9}$$

$W_\pm$ are called *wave operators* (associated with I) and S the *scattering operator*. Theorem 3.13 shows that the so-called *invariance principle* for wave operators holds. It also ensures that $W_\pm$ and hence $F_\pm$ do not depend on the choice of A, B, and C so long as they satisfy Assumptions 3.1-3.5.

3.3. Example. The following example of a Sturm-Liouville operator in R^1 will be used occasionally for the purpose of illustration and motivation. However, the results to be derived on the spectral properties of this operator is well-known. Applications to partial differential operators will be given in subsequent publications (see [**15**]).

EXAMPLE 3.14. Let $\mathfrak{H}=L^2=L^2(R^1)$ and let

$$L_s^2=L_s^2(R^1)=\{u\,|\,(1+|x|^2)^{s/2}u(x)\in L^2\}\,,$$

$$H_s^1=H_s^1(R^1)=\{u\in L_s^2\,|\,u'\in L_s^2\}\,,$$

where $u'=du/dx$ is taken in the sense of distribution and s is a real number. We write $H^1=H_0^1$. Put

$$h_1[u, v]=\int_{-\infty}^{\infty} u'\bar{v}'dx, \qquad u, v\in H^1,$$

$$h_2[u, v]=h_1[u, v]+\int_{-\infty}^{\infty}(pu'\bar{v}'+qu\bar{v})dx,$$

where p and q are real bounded measurable functions such that i) $1+p(x)\geqq c_1>0$; ii) $\max\{|p(x)|, |q(x)|\}\leqq c_2(1+|x|^2)^{-\delta/2}$ with $c_2>0$, $\delta>1$. h_2 corresponds to the Sturm-Liouville operator $-((1+p)u')'+qu$.

In this example one has $\theta=1/2$ and $\mathfrak{D}_{1/2}=H^1$. We write $p(x)=c_1(x)(1+|x|^2)^{-\delta/2}$ and $q(x)=c_2(x)(1+|x|^2)^{-\delta/2}$, $c_j\in L^\infty(R^1)$. Let $\mathfrak{K}=L^2\oplus L^2$ and put $Au=Bu=\{(1+|x|^2)^{-\delta/4}u', (1+|x|^2)^{-\delta/4}u\}$, $C\{u_1, u_2\}=\{c_1u_1, c_2u_2\}$, where $\mathfrak{D}(A)=\mathfrak{D}(B)=\{u\in L^2\,|\,u'\in L^2_{-\delta/2}\}$.

We take $I=(0, \infty)$ and $\mathfrak{C}=\boldsymbol{C}^2$, the two-dimensional unitary space. Denoting by $\mathcal{F}$ the one-dimensional Fourier transform and writing for brevity $\nu(x)=(1+|x|^2)^{1/2}$, explicit forms of F and $T(\lambda; A)$ can be written as follows:

$$(Fu)(\lambda)=2^{-1/2}\lambda^{-1/4}\{(\mathcal{F}u)(\lambda^{1/2}), (\mathcal{F}u)(-\lambda^{1/2})\}\in L^2((0, \infty); \boldsymbol{C}^2),$$

$$\begin{aligned}T(\lambda; A)u&=(F(\nu^{-\delta/2}u)')(\lambda)+(F\nu^{-\delta/2}u)(\lambda)\\&=2^{-1/2}\lambda^{-1/4}\{(\mathcal{F}(\nu^{-\delta/2}u)')(\lambda^{1/2}), (\mathcal{F}(\nu^{-\delta/2}u)')(-\lambda^{1/2})\}+\cdots\\&=2^{-1/2}\lambda^{-1/4}\{(1+i\lambda^{1/2})(\mathcal{F}\nu^{-\delta/2}u)(\lambda^{1/2}), (1-i\lambda^{1/2})(\mathcal{F}\nu^{-\delta/2}u)(-\lambda^{1/2})\}.\end{aligned}$$

Note that, since $\delta>1/2$, $(\mathcal{F}\nu^{-\delta/2}u)(\xi)$ is a Hölder continuous function. Thus, Assumptions 5.2 and 5.3 are verified. Other assumptions being checked immediately, we can apply theorems in § 3.2 to this problem.

§ 4. Proof of Theorems 3.9—3.13.

Proofs of Theorems 3.9-3.13 were given essentially, but somewhat scattered, in [9], [11], and [13]. Partly for the convenience of later reference we give the proof of Theorems 3.9-3.12 in a form adapted to the present setting. Principal changes are as follows: there are various simplifications owing to the fact that we are concerned only with the smooth perturbation; an iteration is needed in the proof of Theorem 3.10 because our compactness assumption (Assumption 3.4) is weaker than those in earlier papers; the method of proof of the surjectivity of $F_\pm$ seems to be new. Since Theorem 3.13 as well as its proof will not be referred to later, we omit the proof of Theorem 3.13, which is similar to (but somewhat simpler than) the proof of Theorem 7.1 of [11].

4.1. Proof of Theorems 3.9 and 3.10.

PROPOSITION 4.1. *Let X and Y be operators each of which is equal to either A or B. Let f be a bounded Borel measurable complex valued function* *on I and let $I' \Subset I$. Then,*

$$[Yf(H_1)E_1(I')X^*]^a = \int_{I'} f(\lambda)T(\lambda\,;\,Y)^*T(\lambda\,;\,X)d\lambda\,, \tag{4.1}$$

where the integral on the right side is the Bochner integral in $B(\mathfrak{K})$.

PROOF. Because of the continuity of T-operators the Bochner integral on the right side of (4.1) exists. We denote it by K. Let $u \in \mathfrak{D}(X^*)$, $v \in \mathfrak{D}(Y^*)$, and $[a, \lambda] \subset I$, where a lies left to I'. Then, by virtue of Assumption 3.2 and Proposition 3.7 we have

$$(E_1([a, \lambda])X^*u,\ Y^*v) = \int_a^\lambda (T(\mu\,;\,X)u,\ T(\mu\,;\,Y)v)d\mu \tag{4.2}$$

and hence by differentiation

$$(d/d\lambda)(E_1[a, \lambda]X^*u,\ Y^*v) = (T(\lambda\,;\,Y)^*T(\lambda\,;\,X)u,\ v)\,, \qquad \text{a. e.} \tag{4.3}$$

Since H_1 is absolutely continuous in I, it follows from (4.3) that $(Yf(H_1)E_1(I')X^*u, v) = (Ku, v)$. This proves (4.1) because u and v range over dense subsets. q. e. d.

Among operators $T(\lambda\,;\,Y)^*T(\lambda\,;\,X)$ the following two are most frequently used and are denoted by the special notations:

$$M_1(\lambda) = T(\lambda\,;\,A)^*T(\lambda\,;\,A)\,, \qquad \lambda \in I\,, \tag{4.4}$$

$$\tilde{M}_1(\lambda) = T(\lambda\,;\,B)^*T(\lambda\,;\,A)\,, \qquad \lambda \in I\,. \tag{4.5}$$

PROOF OF THEOREM 3.9. It suffices to verify the assertion with $\Pi_I^\pm$ replaced by $\Pi_{I'}^\pm = \Pi^\pm \cup I'$, $I' \Subset I$. Put $J = R^1 - I'$.

By the resolvent equation we get

$$Q_1(\zeta) = Q_1(i) + (\zeta - i)\{K_1(\zeta) + K_2(\zeta)\}\,,$$

where

$$\begin{aligned} K_1(\zeta) &= CBR_1(\zeta)E_1(J)[R_1(i)A^*]^a \\ &= \{CBR_1(i)E_1(J) + (\zeta - i)CBR_1(i)R_1(\zeta)E_1(J)\}[R_1(i)A^*]^a\,, \\ K_2(\zeta) &= CBR_1(\zeta)E_1(I')[R_1(i)A^*]^a \\ &= C\int_{I'} \frac{1}{(\mu-\zeta)(\mu-i)} \tilde{M}_1(\mu)d\mu \end{aligned}$$

(cf. Proposition 4.1 and (4.5)). Since $\tilde{M}_1$ is Hölder continuous, $K_2(\zeta)$ can be extended to $\Pi_{I'}^\pm$ as required in the theorem (Privalov's theorem), while $K_1(\zeta)$ has an analytic extension to $\Pi^+ \cup I' \cup \Pi^-$. q. e. d.

PROOF OF THEOREM 3.10. As before we obtain

$$\begin{aligned} G_1(\zeta) &= G_1(i)\{1 + K(\zeta)\}\,, \\ K(\zeta) &= (\zeta - i)G_1(i)^{-1}CBR_1(\zeta)[R_1(i)A^*]^a\,, \qquad \zeta \in \Pi^\pm\,. \end{aligned} \tag{4.6}$$

$K(\zeta)$ has the following properties: i) $K(\zeta)$ is holomorphic in $\Pi^{\pm}$ as can be seen by writing $BR_1(\zeta)=BR_1(i)+(\zeta-i)BR_1(i)R_1(\zeta)$; ii) $K: \Pi^{\pm}\to B(\mathfrak{K})$ can be extended to a continuous function $K_{\pm}: \Pi_I^{\pm}\to B(\mathfrak{K})$, because $K(\zeta)=G_1(i)^{-1}G_1(\zeta)-1$; iii) $K(\zeta)\in B_\infty(\mathfrak{K})$ if $\zeta\in\Pi^{\pm}$ (Assumption 3.4); and iv) $1+K(\zeta)$ is invertible in $B(\mathfrak{K})$ if $\zeta\in\Pi^{\pm}$. It follows from i)-iv) (cf. Lemma 6.2 of [**12**]) that $e'_{\pm}=\{\lambda\in I\,|\,1+K_{\pm}(\lambda)$ is not one-to-one$\}$ is a closed set with the (one-dimensional) Lebesgue measure 0. However, $e'_{\pm}=e_{\pm}$ because (4.6) remains true for the boundary values $G_{1\pm}(\lambda)$ etc. Furthermore, the complete continuity of $K_{\pm}(\lambda)$ implies $G_{1\pm}(\lambda)^{-1}\in B(\mathfrak{K})$, $\lambda\in I-e_{\pm}$. Other assertions are direct consequences of the continuity of the inverse in $B(\mathfrak{K})$. q. e. d.

For later use we add the following remark. Since $K_{\pm}(\lambda)\in B_\infty(\mathfrak{K})$, it follows from (4.6) that $\dim\mathfrak{N}(G_{1\pm}(\lambda))=\dim\mathfrak{N}(G_{1\pm}(\lambda)^*)$. In particular, $\lambda_0\in e_{\pm}$ if and only if $G_{1\pm}(\lambda_0)^*$ is not one-to-one.

4.2. Proof of Theorems 3.11 and 3.12. Theorem 3.12 is an immediate consequence of Theorem 3.11.

In order to prove Theorem 3.11, we introduce a perturbed form of T-operator as follows:

$$T_{\pm}(\lambda;A)=T(\lambda;A)G_{2\pm}(\lambda)\in B(\mathfrak{K},\mathfrak{C}), \qquad \lambda\in I-e. \tag{4.7}$$

It is clear that $T_{\pm}(\lambda;A)$ is a locally Hölder continuous $B(\mathfrak{K},\mathfrak{C})$-valued function on $I-e$.

PROPOSITION 4.2. i) *For every* $\lambda\in I-e$ *we have*

$$T_+(\lambda;A)^*T_+(\lambda;A)=T_-(\lambda;A)^*T_-(\lambda;A)\equiv M_2(\lambda). \tag{4.8}$$

ii) *Let* f *be as in Proposition* 4.1 *and let* $I'\Subset I-e$. *Then,*

$$[Af(H_2)E_2(I')A^*]^a=\int_{I'}f(\lambda)M_2(\lambda)d\lambda. \tag{4.9}$$

PROOF. For any $\lambda\in R^1$ and $\varepsilon>0$ put

$$\delta_\varepsilon(H_j-\lambda)=(2\pi i)^{-1}\{R_j(\lambda+i\varepsilon)-R_j(\lambda-i\varepsilon)\}, \qquad j=1,2. \tag{4.10}$$

For brevity we write $\zeta=\lambda\pm i\varepsilon$. (The following arguments are valid for $\zeta=\lambda+i\varepsilon$ as well as for $\zeta=\lambda-i\varepsilon$.) Let $u, v\in\mathfrak{D}(A^*)$. Then, by the resolvent equation and (2.10) we get

$$\begin{aligned}(\delta_\varepsilon(H_2-\lambda)A^*u, A^*v)&=\pi^{-1}\varepsilon(R_2(\zeta)A^*u, R_2(\zeta)A^*v)\\&=\pi^{-1}\varepsilon([R_1(\zeta)A^*]^aG_2(\zeta)u, [R_1(\zeta)A^*]^aG_2(\zeta)v)\\&=([A\delta_\varepsilon(H_1-\lambda)A^*]^aG_2(\zeta)u, G_2(\zeta)v).\end{aligned} \tag{4.11}$$

Let I' and I'' be intervals such that $I''\Subset I'\Subset I-e$ and let $\lambda\in\bar{I}''$. Put $J=R^1-I'$. Then, Proposition 4.1 implies that

(4.12) $$[A\delta_\varepsilon(H_1-\lambda)A^*]^a = \frac{\varepsilon}{\pi}\int_{I'}\frac{1}{(\mu-\lambda)^2+\varepsilon^2}M_1(\mu)d\mu + [AE_1(J)\delta_\varepsilon(H_1-\lambda)A^*]^a .$$

As $\varepsilon \downarrow 0$ the second term on the right side tends to 0 uniformly for $\lambda \in I''$. Applying the well-known result concerning the boundary value of integrals of Poisson type to the first term, we therefore see that

(4.13) $$\lim_{\varepsilon\downarrow 0}[A\delta_\varepsilon(H_1-\lambda)A^*]^a = M_1(\lambda)$$

uniformly for $\lambda \in I''$. Since $G_2(\lambda \pm i\varepsilon)$ also converge uniformly, the right side of (4.11) converges uniformly for $\lambda \in I''$. Thus, letting $\varepsilon \downarrow 0$ in (4.11) we get

(4.14) $$\frac{d}{d\lambda}(E_2([a, \lambda))A^*u, A^*v) = (M_1(\lambda)G_{2\pm}(\lambda)u, G_{2\pm}(\lambda)v) = (T_\pm(\lambda; A)^*T_\pm(\lambda; A)u, v), \quad \text{a.e. in } I-e .$$

(4.8) follows from this and the continuity of $T_\pm(\lambda; A)$.

The uniform convergence mentioned above yields in particular the boundedness of $(A\delta_\varepsilon(H_1-\lambda)A^*u, v)$ in a (complex) neighborhood of $\bar{I}''$. As is well-known, this implies that the set function $(E_2(\cdot)A^*u, A^*v)$ is absolutely continuous in I''. Noting this fact, (4.9) is derived from (4.14) in the same way as (4.1) was derived from (4.3). q. e. d.

PROPOSITION 4.3. *There exists a uniquely determined isometric operator $F_\pm$ from $E_2(I-e)\mathfrak{H}$ into $L^2(I; \mathfrak{C})$ satisfying* (3.7). *Furthermore,* $(F_\pm E_2(I')F_\pm^{-1}u)(\lambda) = \chi_{I'}(\lambda)u(\lambda)$, *a.e., for any $u \in F_\pm E_2(I-e)\mathfrak{H}$ and any Borel set $I' \subset I-e$.*

PROOF. By the definition of $T_\pm$ formula (3.7) takes the form

(4.15) $$(F_\pm E_2(I')A^*u)(\lambda) = \chi_{I'}(\lambda)T_\pm(\lambda; A)u , \quad \text{a.e. in } I .$$

(4.15) will be used as the basis to define $F_\pm$.

For any $u \in E_2(I-e)\mathfrak{H}$ which can be expressed as

(4.16) $$u = \sum_{k=1}^{l} E_2(I_k)A^*u_k , \quad u_k \in \mathfrak{D}(A^*) , \quad I_k \Subset I-e ,$$

define the $\mathfrak{C}$-valued function $u_\pm$ by

(4.17) $$u_\pm(\lambda) = \sum_{k=1}^{l} \chi_{I_k}(\lambda)T_\pm(\lambda; A)u_k .$$

Evidently $u_\pm \in L^2(I; \mathfrak{C})$. By virtue of Proposition 4.2 we see that

(4.18) $$\|u_\pm\|^2 = \sum_{j,k=1}^{l}\int_{I_j\cap I_k}(T_\pm(\lambda; A)u_j, T_\pm(\lambda; A)u_k)d\lambda = \sum_{j,k=1}^{l}(E_2(I_j)A^*u_j, E_2(I_k)A^*u_k) = \|u\|^2 .$$

Hence, $u_\pm$ is determined by u uniquely as an element of $L^2(I;\mathfrak{C})$, being independent of the way of expressing u in the form (4.16). Since the set of all u admitting expression (4.16) forms a dense set in $E_2(I-e)\mathfrak{H}$ (cf. Assumption 3.5), the correspondence $u\rightarrow u_\pm$ can be extended to an isometric operator $F_\pm$ from $E_2(I-e)\mathfrak{H}$ into $L^2(I;\mathfrak{C})$.

(4.15) with $I'\Subset I$ is a special case of (4.17). For a general I' (4.15) is derived by the limit procedure. The uniqueness of $F_\pm$ and the last assertion of the proposition follow from (4.15) at once. q. e. d.

What is left is to prove that $F_\pm$ is onto $L^2(I;\mathfrak{C})$. The proof will be based on the following three lemmas. In these lemmas $\mathfrak{K}$ and $\mathfrak{C}$ are assumed to be Hilbert spaces and $K(\lambda)$ is a (norm) continuous $B(\mathfrak{K},\mathfrak{C})$-valued function on I.

LEMMA 4.4. *Let at least one of $\mathfrak{K}$ and $\mathfrak{C}$ be separable and let $K(\lambda)\mathfrak{K}$ be dense in $\mathfrak{C}$ for a. e. $\lambda\in I$. Suppose that $h\in L^2(I;\mathfrak{C})$ satisfies*

$$\int_{I'}(K(\lambda)u,\ h(\lambda))d\lambda=0 \tag{4.19}$$

for every $u\in\mathfrak{K}$ and every $I'\Subset I$. Then, $h(\lambda)=0$ a. e.

PROOF. The proof can be reduced to the case that $\mathfrak{K}$ is separable. In fact, if $\mathfrak{C}$ is separable, there exists a separable subspace $\mathfrak{K}_1$ of $\mathfrak{K}$ such that $K(\lambda)^*\mathfrak{C}\subset\mathfrak{K}_1$ for every $\lambda\in I$. Let $K_1(\lambda)$ be the restriction of $K(\lambda)$ to $\mathfrak{K}_1$ and P the projection on $\mathfrak{K}_1$. Then, $K(\lambda)u=K_1(\lambda)Pu$ and hence the assumptions of the lemma are satisfied by $\mathfrak{K}_1$ and $K_1(\lambda)$.

Assume that $\mathfrak{K}$ is separable and take a countable dense subset $\{u_n\}$ of $\mathfrak{K}$. Then, $(K(\lambda)u_n,\ h(\lambda))=0$ a. e. by (4.19). Since $\{K(\lambda)u_n\}$ is dense in $\mathfrak{C}$ a. e. by the assumption, we get $h(\lambda)=0$ a. e. q. e. d.

LEMMA 4.5. *Let $P(\lambda)$ be the projection in $\mathfrak{C}$ on the closure of $K(\lambda)\mathfrak{K}$ and $Q(\lambda)$ the projection in $\mathfrak{K}$ on $\mathfrak{N}(K(\lambda))$. Then, $P(\lambda)$ and $Q(\lambda)$ are strongly measurable in I.*

PROOF. Since $P(\lambda)$ is the projection on $\mathfrak{N}(K(\lambda)^*)$, it suffices to prove the assertion for $Q(\lambda)$. Put $F(\lambda)=K(\lambda)^*K(\lambda)$. Let $F(\lambda)=\int_{0-}^{\infty}\mu E(d\mu;\lambda)$ be the spectral resolution of $F(\lambda)$ and let $R(\zeta;\lambda)=(F(\lambda)-\zeta)^{-1}$. Then

$$Q(\lambda)u=E(\{0\};\lambda)u=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi i}\int_{-1}^{0}\{R(\mu-i\varepsilon;\lambda)-R(\mu+i\varepsilon;\lambda)\}ud\mu\,.$$

The strong measurability of the right side is obvious. q. e. d.

LEMMA 4.6. *Let $\mathfrak{C}$ be separable. Assume that there exists a subset $\mathfrak{D}$ of $\mathfrak{K}$ such that the linear hull of $\{\chi_{I'}(\cdot)K(\cdot)u\,|\,u\in\mathfrak{D},\ I'\Subset I\}$ is dense in $L^2(I;\mathfrak{C})$. Then $K(\lambda)\mathfrak{K}$ is dense in $\mathfrak{C}$ for a. e. $\lambda\in I$.*

PROOF. Let $P(\lambda)$ be the projection in $\mathfrak{C}$ on the closure of $K(\lambda)\mathfrak{K}$. Take a countable dense subset $\{c_n\}$ of $\mathfrak{C}$ and put $c_n(\lambda)=(1-P(\lambda))c_n$. Then, if the conclusion of the lemma were false, there would exist $c_{n_0}(\lambda)$ which does not vanish on a set of positive measure. We write $c(\lambda)=c_{n_0}(\lambda)$. Then: i) by Lemma 4.5 c is strongly measurable and hence $c\in L^2(I;\mathfrak{C})$; ii) $c(\lambda)$ is orthogonal to $K(\lambda)\mathfrak{K}$, $\lambda\in I$; iii) $c\neq 0$ as an element of $L^2(I;\mathfrak{C})$. Property ii) implies that $(\chi_{I'}(\cdot)K(\cdot)u, c(\cdot))_{L^2(I;\mathfrak{C})}=0$ for every $u\in\mathfrak{D}$ and $I'\Subset I$. Hence, by the assumption we have $c=0$ in $L^2(I;\mathfrak{C})$, which contradicts property iii).

q. e. d.

Completion of the proof of Theorem 3.11. We assume that $h\in L^2(I;\mathfrak{C})$ is orthogonal to $F_+E_2(I-e)\mathfrak{H}$ and prove that $h=0$. For brevity we put

$$T(\lambda)=T(\lambda;A), \qquad T_+(\lambda)=T_+(\lambda;A)=T(\lambda)G_{2+}(\lambda). \tag{4.20}$$

Let $\mathfrak{C}_1$ be a separable closed subspace of $\mathfrak{C}$ such that $h(\lambda)\in\mathfrak{C}_1$ a. e. and let P be the projection in $\mathfrak{C}$ on $\mathfrak{C}_1$. It follows by virtue of (4.15) that

$$\int_{I'}(PT_+(\lambda)u, h(\lambda))d\lambda=\int_{I'}(T_+(\lambda)u, h(\lambda))d\lambda=0$$

for any $u\in\mathfrak{D}(A^*)$ and $I'\Subset I-e$. Since $T_+(\lambda)$ is uniformly bounded on I', this relation holds for all $u\in\mathfrak{K}$.

We now want to apply Lemma 4.4. to $K(\lambda)=PT_+(\lambda)$ and $\mathfrak{C}=\mathfrak{C}_1$ and conclude $h=0$. For this it suffices to show that $PT_+(\lambda)\mathfrak{K}$ is dense in $\mathfrak{C}_1$ a.e. However, since $G_{2+}(\lambda)$, $\lambda\in I-e$, is onto, it suffices by (4.20) to show that

$$PT(\lambda)\mathfrak{K} \text{ is dense in } \mathfrak{C}_1 \text{ for a. e. } \lambda\in I. \tag{4.21}$$

(4.21) will be verified by applying Lemma 4.6 to $K(\lambda)=PT(\lambda)$ and $\mathfrak{D}=\mathfrak{D}(A^*)$. Namely, by virtue of (3.1) and Assumption 3.2 we have a. e.

$$\chi_{I'}(\lambda)PT(\lambda)u=P(FE_1(I')A^*u)(\lambda), \qquad u\in\mathfrak{D}(A^*), \qquad I'\Subset I. \tag{4.22}$$

However, Assumption 3.5 and the unitarity of F imply that the linear hull of $\{P(FE(I')A^*u)(\cdot)\}$ is dense in $L^2(I;\mathfrak{C}_1)$. Hence, (4.21) is proved by (4.22) and Lemma 4.6.

The assertion for F_- is proved in the same way. q. e. d.

§ 5. Discreteness of the singular spectrum of H_2.

5.1. Introduction. It follows from Theorem 3.12 that the singular spectrum of H_2 in I is confined in a closed null set $e=e_+\cup e_-$. The next problem is to obtain more information about the singular spectrum, desirably to conclude that it is discrete in I. Here, the discreteness of the singular spectrum means that $E_2(e\cap\Gamma)$ is finite-dimensional for any compact subset Γ of I.

We shall try to investigate this problem still in the framework of abstract operator theory. In order to simulate typical situations in differential operators, several additional assumptions will have to be introduced whose meaning can only be clarified by examples. Thus, it may seem that more direct approaches in concrete problems are preferable. It is hoped, however, that the present formulation may have an advantage of enabling us to avoid the repetition of similar arguments in applications.

In this section we shall be concerned solely with the situation described in § 2.3. Namely, we assume (2.13) and (2.14) and define quadratic forms $h_j^{(\theta)}$ by (2.12). For brevity, we write h_j instead of $h_j^{(\theta)}$. Then, H_1 and H_2 are related through the relation

$$h_2[u, v]=h_1[u, v]+(CBu, Av), \qquad u\in\mathfrak{D}_\theta,\ v\in\mathfrak{D}_{1-\theta}. \tag{5.1}$$

If $\lambda_0\in e_\pm$, there exists $w_\pm\in\mathfrak{K}$ such that $w_\pm\neq 0$ and

$$w_\pm+Q_{1\pm}(\lambda_0)^*w_\pm=0 \tag{5.2}$$

(cf. the remark at the end of § 4.1). Our argument consists of the following three steps:

(1) To derive from (5.2) that λ_0 is an eigenvalue of h_2 in a generalized sense with an eigenvector related to $w_\pm$;

(2) To show that $w_\pm$ has a "decaying property";

(3) To show that the singular spectrum of H_2 is discrete.

5.2. Generalized eigenvalue equation. Let $\mathfrak{Y}_0$ be the quotient space $\mathfrak{D}(A)/\mathfrak{N}(A)$ with the norm $\|[u]\|=Au$, where $[u]$ is the coset[6] determined by $u\in\mathfrak{D}(A)$, and let $\mathfrak{Y}$ be the completion of $\mathfrak{Y}_0$. Let $\mathfrak{R}$ be the closed subspace of $\mathfrak{K}$ defined as $\mathfrak{R}=\mathfrak{R}(A)^a=\mathfrak{K}\ominus\mathfrak{N}(A^*)$. Then the mapping $[u]\to Au$ from $\mathfrak{Y}_0$ to $\mathfrak{R}$ can be extended uniquely to a unitary operator from $\mathfrak{Y}$ onto $\mathfrak{R}$.

DEFINITION 5.1. The unitary operator mentioned above is denoted by $\boldsymbol{A}$: $\mathfrak{Y}\to\mathfrak{R}$.

PROPOSITION 5.2. *$\mathfrak{R}(Q_{1\pm}(\zeta)^*)\subset\mathfrak{R}$ for any $\zeta\in\Pi_I^\pm$.*

PROOF. If $\operatorname{Im}\zeta\neq 0$, then $\mathfrak{R}(Q_{1\pm}(\zeta)^*)=\mathfrak{R}([AR_1(\bar\zeta)B^*]^aC^*)\subset\mathfrak{R}(AR_1(\bar\zeta)B^*)^a\subset\mathfrak{R}$. For real ζ we take the limit. q. e. d.

By virtue of this proposition we can multiply both sides of (5.2) by $\boldsymbol{A}^{-1}$ and obtain

$$\boldsymbol{A}^{-1}w_\pm+\boldsymbol{A}^{-1}Q_{1\pm}(\lambda_0)^*w_\pm=0. \tag{5.3}$$

It will be shown that $\boldsymbol{A}^{-1}w_\pm\in\mathfrak{Y}$ satisfies a generalized eigenvalue equation. For this purpose we first define functionals on $\mathfrak{Y}$ induced by h_1 etc.

For the motivation we briefly consider Example 3.14. There, A is one-to-one so that no quotient space appears. $\mathfrak{Y}$ coincides with $H^1_{-\delta/2}(R^1)$. There-

6) This notation of expressing the coset by $[u]$ will be used throughout the rest of the paper.

fore, if $\phi \in H^1_{\delta/2}(R^1)$, then the functional $h_j[\phi, \cdot]$ on $\mathfrak{D} = H^1(R^1)$ can be extended continuously to a functional on $\mathfrak{Y}$. Similar consideration applies to the functional $(\phi, \cdot)$ defined by the inner product of $\mathfrak{H}$.

Returning to the general situation, we introduce the following definition.

DEFINITION 5.3. The set $\mathfrak{S} \subset \mathfrak{D}_\theta$ is defined to be the set of all $\phi \in \mathfrak{D}_\theta$ such that there exists $c = c_\phi > 0$ satisfying

$$(5.4) \qquad |h_1[\phi, v]| \leqq c\|[v]\|_{\mathfrak{Y}}, \qquad |(\phi, v)| \leqq c\|[v]\|_{\mathfrak{Y}}$$

for all $v \in \mathfrak{D}_{1-\theta}$.

We need the following auxiliary assumption.

ASSUMPTION 5.4. $\mathfrak{D}_{1-\theta}/\mathfrak{N}(A)$, which is a subset of $\mathfrak{Y}_0$ and hence of $\mathfrak{Y}$, is dense in $\mathfrak{Y}$[7].

Let $\phi \in \mathfrak{S}$. Then, Assumption 5.4 and (5.4) imply that the linear functional $h_1[\phi, \cdot]$ (or $(\phi, \cdot)$) on $\mathfrak{D}_{1-\theta}$ induces uniquely a bounded linear functional $\boldsymbol{h}_1[\phi, \cdot]$ (or $(\phi, \cdot)$) on $\mathfrak{Y}$:

$$(5.5) \qquad \boldsymbol{h}_1[\phi, [v]] = h_1[\phi, v], \qquad (\phi, [v]) = (\phi, v), \qquad v \in \mathfrak{D}_{1-\theta}.$$

Since $|(CB\phi, \boldsymbol{A}y)_{\mathfrak{K}}| \leqq \|CB\phi\|_{\mathfrak{K}}\|y\|_{\mathfrak{Y}}$, the functional $\boldsymbol{h}_2[\phi, \cdot]$ defined by

$$(5.6) \qquad \boldsymbol{h}_2[\phi, y] = \boldsymbol{h}_1[\phi, y] + (CB\phi, \boldsymbol{A}y), \qquad y \in \mathfrak{Y},$$

is also a bounded linear functional on $\mathfrak{Y}$. Clearly $\boldsymbol{h}_2[\phi, \cdot]$ is induced by $h_2[\phi, \cdot]$:

$$(5.7) \qquad \boldsymbol{h}_2[\phi, [v]] = h_2[\phi, v], \qquad v \in \mathfrak{D}_{1-\theta}.$$

THEOREM 5.5. *Suppose that the assumptions of Theorem* 2.9 *and Assumptions* 3.1–3.3 *are satisfied together with Assumption* 5.4. *Then, any* $w_\pm \in \mathfrak{K}$ *satisfying* (5.2) *and hence* (5.3) *satisfies*

$$(5.8) \qquad \boldsymbol{h}_2[\phi, \boldsymbol{A}^{-1}w_\pm] = \lambda_0(\phi, \boldsymbol{A}^{-1}w_\pm) \qquad \textit{for any } \phi \in \mathfrak{S}.$$

PROOF. Put

$$(5.9) \qquad K(\zeta) = [BR_1(\zeta)A^*]^a, \ \operatorname{Im}\zeta \neq 0.$$

The proof of Theorem 3.9 shows that K can be extended to $K_\pm$ defined on $\Pi^\pm_I$. We have $\mathfrak{R}(K_\pm(\zeta)^*) \subset \mathfrak{R}$ (cf. the proof of Proposition 5.2) and $K_\pm(\zeta)^*C^* = Q_{1\pm}(\zeta)^*$. Putting $(\boldsymbol{h}_j - \zeta)[\phi, v] = \boldsymbol{h}_j[\phi, v] - \zeta(\phi, v)$, we first prove that

$$(5.10) \qquad (\boldsymbol{h}_1 - \zeta)[\phi, \boldsymbol{A}^{-1}K_\pm(\zeta)^*u] = (B\phi, u), \qquad u \in \mathfrak{K}, \ \phi \in \mathfrak{S}, \ \zeta \in \Pi^\pm_I.$$

Clearly it suffices to prove (5.10) for $u \in \mathfrak{D}(B^*)$ and ζ with $\operatorname{Im}\zeta \neq 0$. But then the left side is equal to $(\boldsymbol{h}_1 - \zeta)[\phi, \boldsymbol{A}^{-1}AR_1(\bar\zeta)B^*u] = (h_1 - \zeta)[\phi, R_1(\bar\zeta)B^*u] = (\phi, B^*u) = (B\phi, u)$, where the relations $\boldsymbol{A}^{-1}Av = [v]$ and (5.5) are used.

It follows from (5.6) and (5.3) that

7) This assumption is satisfied if $A\mathfrak{D}_{1-\theta}$ is dense in $\mathfrak{R}(A)$.

$$\boldsymbol{h}_2[\phi, \boldsymbol{A}^{-1}w_\pm] = (\boldsymbol{h}_1-\lambda_0)[\phi, \boldsymbol{A}^{-1}w_\pm] + \lambda_0(\phi, \boldsymbol{A}^{-1}w_\pm) + (CB\phi, w_\pm)$$
$$= \lambda_0(\phi, \boldsymbol{A}^{-1}w_\pm) - (\boldsymbol{h}_1-\lambda_0)[\phi, \boldsymbol{A}^{-1}K_\pm(\lambda_0)^*C^*w_\pm] + (CB\phi, w_\pm).$$

Applying (5.10) to the right side with $u = C^*w_\pm$ and $\zeta = \lambda_0$, we see that the last two terms of the right side cancel each other. Thus, (5.8) is proved.

q. e. d.

The converse of Theorem 5.5 is not necessarily true. (In Lemma 5.22, however, the converse is proved in a restricted form.) Here we will only prove the following theorem.

THEOREM 5.6. *Let $y \in \mathfrak{Y}$ and $\phi \in \mathfrak{S}$ satisfy $\boldsymbol{h}_2[\phi, y] = \lambda_0(\phi, y)$. Then, $z_\pm = (1+\boldsymbol{A}^{-1}Q_{1\pm}(\lambda_0)^*\boldsymbol{A})y$ satisfies $\boldsymbol{h}_1[\phi, z_\pm] = \lambda_0(\phi, z_\pm)$.*

PROOF. By the assumption and (5.10) with $u = C^*Ay$ and $\zeta = \lambda_0$ we get

$$0 = (\boldsymbol{h}_1-\lambda_0)[\phi, y] + (CB\phi, \boldsymbol{A}y)$$
$$= (\boldsymbol{h}_1-\lambda_0)[\phi, z_\pm] - (\boldsymbol{h}_1-\lambda_0)[\phi, \boldsymbol{A}^{-1}K_\pm(\lambda_0)^*C^*\boldsymbol{A}y] + (CB\phi, \boldsymbol{A}y)$$
$$= (\boldsymbol{h}_1-\lambda_0)[\phi, z_\pm]. \qquad \text{q. e. d.}$$

In Example 3.14 we take $\mathfrak{S} = H^1_{\delta/2}$ (or we may take $\mathfrak{S} = \mathcal{S}(R^1)$). Equation (5.8) for $y_\pm = A^{-1}w_\pm \in H^1_{-\delta/2}$ becomes

$$-\int_{-\infty}^{\infty}(1+p(x))\phi'(x)\overline{y'_\pm(x)}dx + \int_{-\infty}^{\infty}q(x)\phi(x)\overline{y_\pm(x)}dx$$
$$= \lambda_0\int_{-\infty}^{\infty}\phi(x)\overline{y_\pm(x)}dx, \qquad \phi \in H^1_{\delta/2}.$$

5.3. A lemma. Our aim in this subsection is to prove the following lemma for later use.

LEMMA 5.7. *Assume that the assumptions of Theorem 2.9 and Assumptions 3.1-3.3 are satisfied. Then, if $w_\pm \in \mathfrak{K}$ satisfies (5.2) for a $\lambda_0 \in I$, we have $T(\lambda_0; B)C^*w_\pm = 0$.*

The proof will be based on the following propositions.

PROPOSITION 5.8. *We have $L(\zeta) = [BR_1(\zeta)B^*]^a \in B(\mathfrak{K})$. Furthermore, $L: \Pi^\pm \to B(\mathfrak{K})$ can be extended to a locally Hölder continuous function $L_\pm: \Pi^\pm_I \to B(\mathfrak{K})$.*

PROOF. Since $\theta \leqq 1/2$, one has $\mathfrak{D}(B) \supset \mathfrak{D}_\theta \supset \mathfrak{D}(|H_1|^{1/2})$. Hence, $BR_1(\zeta)B^* = B|R_1(\zeta)|^{1/2}W(\zeta)|R_1(\zeta)|^{1/2}B^*$ is bounded, where $W(\zeta)$ is as in the proof of Theorem 2.9. The rest of the proof is the same as that of Theorem 3.9.

q. e. d.

PROPOSITION 5.9. *Let $K_\pm$ and $L_\pm$ be as above. Then*

$$(5.11) \qquad (C^*K_\pm(\zeta)^*u, L_\pm(\zeta)^*v) = (L_\pm(\zeta)^*u, C^*K_\pm(\zeta)^*v), \qquad u, v \in \mathfrak{K}, \ \zeta \in \Pi^\pm_I.$$

PROOF. It suffices to prove (5.11) for $u, v \in \mathfrak{D}(B^*)$ and $\zeta \in \Pi^\pm$. By virtue of the symmetry relation $(CBu_1, Au_2) = (Au_1, CBu_2)$, $u_j \in \mathfrak{D}(H_1) \subset \mathfrak{D}_\theta \cap \mathfrak{D}_{1-\theta}$, which follows from the Hermitian property of h_1 and h_2, we see that the left side of (5.11) is equal to $(AR_1(\bar\zeta)B^*u, CBR_1(\bar\zeta)B^*v)=(BR_1(\bar\zeta)B^*u, C^*AR_1(\bar\zeta)B^*v)$ $=(L_\pm(\zeta)^*u, C^*K_\pm(\zeta)^*v)$. q. e. d.

PROPOSITION 5.10. *We have*

$$(2\pi i)^{-1}(L_\pm(\lambda_0)-L_\pm(\lambda_0)^*)=\pm T(\lambda_0\,;B)^*T(\lambda_0\,;B)\,.$$

The proof is similar to the arguments given in § 4.2, especially to the proof of (4.13).

PROOF OF LEMMA 5.7. Multiplying (5.2) by C^* and taking the inner product with $L_\pm(\lambda_0)^*C^*w_\pm$ we obtain

$$(C^*w_\pm,\ L_\pm(\lambda_0)^*C^*w_\pm)+(C^*K_\pm(\lambda_0)^*C^*w_\pm,\ L_\pm(\lambda_0)^*C^*w_\pm)=0\,.$$

The second term on the left side is real by Proposition 5.9. Hence, by taking the imaginary part and noting Proposition 5.10, we obtain the lemma. q. e. d.

5.4. Spaces $\mathfrak{K}_\gamma$. Step (2), a decaying property of $w_\pm$, will be handled from this subsection through § 5.6. Our aim is to assert that $w_\pm$ (or $y_\pm=\boldsymbol{A}^{-1}w_\pm$) actually lies in a space smaller than $\mathfrak{K}$ (or $\mathfrak{Y}$). For instance we know in Example 3.14 that $y_\pm\in H^1_{-\delta/2}$ with $\delta/2>1/2$. But we want to assert that $y_\pm\in H^1_{-t}$ for some $t<1/2$ or even $y_\pm\in H^1_0\subset\mathfrak{H}$. This is especially desired for higher dimensional problems, because one can then either apply Kato's theorem (see [7]) to equation (5.8) concluding $y_\pm=0$ and hence $e=\phi$ or use other means to prove, as will be done in § 5.7, the discreteness of the singular spectrum.

In order to see that $w_\pm$ lies in a smaller space, we need to have an interpolating family of Hilbert spaces. Interpolating $\mathfrak{H}$ and $\mathfrak{Y}$ will not be adequate. A convenient family will be found by decomposing A as $A=\hat{A}_1D$. In Example 3.14 this decomposition is given as follows. D is from $\mathfrak{H}$ to $L^2_{-\delta/2}\oplus L^2_{-\delta/2}$ $\equiv\mathfrak{K}_{-1}$ and is defined by $Du=\{u', u\}$, $u\in\mathfrak{D}(D)=\mathfrak{D}(A)$. $\hat{A}_1$ is the diagonal operator from $\mathfrak{K}_{-1}$ to $\mathfrak{K}$ determined by the multiplication by $(1+|x|^2)^{-\delta/4}$. Regard now $\mathfrak{K}_{-1}\supset\mathfrak{K}$ and let A_1 be the same multiplication operator viewed as acting in $\mathfrak{K}$. Then the space $\mathfrak{K}_\gamma=L^2_{\delta\gamma/2}\oplus L^2_{\delta\gamma/2}$, $\gamma\in R^1$, which forms a nice interpolating family, is the space determined by $A_1^{-\gamma}$-norm in $\mathfrak{K}$. Furthermore, $\hat{A}_1$ is recaptured as an operator naturally induced by A_1.

After this motivation let us start a new and first fix notations concerning the space with $A_1^{-\gamma}$-norm. Let $A_1\in B(\mathfrak{K})$ be a bounded positive definite self-adjoint operator in $\mathfrak{K}$: $A_1>0$. In particular, A_1 is one-to-one. Put $L=A_1^{-1}$. Then, $L\geqq c>0$. Let γ be a real number and denote by $\mathfrak{K}_\gamma$ the Hilbert space obtained by the completion of $\mathfrak{D}(L^\gamma)$ with respect to the norm $\|u\|_\gamma=\|L^\gamma u\|$. If $\gamma>0$, taking completion is not necessary and hence $\mathfrak{K}_\gamma=\mathfrak{D}(L^\gamma)$. If $\gamma<0$, $\mathfrak{D}(L^\gamma)=\mathfrak{K}$ but it is not necessarily complete with respect to $\|\ \|_\gamma$. For $\gamma=0$ one has $\mathfrak{K}_0=\mathfrak{K}$. Naturally, $\mathfrak{K}_\gamma\subset\mathfrak{K}_{\gamma'}$, if $\gamma'\leqq\gamma$.

For real γ and γ' we denote by $A_1^{\gamma',\gamma}$ the (canonical) unitary operator from $\mathfrak{R}_\gamma$ onto $\mathfrak{R}_{\gamma'}$ determined by the relation $A_1^{\gamma',\gamma}u = L^{\gamma-\gamma'}u$, $u \in \mathfrak{D}(L^{-\gamma'}\cdot L^{\gamma})$. It is not difficult to see that

$$A_1^{\gamma',\gamma}\mathfrak{R}_{\gamma+\beta} = \mathfrak{R}_{\gamma'+\beta}, \qquad \beta > 0, \tag{5.12}$$

$$\|A_1^{\gamma',\gamma}u\|_{\gamma'+\beta} = \|u\|_{\gamma+\beta}, \qquad u \in \mathfrak{R}_{\gamma+\beta}. \tag{5.13}$$

It is well known that the family $\mathfrak{R}_\gamma$ satisfies the interpolation relation. In particular, the following lemma will be used later.

LEMMA 5.11. *Relying on the complex method of constructing interpolation spaces and using the customary notation, we have* $\mathfrak{R}_{-\gamma} = [\mathfrak{R}_0, \mathfrak{R}_{-1}]_\gamma$, $0 \leqq \gamma \leqq 1$.

5.5. A decaying property of $w_\pm$. We list up all the assumptions and make some comments about their meanings afterwards.

ASSUMPTION 5.12. There exist a positive self-adjoint operator $A_1 \in B(\mathfrak{R})$, $A_1 > 0$, and a (not necessarily bounded) operator D from $\mathfrak{D}(D) = \mathfrak{D}(A) \subset \mathfrak{H}$ into $\mathfrak{R}_{-1}$ such that

$$A = A_1^{0,-1}D. \tag{5.14}$$

(Here and in what follows, $\mathfrak{R}_\gamma$ etc. are defined in reference to A_1 appearing in this assumption.)

ASSUMPTION 5.13. D maps $\mathfrak{D}_{1-\theta}$ boundedly in $\mathfrak{R}_0$; namely there exists $c > 0$ such that

$$\|Du\|_{\mathfrak{R}_0} \leqq c(\|\,|H_1|^{1-\theta}u\| + \|u\|), \qquad u \in \mathfrak{D}_{1-\theta}. \tag{5.15}$$

It follows from Assumption 5.13 that for any interval $I' \Subset I$, D maps $E_1(I')\mathfrak{H}$ boundedly in $\mathfrak{R}_0$; namely there exists $c_{I'} \geqq 0$ such that

$$\|Du\|_{\mathfrak{R}_0} \leqq c_{I'}\|u\|_{\mathfrak{H}}, \qquad u \in E_1(I')\mathfrak{H}. \tag{5.16}$$

(Note that $\mathfrak{D}(D) = \mathfrak{D}(A) \supset \mathfrak{D}_{1-\theta} \supset \mathfrak{D}(H_1) \supset E_1(I')\mathfrak{H}$.)

DEFINITION 5.14. The family of operators $\Phi(\lambda) \in B(\mathfrak{R}_1, \mathfrak{G})$, $\lambda \in I$, is defined by the formula

$$T(\lambda\,; A) = \Phi(\lambda)A_1^{1,0}. \tag{5.17}$$

ASSUMPTION 5.15. If $w \in \mathfrak{R}_\gamma$, $\gamma \geqq 0$, then $T(\lambda\,; B)C^*w$ is locally Hölder continuous with exponent $\theta = \min(\gamma/2 + \rho_0, 1)$, where $\rho_0 > 0$ is a constant independent of γ, uniformly in the following sense: for any compact interval $I' \Subset I$ and any $\gamma \geqq 0$, there exists a constant $c = c_{I',\gamma} > 0$ such that

$$\|\{T(\lambda\,; B) - T(\lambda'\,; B)\}C^*w\|_{\mathfrak{G}} \leqq c\|w\|_{\mathfrak{R}_\gamma}|\lambda - \lambda'|^\theta, \tag{5.18}$$

$$\lambda,\ \lambda' \in I', \quad w \in \mathfrak{R}_\gamma.$$

The meaning of Assumption 5.12 was already explained. $A_1^{0,-1}$ is the correct interpretation of what we wrote $\hat{A}_1$ previously.

Assumption 5.13 is a technical one and can easily be verified in Example 3.14.

In order to explain the meaning of Definition 5.14, we note that formally $T(\lambda; A)u=(FA^*u)(\lambda)=(FD^*\hat{A}_1^*u)(\lambda)$. In Example 3.14, D^* is a differential operator and may be taken out in front of F, a modified Fourier transform, as a multiplication operator $D(\lambda)$. Furthermore, we may write $(Fv)(\lambda)=F(\lambda)v$ with $F(\lambda)$ being a sort of trace operator from $K_1=L^2_{\delta/2}\oplus L^2_{\delta/2}$ to $\mathfrak{C}\oplus\mathfrak{C}$. Now, $\Phi(\lambda)$ in Assumption 5.14 corresponds to $D(\lambda)F(\lambda)$. The correct interpretation of $\hat{A}_1^*$ in this formula turns out to be $A_1^{\prime,0}$. (The reasoning given above is rather formal. The correct one will be given in [**15**].)

Referring still to Example 3.14, we can write similarly $T(\lambda; B)C^*w=D(\lambda)F(\lambda)A_1^{\prime,0}C^*w$ (note that $B=A$ in this example). C^* maps $\mathfrak{K}_\gamma$ into $\mathfrak{K}_\gamma$ so that $A_1^{\prime,0}C^*w\in\mathfrak{K}_{1+\gamma}$ if $w\in\mathfrak{K}_\gamma$. Now, $F(\lambda)$ can be regarded as the trace operator from $\mathfrak{K}_{1+\gamma}=L^2_{(1+\gamma)\delta/2}\oplus L^2_{(1+\gamma)\delta/2}$ to $\mathfrak{C}\oplus\mathfrak{C}$. It is well-known that this $F(\lambda)$ is Hölder continuous with exponent $\gamma\delta/2+(\delta-1)/2$. Since $\delta>1$, Assumption 5.15 is satisfied with $\rho_0=(\delta-1)/2$.

The following theorem describes the decaying property of $w_\pm$ and is fundamental in the present section.

THEOREM 5.16. *Assume all the assumptions of Lemma* 5.7 *and Assumptions* 5.12, 5.13, *and* 5.15. *Let* $w_\pm\in\mathfrak{K}$ *satisfy* (5.2). *Then* $w_\pm\in\mathfrak{K}_1$.

5.6. Proof of Theorem 5.16. We prove the theorem for w_+. The proof is the same for w_-. For simplicity we write w instead of w_+.

LEMMA 5.17. *Suppose that in a neighbourhood of* λ_0 *the exponent of Hölder continuity of* $T(\lambda; B)C^*w$ *can be taken as* ρ, $0<\rho\leqq 1$. *Then*: *if* $\rho\leqq 1/2$, *one has* $w\in\mathfrak{K}_\gamma$, $0\leqq\gamma<2\rho$; *and if* $\rho>1/2$, *one has* $w\in\mathfrak{K}_1$.

PROOF. Let us fix an open interval I' such that $\lambda_0\in I'\Subset I$. We put $J=R^1-I'$ and

$$R_J(\lambda_0)=\int_J\frac{1}{\lambda-\lambda_0}E_1(d\lambda)\in B(\mathfrak{H}).$$

Under the situation of Theorem 2.9 the operator $AR_J(\lambda_0)B^*$ is densely defined and bounded. More precisely, we have

$$[AR_J(\lambda_0)B^*]^a=A|R_J(\lambda_0)|^{1-\theta}W(\lambda_0)[|R_J(\lambda_0)|^\theta B^*]^a, \tag{5.19}$$

where $W(\lambda_0)=\int_J e^{-\arg(\lambda-\lambda_0)}E_1(d\lambda)$. Therefore, noting (5.2), (2.2), and Proposition 4.1, we can express w as

$$\begin{cases} w=w_1+w_2, \\ w_1=-[AR_J(\lambda_0)B^*]^aC^*w, \\ w_2=-\lim\limits_{\varepsilon\downarrow 0}\displaystyle\int_{I'}\frac{1}{\lambda-\lambda_0-i\varepsilon}T(\lambda; A)^*T(\lambda; B)C^*wd\lambda. \end{cases}\tag{5.20}$$

By virtue of (5.19) w_1 can be written as $w_1 = Au_1 = A_1^{0,-1}Du_1$, where $u_1 \in \mathfrak{D}_{1-\theta}$. Therefore, it follows from Assumption 5.13 and (5.12) that $w_1 \in \mathfrak{K}_1$ and hence $w_1 \in \mathfrak{K}_\gamma$ for any γ, $0 \leqq \gamma \leqq 1$.

To handle w_2 we recall that $T(\lambda; A)^*$ and $T(\lambda; B)$ are Hölder continuous and that $T(\lambda_0; B)C^*w = 0$ (Lemma 5.7). Therefore, if we put

$$\phi(\lambda) = \frac{1}{\lambda - \lambda_0} T(\lambda; B)C^*w \in L^1(I'; \mathfrak{C}), \tag{5.21}$$

then the well-known formula for the limit of the integral of Cauchy type gives that

$$\begin{aligned} w_2 &= -\int_{I'} T(\lambda; A)^*\phi(\lambda)d\lambda \\ &= -A_1^{0,1}\int_{I'} \Phi(\lambda)^*\phi(\lambda)d\lambda . \end{aligned} \tag{5.22}$$

We now introduce the operator Λ acting on $\mathfrak{C}$-valued functions on I' and defined formally as

$$\Lambda\phi = A_1^{0,1}\int_{I'} \Phi(\lambda)^*\phi(\lambda)d\lambda .$$

Since $\Phi(\lambda) = T(\lambda; A)A_1^{0,1}$ is continuous, it is clear that

$$\Lambda \in B(L^1(I'; \mathfrak{C}), \mathfrak{K}_0) \qquad (\mathfrak{K}_0 = \mathfrak{K}) . \tag{5.23}$$

We claim that

$$\Lambda \in B(L^2(I'; \mathfrak{C}), \mathfrak{K}_1) . \tag{5.24}$$

To prove this, let $\phi \in L^2(I'; \mathfrak{C}) \subset L^1(I'; \mathfrak{C})$. Then, ϕ can be written as $\phi(\lambda) = (Fu)(\lambda)$, $u \in E_1(I')\mathfrak{H}$ (Assumption 3.2). Therefore, for any $w \in \mathfrak{D}(A^*)$ the following manipulation can be carried out:

$$\begin{aligned} (\Lambda\phi, w)_{\mathfrak{K}} &= \int_{I'} (\phi(\lambda), \Phi(\lambda)A_1^{1,0}w)_{\mathfrak{C}}d\lambda \\ &= \int_{I'} (\phi(\lambda), T(\lambda; A)w)_{\mathfrak{C}}d\lambda \\ &= \int_{I'} ((Fu)(\lambda), (FE_1(I)A^*w)(\lambda))_{\mathfrak{C}}d\lambda \\ &= (u, A^*w)_{\mathfrak{H}} = (Au, w)_{\mathfrak{K}} , \end{aligned}$$

where $u \in \mathfrak{D}(H_1) \subset \mathfrak{D}(A)$ is used in the last step. Since $\mathfrak{D}(A^*)$ is dense in $\mathfrak{K}_0 = \mathfrak{K}$, we get $\Lambda\phi = Au = A_1^{0,-1}Du$. From (5.16), (5.12) and (5.13) it now follows that $\Lambda\phi \in \mathfrak{K}_1$ and

$$\|\Lambda\phi\|_{\mathfrak{K}_1} = \|Du\|_{\mathfrak{K}_0} \leqq c_{I'}\|E(I')u\|_{\mathfrak{H}} = c_{I'}\|\phi\|_{L^2(I'; \mathfrak{C})} .$$

Thus (5.24) is proved.

By the interpolation it follows from (5.23), (5.24) and Lemma 5.11 that

$$A \in B(L^p(I'; \mathfrak{C}); \mathfrak{R}_\gamma), \qquad 1 \leqq p \leqq 2, \ \gamma = 2(1-p^{-1}). \tag{5.25}$$

On the other hand $w_2 = -A\psi$ with ψ given by (5.21). From the assumption of the lemma and the relation $T(\lambda; B)C^*w = 0$ it follows that

$$\psi \in L^p(I'; \mathfrak{C}) \qquad \text{for any } p, \ 1 \leqq p < (1-\rho)^{-1}. \tag{5.26}$$

Therefore, if $\rho > 1/2$ one can take $p = 2$ and conclude $w_2 \in \mathfrak{R}_1$. If $\rho \leqq 1/2$, then p^{-1} in (5.25) ranges over $(1-\rho, 1]$ and correspondingly γ ranges over $[0, 2\rho)$. This proves the lemma. q. e. d.

PROOF OF THEOREM 5.16. Suppose that $w \in \mathfrak{R}_\gamma$ has been shown for some γ, $0 \leqq \gamma < 1$. Then, by Assumption 5.15 and Lemma 5.17 it follows that $w \in \mathfrak{R}_{\gamma'}$, for any γ' satisfying $0 \leqq \gamma' < \gamma + 2\rho_0$ if $\gamma + 2\rho_0 \leqq 1$ and $w \in \mathfrak{R}_1$ if $\gamma + 2\rho_0 > 1$. Hence, beginning with $w \in \mathfrak{R} = \mathfrak{R}_0$ and applying the procedure given above repeatedly, we can conclude $w \in \mathfrak{R}_1$ after a finite number of steps. q. e. d.

Before finishing step (2) we record the following lemma which follows immediately from Assumption 5.15 and the proof of Lemma 5.17.

LEMMA 5.18. *If $w \in \mathfrak{R}_1$ and $T(\lambda_0; B)C^*w = 0$, then $Q_{1\pm}(\lambda_0)^*w \in \mathfrak{R}_1$.*

5.7. Discreteness of the singular spectrum. For simplicity we assume in this subsection that A is one-to-one. Then, $\mathfrak{D}(A)$ can be regarded as a linear subspace of $\mathfrak{Y}$ and on $\mathfrak{D}(A)$ the operator $\boldsymbol{A}: \mathfrak{Y} \to \mathfrak{R}$ introduced in § 5.2 coincides with A.

ASSUMPTION 5.19. i) A is one-to-one; ii) if $\boldsymbol{A}y \in \mathfrak{R}_1$, $y \in \mathfrak{Y}$, then $y \in \mathfrak{D}_{1-\theta}$ (and hence $\boldsymbol{A}y = Ay$); and iii) there exists $c > 0$ such that

$$\|y\|_{\mathfrak{H}} \leqq c\|\boldsymbol{A}y\|_{\mathfrak{R}_1} \qquad \text{if } \boldsymbol{A}y \in \mathfrak{R}_1. \tag{5.27}$$

ASSUMPTION 5.20. For any $u \in \mathfrak{D}_\theta$ there exist $\phi_n^{(j)} \in \mathfrak{S}$, $n = 1, 2, \cdots$, $j = 1, 2$, such that $\|\phi_n^{(j)} - u\|_{\mathfrak{H}} \to 0$ and

$$h_j[u, v] = \lim_{n\to\infty} h_j[\phi_n^{(j)}, v], \qquad v \in \mathfrak{D}_{1-\theta}.$$

In Example 3.14 $\boldsymbol{A}y \in \mathfrak{R}_1$ means that $\{y', y\} \in L^2 \oplus L^2$ and hence $y \in H^1 = \mathfrak{D}_{1/2}$. Hence Assumption 5.19 is fulfilled. Since $\mathfrak{S} = H^1_{3/2}$ and $\mathfrak{D}_\theta = H^1$, the validity of Assumption 5.20 is obvious.

We can now formulate our main theorem.

THEOREM 5.21. *Assume the following set of assumptions*: i) *all the assumptions of Theorem* 2.9; ii) *Assumptions* 3.2–3.5 *and Assumption* 5.4; iii) *Assumptions* 5.12, 5.13, *and* 5.15; *and* iv) *Assumptions* 5.19 *and* 5.20. *Then, $e_+ = e_- = e$ and $E_2(\Gamma \cap e)$ is finite-dimensional projection for any compact interval $\Gamma \subset I$. (In other words, the singular spectrum of H_2 in I consists of eigenvalues with finite multiplicity and has no points of accumulation in I.)*

For the proof we first prove a lemma.

LEMMA 5.22. *Let $\lambda_0 \in I$. Then, A maps the eigenspace $\mathfrak{M}(\lambda_0)=\{y \in \mathfrak{H} \mid H_2 y=\lambda_0 y\}$ of H_2 associated with λ_0 onto the null space of $1+Q_{1\pm}(\lambda_0)^*$: $A\mathfrak{M}(\lambda_0)=\mathfrak{N}(1+Q_{1\pm}(\lambda_0)^*)$. (Note that $\mathfrak{M}(\lambda_0) \subset \mathfrak{D}(H_2) \subset \mathfrak{D}_{1-\theta} \subset \mathfrak{D}(A)$.)*

PROOF. First let $y \in \mathfrak{M}(\lambda_0)$ and put $w=Ay$. Since $y \in \mathfrak{D}_{1-\theta}$, we have $w=A_1^{0,-1}Dy \in \mathfrak{K}_1$ by Assumption 5.13 and (5.12).

For the purpose of applying Lemma 5.18 we will first prove that $T(\lambda; B)C^*w=0$. The relation $y \in \mathfrak{M}(\lambda_0)$ implies $h_2[v, y]=\lambda_0(v, y)$ for any $v \in \mathfrak{D}_\theta$. Hence, putting $v=R_1(\zeta)B^*u \in \mathfrak{D}(H_1) \subset \mathfrak{D}_\theta$, one gets

$$(h_1-\lambda_0)[R_1(\zeta)B^*u, y]+(CBR_1(\zeta)B^*u, Ay)=0\,, \qquad u \in \mathfrak{D}(B^*)\,.$$

By subtracting from this the similar relation for $\bar{\zeta}$, we obtain

$$(h_1-\lambda_0)[\{R_1(\zeta)-R_1(\bar{\zeta})\}B^*u, y]+(\{L(\zeta)-L(\bar{\zeta})\}u, C^*Ay)=0\,, \tag{5.28}$$

where L is as defined in Proposition 5.8. Put $\zeta=\lambda_0+i\varepsilon$ and let $\varepsilon \downarrow 0$. By Propositions 5.8 and 5.10 we see that the second term on the left tends to $2\pi i(T(\lambda_0; B)^*T(\lambda_0; B)u, C^*Ay)$. For brevity write $\tilde{E}_1(\lambda)=E_1((-\infty, \lambda])$. Then the first term is equal to

$$\begin{aligned}((H_1-\lambda_0)\{R_1(\zeta)-R_1(\bar{\zeta})\}B^*u, y) &\\ =\int_{I'}\frac{2i\varepsilon(\lambda-\lambda_0)}{(\lambda-\lambda_0)^2+\varepsilon^2}\frac{d}{d\lambda}(\tilde{E}_1(\lambda)B^*u, y)d\lambda &\\ +\text{remainder}\,,&\end{aligned} \tag{5.29}$$

where $I' \Subset I$ is open and $\lambda_0 \in I'$. The remainder is seen to converge to 0 as $\varepsilon \downarrow 0$. To handle the other term we will show that

$$\frac{d}{d\lambda}(\tilde{E}_1(\lambda)B^*u, y) \in L^2(I')\,. \tag{5.30}$$

In fact, by using Assumption 3.2 and equation (3.2) together with Proposition 3.7, one has $(d/d\lambda)\|\tilde{E}_1(\lambda)B^*u\|^2=\|(FE_1(I)B^*u)(\lambda)\|_{\mathfrak{C}}^2=\|T(\lambda; B)u\|_{\mathfrak{C}}^2$, $\lambda \in I$. Since this is bounded in $\bar{I'}$ because of the continuity of $T(\lambda; B)$, (5.30) follows from the inequality $|(d/d\lambda)(\tilde{E}_1(\lambda)B^*u, y)|^2 \leqq (d/d\lambda)\|\tilde{E}_1(\lambda)B^*u\|^2 \cdot (d/d\lambda)\|\tilde{E}_1(\lambda)y\|^2$.

(5.30) implies that the first term on the right of (5.29) tends to 0 as $\varepsilon \downarrow 0$. Therefore, by letting $\varepsilon \downarrow 0$ in (5.28) we obtain

$$(T(\lambda_0; B)^*T(\lambda_0; B)u, C^*Ay)=0\,, \qquad u \in \mathfrak{D}(B^*)\,.$$

This result can be extended to an arbitrary $u \in \mathfrak{K}$. In particular, we can take $u=C^*Ay$ and conclude $T(\lambda_0; B)C^*w=0$, $w=Ay$.

Since $w=Ay \in \mathfrak{K}_1$ as shown before, Lemma 5.18 tells us that $Q_{1\pm}(\lambda_0)^*w \in \mathfrak{K}_1$. Therefore, by Proposition 5.2 and ii) of Assumption 5.19, we see that $Q_{1\pm}(\lambda_0)^*w \in \mathfrak{D}(A^{-1})$ and $z_\pm=y+A^{-1}Q_{1\pm}(\lambda_0)^*Ay=A^{-1}(1+Q_{1\pm}(\lambda_0)^*)w \in \mathfrak{D}_{1-\theta}$. On the other hand, $h_2[\phi, y]=\lambda_0(\phi, y)$, $\phi\in\mathfrak{S}$, because $y\in\mathfrak{M}(\lambda_0)$. We now apply Theorem 5.6. Noting $\boldsymbol{h}_2[\phi, y]=h_2[\phi, y]$ and $\boldsymbol{h}_1[\phi, z_\pm]=h_1[\phi, z_\pm]$ as y, $z_\pm\in\mathfrak{D}_{1-\theta}$, we then obtain $h_1[\phi, z_\pm]=\lambda_0(\phi, z_\pm)$, $\phi\in\mathfrak{S}$. By Assumption 5.20 this relation remains true for any $\phi\in\mathfrak{D}_\theta$. Hence $z_\pm\in\mathfrak{D}(H_1)$ and $H_1z_\pm=\lambda_0z_\pm$. However, H_1 is absolutely continuous in I (Assumption 3.2). Hence $z_\pm=0$. Recalling the definition of $z_\pm$, we see that $(1+Q_{1\pm}(\lambda_0)^*)w=0$.

Conversely, suppose that $(1+Q_{1\pm}(\lambda_0)^*)w_\pm=0$, $w_\pm\in\mathfrak{K}$. As shown before, $w_\pm\in\mathfrak{R}(A)$ and $y_\pm=A^{-1}w_\pm$ satisfies (5.8). On the other hand Theorem 5.16 shows that $w_\pm\in\mathfrak{K}_1$. Hence, by ii) of Assumption 5.19 one sees that $y_\pm\in\mathfrak{D}_{1-\theta}$ and $y_\pm=A^{-1}w_\pm$. Then, (5.8) can be written as

$$h_2[\phi, y_\pm]=\lambda_0(\phi, y_\pm), \qquad \phi\in\mathfrak{S}.$$

By virtue of Assumption 5.20 this relation remains true for any $\phi\in\mathfrak{D}_\theta$. But then it follows that $y_\pm\in\mathfrak{D}(H_2)$ and $H_2y_\pm=\lambda_0y_\pm$. q. e. d.

From Lemma 5.22 it follows that i) $e_+=e_-=e$; ii) $e=\sigma_{\text{sing}}(H_2)\cap I$, where $\sigma_{\text{sing}}(H_2)$ denotes the singular spectrum of H_2; and iii) every point of e is an eigenvalue of H_2.

PROOF OF THEOREM 5.21. Suppose there exists an infinite sequence $\lambda_n\in\Gamma\cap e$, $n=1, 2, \cdots$, with a corresponding *orthonormal* system $\{y_n\}$, $y_n\in\mathfrak{H}$, of eigenvectors of H_2: $H_2y_n=\lambda_ny_n$ (some of λ_n's may be equal). The theorem will be proved if we can derive a contradiction from this supposition.

Put $w_n=Ay_n$. Writing $y_n=(|\lambda_n|^{1-\theta}+i)(|H_2|^{1-\theta}+i)^{-1}y_n$, we see by (5.13) and (5.15) that

$$\begin{aligned}(5.31)\qquad \|w_n\|_{\mathfrak{K}_1}&=\|A_1^{0,-1}Dy_n\|_{\mathfrak{K}_1}=\|Dy_n\|_{\mathfrak{K}_0}\\ &\leqq c(|\lambda_n|^{1-\theta}+i)\{\||H_1|^{1-\theta}(|H_2|^{1-\theta}+i)^{-1}\|\\ &\qquad+\|(|H_2|^{1-\theta}+i)^{-1}\|\}\|y_n\|_{\mathfrak{H}}\\ &\leqq M,\end{aligned}$$

where M is a constant independent of n (note $\|y_n\|_{\mathfrak{H}}=1$). Hence $\{w_n\}$ is bounded in $\mathfrak{K}_1$ and a fortiori in $\mathfrak{K}_0=\mathfrak{K}$.

We will show that $\{w_n\}$ contains a subsequence convergent in $\mathfrak{K}$. For this purpose we may assume that $\lambda_n\to\lambda_0\in\Gamma$. Lemma 5.22 shows that $(1+Q_{1+}(\lambda_n)^*)w_n=0$. On the other hand, (4.6) and the discussions that follow it show that $1+Q_{1+}^*$ can be written as $1+Q_{1+}(\lambda)^*=(1+K(\lambda))S$, where $S\in B(\mathfrak{K})$ is invertible in $B(\mathfrak{K})$ and $K(\lambda)$ is a $B_\infty(\mathfrak{K})$-valued continuous function ($K(\lambda)$ is $K_+(\lambda)^*$ in the notation of (4.6)). Therefore, $Sw_n=-K(\lambda_n)Sw_n=-\{K(\lambda_n)-K(\lambda_0)\}Sw_n-K(\lambda_0)Sw_n$ contains a subsequence convergent in $\mathfrak{K}$, and so does $w_n=S^{-1}(Sw_n)$.

Switching to a subsequence, we assume from now on that $\lambda_n\to\lambda_0$ and $w_n\to w_0$ (in $\mathfrak{K}$). By Theorem 5.16 $w_0\in\mathfrak{K}_1$, because $(1+Q_{1+}(\lambda_0)^*)w_0=0$ holds by continuity. We now claim that

$$\|w_n - w_0\|_{\mathfrak{K}_1} \longrightarrow 0, \qquad n \to \infty. \tag{5.32}$$

For the time being, suppose that (5.32) has been proved. Then, by (5.27) we get $\|y_m - y_n\|_{\mathfrak{H}} \leqq c\|w_m - w_n\|_{\mathfrak{K}_1} \to 0$, $m, n \to \infty$. Hence, $\{y_n\}$ is a Cauchy sequence in $\mathfrak{H}$, which contradicts the orthonormality of $\{y_n\}$.

The proof of (5.32) somehow follows that of Lemma 5.17. Whenever an object, which was introduced in the proof of Lemma 5.17 related to w, is used related to w_n, we will use the same notation with suitable subscripts. As in that proof w_n, $n = 0, 1, \cdots$, are decomposed as $w_n = w_{1,n} + w_{2,n}$.

By writing $w_{1,n} = Au_{1,n}$, $u_{1,n} \in \mathfrak{D}_{1-\theta}$ as in (5.20) and using an argument similar to (5.31) we get $\|w_{1,n} - w_{1,0}\|_{\mathfrak{K}_1} \leqq c(\||H_1|^{1-\theta}(u_{1,n} - u_{1,0})\|_{\mathfrak{H}} + \|u_{1,n} - u_{1,0}\|_{\mathfrak{H}})$. It is clear by (5.19) and the definition of $u_{1,k}$ that the right side is majorized by $c\|w_n - w_0\|_{\mathfrak{K}}$ (we use the same letter c to denote various constants). Hence, $\|w_{1,n} - w_{1,0}\|_{\mathfrak{K}_1} \to 0$.

We next show that $\|w_{2,n} - w_{2,0}\|_{\mathfrak{K}_1} \to 0$. The proof of Lemma 5.17 shows that it suffices for this purpose to prove $\|\psi_n - \psi_0\|_{L^2(I'; \mathfrak{C})} \to 0$, where $\psi_n(\lambda) = (\lambda - \lambda_n)^{-1} T(\lambda; B) C^* w_n$. It is clear that $\psi_n(\lambda) \to \psi_0(\lambda)$ for each $\lambda \in I' - \{\lambda_0\}$. On the other hand, it follows from (5.18) and $T(\lambda_n; B) C^* w_n = 0$ that $\|\psi_n(\lambda)\|_{\mathfrak{C}} \leqq c\|w_n\|_{\mathfrak{K}_1} |\lambda - \lambda_n|^{\theta - 1}$, $\lambda \in I'$, where $\theta > 1/2$. Since $\|w_n\|_{\mathfrak{K}_1} \leqq M$ by (5.31), we see that the $\mathfrak{C}$-valued set functions $E \to \int_E \|\psi_n(\lambda)\|^2 d\lambda$, $E \subset I$, $n = 1, 2, \cdots$, are uniformly absolutely continuous. This fact combined with the pointwise convergence mentioned above establishes the desired L^2-convergence of ψ_n (the Vitali convergence theorem). q. e. d.

§ 6. Supplementary remarks.

6.1. Principle of limiting absorption. The principle of limiting absorption may be considered as the assertion of the convergence as $\varepsilon \downarrow 0$ of the resolvent $R_2(\lambda \pm i\varepsilon)$ with respect to the operator norm between suitable spaces. In our approach this principle may be formulated as in the following theorem. For simplicity we assume that A is one-to-one.

THEOREM 6.1. *Let all the assumptions of Theorem* 5.21 *be satisfied*[8]. *As before, let* $\mathfrak{Y}$ *be the completion of* $\mathfrak{D}(A)$ *with respect to the norm* $\|u\|_{\mathfrak{Y}} = \|Au\|_{\mathfrak{H}}$. *Let* $\mathfrak{K}'$ *be a subspace of* $\mathfrak{D}(A^*) \subset \mathfrak{K}$ *such that* A^* *is one-to-one on* $\mathfrak{K}'$ *and let* $\mathfrak{X}_0$ *be the inner product space* $A^*\mathfrak{K}'$ *with the norm* $\|A^*u\|_{\mathfrak{X}} = \|u\|_{\mathfrak{K}}$, $u \in \mathfrak{K}'$. *Let* $\mathfrak{X}$ *be the completion of* $\mathfrak{X}_0$. *Furthermore, assume either one of the following conditions* (1) *and* (2): (1) $[AR_1(\zeta)A^*]^a \in B(\mathfrak{K})$, $\zeta \in \rho(H_1)$; (2) $\mathfrak{X}$ *and* $\mathfrak{H}$ *are con-*

8) However, the arguments in this subsection depend essentially on the arguments up to § 4. 1.

tinuously imbedded in $\mathfrak{H}$ and $\mathfrak{Y}$, respectively. Then, $R_2(\zeta)$ restricted to $\mathfrak{X}_0$ and regarded as an operator from $\mathfrak{X}_0$ to $\mathfrak{D}(A)$ can be extended to a bounded operator $\tilde{R}_2(\zeta)\in B(\mathfrak{X}, \mathfrak{Y})$ from $\mathfrak{X}$ to $\mathfrak{Y}$. Furthermore, the operator valued function R_2: $\Pi^\pm\to B(\mathfrak{X}, \mathfrak{Y})$ can be extended uniquely to a locally Hölder continuous operator valued function $\tilde{R}_{2\pm}$: $\Pi^\pm_{I-e}\to B(\mathfrak{X}, \mathfrak{Y})$.

PROOF. In the case that (1) is satisfied we have $X(\zeta)\equiv[AR_2(\zeta)A^*]^a\in B(\mathfrak{R})$ (cf. (2.4)). Furthermore, X: $\Pi^\pm\to B(\mathfrak{R})$ can be extended to $X_\pm$: $\Pi^\pm_{I-e}\to B(\mathfrak{R})$ (cf. (4.9) and the proof of Theorem 3.9). On the other hand, let $A^\dagger$ (resp. $\boldsymbol{A}$) be the unitary operator from $\mathfrak{R}'^a$ to $\mathfrak{X}$ (resp. from $\mathfrak{Y}$ to $\mathfrak{R}(A)^a$) determined by $A^\dagger u=A^*u$, $u\in\mathfrak{R}'$ (resp. $\boldsymbol{A}u=Au$, $u\in\mathfrak{D}(A)$). Then, it is easy to see that the operator valued function $\tilde{R}_2(\zeta)=\boldsymbol{A}^{-1}X(\zeta)A^{\dagger-1}$ satisfies the requirement of the theorem.

In the case that (2) is satisfied we put $X(\zeta)=[AE_2(I')R_2(\zeta)A^*]^a\in B(\mathfrak{R})$, where $I'\subset I-e$ is arbitrarily fixed. X: $\Pi^\pm\to B(\mathfrak{R})$ can be extended to $X_\pm$: $\Pi^\pm_{I'}\to B(\mathfrak{R})$. Put $\tilde{R}_2(\zeta)=E_2(R^1-I')R_2(\zeta)|_{\mathfrak{X}}+A^{-1}X_\pm(\zeta)A^{\dagger-1}$. The second term on the right side satisfies the requirement of the theorem with $I-e$ replaced by I'. As to the first term we note that $E_2(R^1-I')R_2(\zeta)$ is analytic in $\Pi^+\cup\Pi^-\cup I'$ as a $B(\mathfrak{H})$-valued function. By virtue of (2), therefore, the first term is analytic there as a $B(\mathfrak{X}, \mathfrak{Y})$-valued function. Since I' is arbitrary, the theorem is proved. q. e. d.

COROLLARY 6.2. *Suppose that either* (1) *is satisfied or $\mathfrak{H}$ is continuously imbedded in $\mathfrak{Y}$. Let $v\in\mathfrak{R}(A^*)$ and let $u(\zeta)$, $\zeta\in\rho(H_2)$, be the unique solution in $\mathfrak{H}$ of the equation $H_2u(\zeta)-\zeta u(\zeta)=v$. Then, u : $\Pi^\pm\to\mathfrak{Y}$[9] can be extended to a locally Hölder continuous function $u_\pm$: $\Pi^\pm_{I-e}\to\mathfrak{Y}$. For any $\lambda\in I-e$, $u_\pm$ satisfies*

$$\boldsymbol{h}_2[\phi, u_\pm(\lambda)]-(\phi, u_\pm(\lambda))=(\phi, v)\,, \qquad \phi\in\mathfrak{S}\,. \tag{6.1}$$

PROOF. We may suppose that $v\neq 0$. The first statement follows from Theorem 6.1 by taking $\mathfrak{R}'=\{\alpha u\}$, $v=A^*u$. Let $\zeta\in\Pi^\pm$. Then,

$$\begin{aligned}(\boldsymbol{h}_2-\zeta)[\phi, u_\pm(\zeta)]&=(\boldsymbol{h}_2-\zeta)[\phi, u(\zeta)]=(h_2-\zeta)[\phi, u(\zeta)]\\&=(h_2-\zeta)[\phi, R_2(\zeta)v]=(\phi, v)\,, \qquad \phi\in\mathfrak{S}\,.\end{aligned}$$

(6.1) is derived from this by letting $\zeta\to\lambda$. q. e .d.

In Example 3.14 we have $\mathfrak{Y}=H^1_{-\delta/2}$. If we take $\mathfrak{R}'=\{0\}\oplus L^2\subset L^2\oplus L^2=\mathfrak{R}$, then $\mathfrak{X}=L^2_{\delta/2}$. Hence, condition (2) is satisfied. Thus, the principle of limiting absorption holds for $R_2(\zeta)$ as operators from $L^2_{\delta/2}$ to $H^1_{-\delta/2}$.

6.2. Scattering matrix. The scattering operator $S=S(H_2, H_1; I)$ introduced in Theorem 3.12 will be a decomposable operator in the spectral representation space $L^2(I; \mathfrak{C})$ of $E_1(I)H_1$. In this subsection an explicit formula for such an operator will be proved. The formula can be used to examine

9) Note that $u(\zeta)\in\mathfrak{D}(H_2)\subset\mathfrak{D}(A)\subset\mathfrak{Y}$.

how quickly the eigenvalues of $\mathcal{S}(\lambda)-1$, the scattering matrix minus the identity, tends to 0 (cf. Corollary 6.4).

The scattering operator S is defined as $S=W_+^*W_-$ where $W_\pm=F_\pm^*F$. Hence $S=F^*F_+F_-^*F$ and S is a unitary operator in $E_1(I)\mathfrak{H}$. We put $\hat{S}=FSF^*=F_+F_-^*$. $\hat{S}$ is the representation of S in $L^2(I;\mathfrak{C})$ and is a unitary operator in $L^2(I;\mathfrak{C})$.

THEOREM 6.3. *For any $\lambda\in I-e$ put*

$$\mathcal{S}(\lambda)=1-2\pi iT(\lambda;A)G_{2+}(\lambda)CT(\lambda;B)^*. \tag{6.2}$$

Then, for any $f\in L^2(I;\mathfrak{C})$ we have

$$(\hat{S}f)(\lambda)=\mathcal{S}(\lambda)f(\lambda), \qquad a.e.\ \lambda\in I, \tag{6.3}$$

$$(\hat{S}^{-1}f)(\lambda)=\mathcal{S}(\lambda)^{-1}f(\lambda), \qquad a.e.\ \lambda\in I, \tag{6.4}$$

where $\mathcal{S}(\lambda)^{-1}$ has the form

$$\mathcal{S}(\lambda)^{-1}=1+2\pi iT(\lambda;A)G_{2-}(\lambda)CT(\lambda;B)^*. \tag{6.5}$$

$\mathcal{S}(\lambda)$ is a unitary operator in $\mathfrak{C}$ and depends locally Hölder continuously on $\lambda\in I-e$ with respect to the operator norm.

PROOF. That $\mathcal{S}(\lambda)^{-1}$ exists and has the form (6.5) can be verified by a direct computation using the relation $CT(\lambda;B)^*T(\lambda;A)=(2\pi i)^{-1}\{Q_{1+}(\lambda)-Q_{1-}(\lambda)\}$.

For $u\in E_2(I)\mathfrak{H}$ put $f_\pm(\lambda)=(F_\pm u)(\lambda)$. Then, (6.3) is equivalent to the relation

$$f_+(\lambda)=\mathcal{S}(\lambda)f_-(\lambda). \tag{6.6}$$

We first claim that (6.6) holds for u having the form $u=E_2(I')A^*w$, $w\in\mathfrak{D}(A^*)$, $I'\subset I-e$. In fact, by (3.7) we get

$$f_\pm(\lambda)=\chi_{I'}(\lambda)T(\lambda;A)G_{2\pm}(\lambda)w \tag{6.7}$$

and hence

$$f_+(\lambda)=f_-(\lambda)+\chi_{I'}(\lambda)T(\lambda;A)\{G_{2+}(\lambda)-G_{2-}(\lambda)\}w. \tag{6.8}$$

Recalling $G_{2\pm}(\lambda)=G_{1\pm}(\lambda)^{-1}$, $\lambda\in I-e$, we have

$$\begin{aligned} G_{2+}(\lambda)-G_{2-}(\lambda)&=G_{2+}(\lambda)\{G_{1-}(\lambda)-G_{1+}(\lambda)\}G_{2-}(\lambda) \\ &=-2\pi iG_{2+}(\lambda)CT(\lambda;B)^*T(\lambda;A)G_{2-}(\lambda). \end{aligned} \tag{6.9}$$

By inserting (6.9) into (6.8) and using (6.7) for f_-, (6.6) is proved for u having the form mentioned above. However, since the set of all such u forms a fundamental set in $E_2(I-e)\mathfrak{H}$ (Assumption 3.5), so is in $L^2(I;\mathfrak{C})$ the set of all f_- arising from such u. Therefore, the validity of (6.6) can be extended to an arbitrary $f_-\in L^2(I;\mathfrak{C})$ by a limit procedure (note that $\mathcal{S}(\lambda)\in B(\mathfrak{C})$). Thus, (6.3) is proved.

From the unitarity of $\hat{S}$ and the relation $E_1(I')S=SE_1(I')$, $I'\subset I-e$, it follows that $\int_{I'}\|\mathcal{S}(\lambda)c\|^2 d\lambda=\int_{I'}\|c\|^2 d\lambda$, $c\in\mathfrak{C}$, $I'\Subset I-e$. Hence, $\|\mathcal{S}(\lambda)c\|=\|c\|$ a.e. Since $\mathcal{S}(\lambda)$ is continuous in λ, we see that $\|\mathcal{S}(\lambda)c\|=\|c\|$, $\lambda\in I-e$, $c\in\mathfrak{C}$. Thus, $\mathcal{S}(\lambda)$ is isometric. Since $\mathcal{S}(\lambda)^{-1}$ exists as an operator in $B(\mathfrak{C})$, $\mathcal{S}(\lambda)$ is unitary. Finally, (6.4) follows from (6.3) at once. q.e.d.

Let $\mathfrak{K}$ and $\mathfrak{C}$ be Hilbert spaces. We denote by $C_p(\mathfrak{K}, \mathfrak{C})$, $0<p<\infty$, the von Neumann-Schatten class of completely continuous operators from $\mathfrak{K}$ to $\mathfrak{C}$. Namely, $T\in C_p(\mathfrak{K}, \mathfrak{C})$ if the sequence of eigenvalues of $(T^*T)^{1/2}\in B_\infty(\mathfrak{K})$ belongs to l^p. Put $C_p(\mathfrak{C})=C_p(\mathfrak{C}, \mathfrak{C})$.

COROLLARY 6.4. *Let p, $q>0$ and suppose that*

$$T(\lambda;A)\in C_p(\mathfrak{K},\mathfrak{C}), \qquad T(\lambda;B)\in C_q(\mathfrak{K},\mathfrak{C}). \tag{6.10}$$

Then one has

$$\mathcal{S}(\lambda)-1\in C_r(\mathfrak{C}), \qquad r^{-1}=p^{-1}+q^{-1}. \tag{6.11}$$

PROOF. (6.11) follows immediately from (6.2) and the relations $C_q(\mathfrak{K}, \mathfrak{C})^*=C_q(\mathfrak{K}, \mathfrak{C})$[10], $B(\mathfrak{K})\cdot C_q(\mathfrak{C}, \mathfrak{K})\subset C_q(\mathfrak{C}, \mathfrak{K})$, and $C_p(\mathfrak{K}, \mathfrak{C})\cdot C_q(\mathfrak{C}, \mathfrak{K})\subset C_r(\mathfrak{K})$ (see, e.g., N. Dunford and J. T. Schwartz, Linear Operators, Part II, Interscience, New York and London, 1963, Chapt. XI).

Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Meguro-ku
Tokyo, Japan

References

[1] S. Agmon, Spectral properties of Schrödinger operators, Actes, Congrès intern. Math., 1970, Tome 2, 679–683.

[2] M. S. Birman, Scattering problems for differential operators by the perturbation of spaces, Izv. Akad. Nauk USSR, **35** (1971), 440–455.

[3] C. Goldstein, Perturbation of non-selfadjoint operators, I, Arch. Rational Mech. Anal., **37** (1970), 268–296.

[4] J. S. Howland, Banach space techniques in the perturbation theory of self-adjoint operators with continuous spectra, J. Math. Anal. Appl., **20** (1967), 22–47.

[5] T. Ikebe, Eigenfunction expansions associated with the Schroedinger operators and their applications to scattering theory, Arch. Rational Mech. Anal., **5** (1960), 1–34.

[6] T. Ikebe, On the eigenfunction expansion connected with the exterior problem for the Schrödinger equation, Japan. J. Math., **36** (1967), 33–55.

[7] T. Kato, Growth properties of solutions of the reduced wave equation with a variable coefficient, Comm. Pure Appl. Math., **12** (1959), 403–425.

10) Here * indicates the set of adjoint operators, not the adjoint space.

[8] T. Kato, Wave operators and similarity for some non-selfadjoint operators, Math. Ann., **162** (1966), 258–279.

[9] T. Kato, Some results on potential scattering, Proc. Intern. Conf. Functional Anal. and Related Topics, Tokyo, 1969, Tokyo Univ. Press, 1970, 206–215.

[10] T. Kato, Scattering theory and perturbation of continuous spectra, Actes Congrès intern. Math., 1970, Tome 1, 135–140.

[11] T. Kato and S. T. Kuroda, Theory of simple scattering and eigenfunction expansions, Functional Analysis and Related Fields, Springer-Verlag, Berlin, Heidelberg, and New York, 1970, 99–131.

[12] S. T. Kuroda, An abstract stationary approach to perturbation of continuous spectra and scattering theory, J. Analyse Math., **20** (1967), 57–117.

[13] S. T. Kuroda, Some remarks on scattering for Schrödinger operators, J. Fac. Sci. Univ. Tokyo Sect. IA, **17** (1970), 315–329.

[14] S. T. Kuroda, Spectral representations and the scattering theory for Schrödinger operators, Actes, Congrès intern. Math., 1970, Tome 2, 441–445.

[15] S. T. Kuroda, Scattering theory for differential operators, II, self-adjoint elliptic operators, J. Math. Soc. Japan, **25** (2) (1973).

[16] P. Rejto, On partly gentle perturbations, I, J. Math. Anal. Appl., **17** (1967), 453–462.

[17] N. Shenk and D. Thoe, Outgoing solutions of $(-\Delta+q-k^2)u=f$ in an exterior domain, J. Math. Anal. Appl., **31** (1970), 81–116.