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A relative Hodge-Kodaira decomposition

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§ 1. Introduction.

Let X be an $m+n$ dimensional oriented compact C^{∞} Riemannian manifold and $\Omega^{p}(X)$ be the space of smooth p-forms on X. Celebrated Hodge theorem says that every cohomology class $H^{p}(X)$ of de Rham is canonically represented by a harmonic p-form (cf. [2] and [5]). The aim of this note is to prove an analogy of this theorem of Hodge also for the cohomology group $H^{p}(X, Y)$ relative to m-dimensional submanifold $Y\subset X$. More precisely, let Y be an m-dimensional compact oriented submanifold of X and $\Omega^{p}(Y)$ be the space of smooth p-forms of Y. The relative cohomology group $H^{*}(X, Y)$ is the cohomology group of the complex $\Omega^{*}(X, Y)$ defined by the exact sequence of complexes

$$0\longrightarrow\Omega^{*}(X, Y)\longrightarrow\Omega^{*}(X)\xrightarrow{\iota}\Omega^{*}(Y)\longrightarrow 0 \tag{1.1}$$

where ι is the restriction mapping. Kodaira [5] proved that every cohomology class of $H^{p}(X, Y)$ can be represented by a square summable harmonic p-forms on open submanifold $X-Y$ of X (cf. [2]). However, this is not convenient when one wants to deal with the long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^{0}(X, Y)\longrightarrow H^{0}(X)\longrightarrow H^{0}(Y)\longrightarrow \\ &\longrightarrow H^{1}(X, Y)\longrightarrow H^{1}(X)\longrightarrow H^{1}(Y)\longrightarrow \\ &\qquad\cdots\cdots\cdots\cdots\cdots \\ &\longrightarrow H^{p}(X, Y)\longrightarrow H^{p}(X)\longrightarrow H^{p}(Y)\longrightarrow \\ &\qquad\cdots\cdots\cdots\cdots\cdots . \end{aligned} \tag{1.2}$$

In this note we prove the following facts:

(i) Every cohomology class of $H^{p}(X, Y)$ can be represented by a current α on X which satisfies the equation

$$\Delta(1+\Delta)^{a}\alpha=0 \qquad \text{on } X-Y \tag{1.3}$$

and $\alpha|_{Y}=0$, where a is the integral part of $n/2$.

(ii) Every cohomology class of $H^{p}(X, Y)$ can be represented by a current β on X which is harmonic in $X-Y$ and has singularity on Y.

(iii) Representations (i) and (ii) of the cohomology class of $H^p(X, Y)$ are compatible with the long exact sequence (1.2).

In proving (iii), we make use of a slight variation of Hodge-Kodaira decomposition of forms on Y.

Our results are obtained through the following standard steps:

(a) We make a Hilbert space $W_a^p(X)$ in which $\Omega^p(X, Y)$ is everywhere dense. This is done in § 3.

(b) We consider the adjoint operator d^* of d in $W_a^p(X)$. This is treated in § 4 and § 5.

(c) We introduce the generalized Laplacian dd^*+d^*d and apply the classical method of Weyl-Kodaira, i.e., make use of the orthogonal decomposition of $W_a^p(X)$ into the sum of the image and the kernel of the self-adjoint operator dd^*+d^*d. We prove that the kernel of dd^*+d^*d is isomorphic to the cohomology group $H^*(X, Y)$. These are proved in § 6 and § 8.

(d) Interpretation of the long exact sequence (1.2) is given in § 7 and § 8.

Crucial point lies in only one point how the space $W_a^p(X)$ should be chosen. This space coincides with the space of square integrable currents on $X-Y$ if $n=1$. In this case, steps (a), (b) and the first part of step (c) were done by several authors. One may consult with, for example, [**1**] and [**6**]. See their bibliographies for further references. However, the space $W_a^p(X)$ is strictly smaller than the space of square integrable currents on $X-Y$ if $n\geqq 2$. The operator d^* is no longer a local operator in this case.

One may feel that the equation (1.3) is not sufficiently natural. We slightly modify discussions to get more natural representative of the cohomology class of $H^p(X, Y)$. It is shown in § 9 that every cohomology class in $H^p(X, Y)$ is uniquely represented by a current α satisfying

$$\Delta^{a+1}\alpha=0 \quad \text{on } X-Y \tag{1.4}$$

and

$$\alpha|_Y=0. \quad \text{(cf. Theorem 9.6, 9.7 and Remark 9.8)}.$$

One can easily see that suitable modification makes similar discussion possible for non compact X. However this is left to the reader.

§ 2. Some lemmas from analysis.

Let $\boldsymbol{R}^{m+n}$ be the Euclidean $m+n$ space. We shall denote an arbitrary point in $\boldsymbol{R}^{m+n}$ by $x=(x', x'')$ with $x'=(x_1, \cdots, x_m)$ and $x''=(x_{m+1}, \cdots, x_{m+n})$. Let $\mathcal{S}(\boldsymbol{R}^{m+n})$ and $\mathcal{S}'(\boldsymbol{R}^{m+n})$ be the space of rapidly decreasing C^∞ functions on $\boldsymbol{R}^{m+n}$ and its dual space (cf. [**7**]). For any function $u(x)$ in $\mathcal{S}(\boldsymbol{R}^{m+n})$ and any a, b in $\boldsymbol{R}$, we define the norm

$$\|u\|_{a,b}=\Big[\int_{R^{m+n}}|\hat{u}(\xi)|^2(1+|\xi'|^2+|\xi''|^2)^a(1+|\xi'|^2)^b d\xi\Big]^{1/2}, \tag{2.1}$$

where $\xi=(\xi',\xi'')$ runs over $\boldsymbol{R}^{m+n}$ and $\hat{u}(\xi)$ is the Fourier transform of u, that is, $\hat{u}(\xi)=\int_{R^{m+n}}u(x)e^{-ix\cdot\xi}dx$, with $x\cdot\xi=\sum_{i=1}^{m+n}x_i\xi_i$. The completion of $\mathcal{S}(\boldsymbol{R}^{m+n})$ by the norm $\|\ \|_{a,b}$ can be identified with

$W_{a,b}(\boldsymbol{R}^{m+n})=\{T\in\mathcal{S}'(\boldsymbol{R}^{m+n})|$ The Fourier transform $\hat{T}$ is a function of ξ satisfying $\int|\hat{T}(\xi)|^2(1+|\xi'|^2+|\xi''|^2)^a(1+|\xi'|^2)^b d\xi<\infty\}$.

The space $W_{a,b}(\boldsymbol{R}^{m+n})$ is a Hilbert space with scalar product

$$\int\hat{S}(\xi)\overline{\hat{T}(\xi)}(1+|\xi'|^2+|\xi''|^2)^a(1+|\xi'|^2)^b d\xi$$

for any S and T in $W_{a,b}(\boldsymbol{R}^{m+n})$. We shall denote $W_a(\boldsymbol{R}^{m+n})$ instead of $W_{a,0}(\boldsymbol{R}^{m+n})$ for the sake of brevity. We know that $W_{a,b}(\boldsymbol{R}^{m+n})$ and $W_{-a,-b}(\boldsymbol{R}^{m+n})$ are mutually dual by the sesquilinear form $(S,T)\to\int_{R^{m+n}}\hat{S}(\xi)\overline{\hat{T}(\xi)}d\xi$.

Let δ_{R^m} be the distribution defined by

$$\langle\varphi,\delta_{R^m}\rangle=\int\varphi(x',0)dx'$$

for any φ in $C_0^\infty(\boldsymbol{R}^{m+n})$. For any multi-index $\nu=(\nu_{m+1},\cdots,\nu_{m+n})$ and any distribution T in $\mathcal{D}'(\boldsymbol{R}^m)$ we define a distribution $T\otimes\delta_{R^m}^{(\nu)}$ by

$$\langle\varphi,T\otimes\delta_{R^m}^{(\nu)}\rangle=\Big\langle\frac{\partial^{|\nu|}}{\partial x''^{\nu}}\varphi(x',0),T\Big\rangle.$$

PROPOSITION 2.1.

(i) $W_{a,b}(\boldsymbol{R}^{m+n})\subset W_{a',b'}(\boldsymbol{R}^{m+n})$ *if* $a'\leqq a$ *and* $b'\leqq b$.

(ii) $W_{a,b}(\boldsymbol{R}^{m+n})\subset W_{a+b}(\boldsymbol{R}^{m+n})$ *if* $b\leqq 0$,

$W_{a+b}(\boldsymbol{R}^{m+n})\subset W_{a,b}(\boldsymbol{R}^{m+n})$ *if* $b\geqq 0$.

In the following, we shall denote $|\nu|=\nu_{m+1}+\cdots+\nu_{m+n}$ for multi-index ν.

PROPOSITION 2.2. *A distribution T in $\mathcal{D}'(\boldsymbol{R}^m)$ belongs to $W_a(\boldsymbol{R}^m)$ if and only if $T\otimes\delta_{R^m}^{(\nu)}$ belongs to $W_{b,a-b-|\nu|-n/2}(\boldsymbol{R}^{m+n})$ for some $b<-|\nu|-n/2$. The mapping $T\to T\otimes\delta_{R^m}^{(\nu)}$ has closed range as a mapping from $W_a(\boldsymbol{R}^m)$ to $W_{b,a-b-|\nu|-n/2}(\boldsymbol{R}^{m+n})$.*

PROOF. T belongs to $W_a(\boldsymbol{R}^m)$ if and only if

$$\int_{R^m}|\hat{T}(\xi')|^2(1+|\xi'|^2)^a d\xi'<\infty. \tag{2.2}$$

On the other hand $T\otimes\delta_{R^m}^{(\nu)}$ belongs to $W_{a',b'}(\boldsymbol{R}^{m+n})$ if and only if

$$\int_{R^{m+n}} |\hat{T}(\xi')|^2 |\xi''^{\nu}|^2 (1+|\xi'|^2+|\xi''|^2)^{a'} (1+|\xi'|^2)^{b'} d\xi' d\xi'' < \infty . \tag{2.3}$$

This is finite if and only if both

$$k(\xi') = \int_{R^n} |\xi''^{\nu}|^2 (1+|\xi'|^2+|\xi''|^2)^{a'} d\xi'' \tag{2.4}$$

and

$$\int_{R^m} |\hat{T}(\xi')|^2 (1+|\xi'|^2)^{b'} k(\xi') d\xi' \tag{2.5}$$

are finite. $k(\xi')$ is finite if and only if $2(|\nu|+a') < -n$. If this holds,

$$k(\xi') = (1+|\xi'|^2)^{a'+|\nu|+n/2} \int_{R^n} |\xi''^{\nu}|^2 (1+|\xi''|^2)^{a'} d\xi'' . \tag{2.6}$$

The integral (2.5) is finite if and only if

$$\int |\hat{T}(\xi')|^2 (1+|\xi'|^2)^{a'+|\nu|+b'+n/2} d\xi' \tag{2.7}$$

is finite. This proves our proposition.

COROLLARY 2.2. *The restriction mapping* $\mathcal{S}(\boldsymbol{R}^{m+n}) \ni \varphi(x', x'') \rightarrow D^{\nu}_{x''}\varphi(x', 0) \in \mathcal{S}(\boldsymbol{R}^m)$ *can be extended as a continuous open mapping from* $W_{a,b}(\boldsymbol{R}^{m+n})$ *to* $W_{a+b-|\nu|-n/2}(\boldsymbol{R}^m)$ *if* $a > n/2+|\nu|$. *This is surjective if* $\nu = 0$.

The following lemma is of fundamental importance in this note.

LEMMA 2.3 (cf. [8]). *Let* T_ν, ν *being multi-indices with* $|\nu| \leqq N$ *for some integer* N, *be in* $W_b(\boldsymbol{R}^m)$ *with some* $b \in \boldsymbol{R}$. *Let* T *be the sum* $T = \sum_{|\nu| \leqq N} T_\nu \otimes \delta^{(\nu)}_{R^m}$. *Assume that* T *belongs to* $W_{a,b'}(\boldsymbol{R}^{m+n})$ *for some* b'. *Then* $T_\nu = 0$ *for all* ν *satisfying* $|\nu| \geqq -a-n/2$.

PROOF. Set $T_M = \sum_{|\nu|=M} T_\nu \otimes \delta^{(\nu)}_{R^m}$. Then the fact that $T \in W_{a,b'}(\boldsymbol{R}^{m+n})$ means that the integral

$$\int_{R^{m+n}} |\hat{T}(\xi', \xi'')|^2 (1+|\xi'|^2+|\xi''|^2)^a (1+|\xi'|^2)^{b'} d\xi' d\xi''$$

is finite. Therefore, for almost all ξ',

$$\int |\hat{T}(\xi', \xi'')|^2 (1+|\xi'|^2+|\xi''|^2)^a d\xi''$$

is finite. That is,

$$\int |\sum_{M \leqq N} \hat{T}_M(\xi', \xi'')|^2 (1+|\xi'|^2+|\xi''|^2)^a d\xi'' < \infty . \tag{2.8}$$

Since $\hat{T}_M(\xi', \xi'') = \sum_{|\nu|=M} \hat{T}_\nu(\xi')\xi''^{\nu}$, (2.8) means that $\hat{T}_M(\xi', \xi'') = 0$ if $2M+2a \geqq -n$. Hence we have $T_\nu = 0$ for any ν; $|\nu| \geqq -a-n/2$.

§ 3. Hilbert spaces.

We denote the Hilbert space of square integrable p-currents by $W_0^p(X)$. The scalar product in it is denoted by

$$(\alpha, \beta)=\int_X \alpha \wedge *\beta, \tag{3.1}$$

for any α and β in $W_0^p(X)$. We denote the exterior and interior differentiation in the sense of current by d_0 and δ_0 respectively. Let Δ_0 be the Laplacian operator $d_0\delta_0+\delta_0 d_0$. Then it is well known that $(1+\Delta_0)^{-1}$ exists and that $(1+\Delta_0)^{-1}W_0^p(X)$ is dense in $W_0^p(X)$. The operator Δ_0 restricted to $(1+\Delta_0)^{-1}W_0^p(X)$ is a non-negative self-adjoint operator which is denoted by Δ. We denote its domain by $W_2^p(X)$. We can define $(1+\Delta)^s$ for any real number s. Its domain is denoted by $W_{2s}^p(X)$. It is very easy to check that every section α in $W_{2s}^p(X)$ can be written by coordinates $(x_1, x_2, \cdots, x_{m+n})$ as

$$\alpha=\sum_{1\leqq i_1<\cdots<i_p\leqq m+n} \alpha_{i_1 i_2\cdots i_p}(x)dx_{i_1}\wedge\cdots\wedge dx_{i_p} \tag{3.2}$$

and coefficients $\alpha_{i_1\cdots i_p}(x)$ are locally identified with functions in $W_{2s}(\boldsymbol{R}^{m+n})$ in § 2. Similarly we may consider $W_{2s}^p(Y)$ and Laplacian Δ' on the submanifold Y. Let B be a tubular neighbourhood of Y. Let φ_1 and φ_2 be a C_0^∞ partition of unity subordinate to the open covering $B\cup X-Y$. Any p-current α can be decomposed into two parts; $\alpha=\alpha_1+\alpha_2$, $\alpha_1=\varphi_1\alpha$, $\alpha_2=\varphi_2\alpha$. We shall denote by $W_{a,b}^p(X)$ the space of those currents that satisfy the following conditions; (i) α_2 belongs to $W_{a+b}^p(X)$. (ii) Every coefficient $\alpha_{i_1\cdots i_p}$ in the coordinate expression (3.2) of α_1 coincides with a distribution in $W_{a,b}(\boldsymbol{R}^{m+n})$ introduced in the previous section. The following Theorem holds.

THEOREM 3.1. *$(1+\Delta_0)$ is an isomorphism from $W_{a,b}^p(X)$ onto $W_{a-2,b}^p(X)$.*

All results in § 2 apply to our spaces $W_a^p(X)$ and $W_{a,b}^p(X)$ with obvious modification. In particular, we have

PROPOSITION 3.2.

(i) $$W_{a,b}^p(X)\subset W_{a',b'}^p(X) \qquad \textit{if } a'\leqq a,\ b'\leqq b\,.$$

Injection is completely continuous if $a'<a$ and $a'+b'<a+b$.

(ii) $$W_{a,b}^p(X)\subset W_{a+b}^p(X) \qquad \textit{if } b\leqq 0\,.$$

$$W_{a+b}^p(X)\subset W_{a,b}^p(X) \qquad \textit{if } b\geqq 0\,.$$

PROOF. Assertion (i) follows from Proposition 2.1 and the fact that X is compact.

The space $W_a^p(X)$ has Hilbert space structure of which scalar product is given by

(3.3) $$(\alpha, \beta)_a=((1+\Delta_0)^{a/2}\alpha, (1+\Delta_0)^{a/2}\beta), \qquad a\in R.$$

The parenthesis in the right is the scalar product (3.1).

In the following we fix $a=[n/2]$. The scalar product in the space $W_a^p(X)$, a being odd, is equivalent to

(3.4) $$\begin{aligned}(\alpha, \beta)_a=&((1+\Delta_0)^{[a/2]}\alpha, (1+\Delta_0)^{[a/2]}\beta)\\ &+(d_0(1+\Delta_0)^{[a/2]}\alpha, d_0(1+\Delta_0)^{[a/2]}\beta)\\ &+(\delta_0(1+\Delta_0)^{[a/2]}\alpha, \delta_0(1+\Delta_0)^{[a/2]}\beta).\end{aligned}$$

The following theorem is interesting.

THEOREM 3.3. *The space $W_a^p(X)$ contains $\Omega^p(X)$. Furthermore, the space $\Omega_0^p(X-Y)$ of smooth p-forms with support contained in $X-Y$ is everywhere dense in $W_a^p(X)$.*

PROOF. We have only to prove that $\Omega_0^p(X-Y)$ is dense in $W_a^p(X)$. Assume that $\alpha\in W_a^p(X)$ is orthogonal to $\Omega_0^p(X-Y)$, i.e., $(\alpha, \beta)_a=0$ for any β in $\Omega_0^p(X-Y)$. Then the current α on X satisfies $(1+\Delta_0)^a\alpha=0$ in $X-Y$. The support of $(1+\Delta_0)^a\alpha$ is contained in Y. For any C^∞ function φ with support contained in a small coordinate patch in X, $\varphi(1+\Delta_0)^a\alpha$ has coordinate expression (3.2), where we may assume that $dx_1, \cdots, dx_m$ are cotangent to Y and $dx_{m+1}, \cdots, dx_{m+n}$ are co-normal to Y if support of φ intersects Y. By Schwartz' Theorem, we have for any $i_1, \cdots, i_p$,

$$\alpha_{i_1\cdots i_p}(x)=\sum_\nu \alpha_{i_1\cdots i_p}^{(\nu)}(x')\otimes\delta_Y^{(\nu)},$$

where $\alpha_{i_1\cdots i_p}^{(\nu)}(x')$ are scalar valued distributions on Y (cf. [7]).

$\delta_Y^{(\nu)}$ is transversal derivative of the δ-function in the $(x_{m+1}, \cdots, x_{m+n})$ space as is introduced in the previous section. Since α belongs to $W_a^p(X)$, $(1+\Delta_0)^a\alpha$ must belong to $W_{-a}^p(X)$. We can apply Lemma 2.3 and obtain that $\alpha_{i_1\cdots i_p}^{(\nu)}(x')=0$. This implies that $\varphi(1+\Delta_0)^a\alpha=0$. Since φ is arbitrary, $(1+\Delta_0)^a\alpha=0$. The operator $(1+\Delta_0)^a$ being invertible, this proves the theorem.

REMARK 3.4. Theorem 3.3 enables us to identify any α in $W_a^p(X)$ with a current γ over $X-Y$. In fact, we identify α with γ by the formula

(3.5) $$(\alpha, \beta)_a=\int_{X-Y}\gamma\wedge*\beta, \qquad \text{for any } \beta \text{ in } \Omega_0^p(X-Y).$$

Considering (3.3) or (3.4), we have

(3.6) $$\gamma=(1+\Delta_0)^a\alpha \qquad \text{restricted on } X-Y.$$

§ 4. Coordinate expression.

Let B be the tubular neighbourhood of Y which consists of points of X with distance less than ε from Y. We fix decomposition of cotangent bundle

$T^*(B)$;

(4.1) $$T^*(B) \cong T^*(Y) + N^*(Y)$$

where $N^*(Y)$ is the conormal bundle of Y. We have isomorphism $\Lambda^p(B) \cong \bigoplus_{k=0}^{p} \Lambda^{p-k}(Y) \otimes \Lambda^k N^*(Y)$, where $\Lambda^p(B)$ and $\Lambda^p(Y)$ are exterior p-products of $T^*(B)$ and $T^*(Y)$ respectively. Any p-form or current of degree p on X has decomposition

$$\alpha = \sum_{k=0}^{p} \alpha_{p-k,k}, \qquad \alpha_{p-k,k} \in \Lambda^{p-k}(Y) \otimes \Lambda^k N^*(Y),$$

in the neighbourhood B of Y. We shall call $\alpha_{p,0}$ the tangential component of α and $\sum_{k=1}^{p} \alpha_{p-k,k}$ the normal component of it. A p-form α in $\Omega^p(X)$ belongs to $\Omega^p(X, Y)$ if and only if its tangential part vanishes on Y.

Let d' and d'' denote the exterior differential operator restricted on tangential and normal components respectively. These are well defined in B. In other words, $d_0\alpha = d'\alpha' + d''\alpha''$ if $\alpha = \alpha' + \alpha''$ be the decomposition of α into the sum of its tangential and normal components. $d'\alpha'$ is again tangential.

We shall denote by $*$ and $*'$ the "star-operations" in $\Omega^*(X)$ and $\Omega^*(Y)$ respectively. Here $*'$ is defined in $\Omega^*(Y)$ with respect to the induced Riemannian metric in Y. Interior differential operators δ_0 and δ' are defined on X and Y respectively (cf. [2]). If $\alpha \in \Omega^p(X)$ is tangential to Y, then we may consider both $(\delta_0\alpha)|_Y$ and $\delta'(\alpha|_Y)$. As to these two operations we can prove

PROPOSITION 4.1. *If $\alpha \in \Omega^p(X)$ is tangential to Y, then $\delta_0\alpha$ is also tangential to Y at every point y in Y and is equal to $\delta'(\alpha|_Y)$.*

PROOF. We choose a coordinate system in an open set of X around a point y_0 in Y as follows. Let y_0 be an arbitrary point of Y. Let $y^1, \cdots, y^{m+n}$ be a normal coordinate of X at y_0. The metric tensor of X is of the form

(4.2) $$g^{ij}(y) = (dy^i, dy^j) = \delta^{ij} + O(|y|^2), \qquad i, j = 1, \cdots, m+n$$

where δ^{ij} is Kronecker index. We may assume that the local equations of the submanifold Y are

(4.3) $$y^r - \varphi^r(y^1, \cdots, y^m) = 0, \qquad r = m+1, \cdots, m+n$$

and that Taylor expansion of φ^r are

$$\varphi^r(y^1, \cdots, y^m) = \frac{1}{2} \sum_{\lambda,\mu=1}^{m} A^r_{\lambda\mu} y^\lambda y^\mu + O(|y|^3).$$

Indices $i, j, k, \cdots$ run from 1 to $m+n$, $\lambda, \mu, \nu, \cdots$ run from 1 to m and $r, s, t, \cdots$ run from $m+1$ to $m+n$ in the following. The quadratic form

$$(4.4)\qquad (y^1, \cdots, y^m) \longrightarrow \sum_{\lambda\mu} A^r_{\lambda\mu} y^\lambda y^\mu$$

is the second fundamental form of Y at x_0. We make the following change of coordinates:

$$(4.5)\qquad \begin{cases} x^\lambda = y^\lambda \\ x^r = y^r - \varphi^r(y^1, \cdots, y^m)\,. \end{cases}$$

Now the equation of Y are $x^r = 0$ in the new coordinates. We have

$$(4.6)\qquad dx^r = dy^r - \sum_{\lambda\mu} A^r_{\lambda\mu} y^\lambda dy^\mu + O(|y|^2)$$

and

$$(4.7)\qquad \begin{cases} (dx^r, dx^s) = \delta^{rs} + O(|x|^2) \\ (dy^\nu, dx^r) = -\sum_\lambda A^r_{\lambda\nu} y^\lambda + O(|x|^2)\,. \end{cases}$$

Setting

$$(4.8)\qquad \begin{cases} \pi^\nu = dy^\nu + \sum_r (dy^\nu, dx^r) dx^r \\ \pi^r = dx^r\,, \end{cases}$$

we have

$$(4.9)\qquad \begin{cases} (\pi^i, \pi^j) = \delta^{ij} + O(|x|^2) \\ (\pi^r, \pi^\lambda) = 0\,. \end{cases}$$

Hence we can choose functions $a^\lambda_\mu(x)$ and $b^r_s(x)$ such that

$$(4.10)\qquad \begin{aligned} \omega^\lambda &= \pi^\lambda + \sum_\mu a^\lambda_\mu \pi^\mu \\ \omega^r &= \pi^r + \sum_s b^r_s \pi^s \end{aligned}$$

satisfy

$$(4.11)\qquad (\omega^i, \omega^j) = \delta^{ij}\,.$$

In fact, we can choose a^λ_μ, b^r_s as $a^\lambda_\mu(x) = O(|x|^2)$, $b^r_s = O(|x|^2)$ because of (4.9). Let $\{X_j\}_{j=1}^{m+n}$ be the orthonormal frame of $T(X)$ which is dual to the frame $\{\omega^j\}_{j=1}^{m+n}$ in $T^*(X)$. $\{X_\lambda\}_{\lambda=1}^m$ is tangent to Y. We have

$$(4.12)\qquad \begin{aligned} d\omega^\nu &= -\sum_{\lambda r} A^r_{\lambda\nu} \omega^\lambda \wedge \omega^r + O(|x|) \\ d\omega^r &= O(|x|) \end{aligned}$$

and $*'1 = \omega^1 \wedge \cdots \wedge \omega^m$, and $*1 = \omega^1 \wedge \cdots \wedge \omega^{m+n}$.

Let us prove Proposition 4.1. We may assume that

$$\alpha = a(x)\omega^1 \wedge \cdots \wedge \omega^p\,, \qquad p \leqq m\,,$$

where $a(x)$ is a function.
We have

$$*\alpha = a(x)\omega^{p+1} \wedge \cdots \wedge \omega^m \wedge \omega^{m+1} \wedge \cdots \wedge \omega^{m+n}$$

and

$$\begin{aligned} d*\alpha &= da \wedge \omega^{p+1} \wedge \cdots \wedge \omega^m \wedge \omega^{m+1} \wedge \cdots \wedge \omega^{m+n} \\ &\quad + ad(\omega^{p+1} \wedge \cdots \wedge \omega^m) \wedge \omega^{m+1} \wedge \cdots \wedge \omega^{m+n} \\ &\quad + (-1)^{m-p} a\omega^{p+1} \wedge \cdots \wedge \omega^m \wedge d(\omega^{m+1} \wedge \cdots \wedge \omega^{m+n}) \\ &= \sum_\lambda (X_\lambda a)\omega^\lambda \wedge \omega^{p+1} \wedge \cdots \wedge \omega^{m+1} \wedge \cdots \wedge \omega^{m+n} + O(|x|). \end{aligned}$$

Therefore

$$\begin{aligned} *d*\alpha(x_0) &= \sum_\lambda X_\lambda a(x_0) * (\omega^\lambda \wedge \omega^{p+1} \wedge \cdots \wedge \omega^{m+n}) \\ &= (-1)^{n(p-1)} \sum_\lambda (X_\lambda a)(x_0) *'(\omega^\lambda \wedge \omega^{p+1} \wedge \cdots \wedge \omega^{m+n}). \end{aligned}$$

On the other hand

$$\alpha|_Y = a|_Y \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^p,$$

$$*'(\alpha|_Y) = a|_Y \omega^{p+1} \wedge \cdots \wedge \omega^m,$$

and

$$\begin{aligned} d'*'(\alpha|_Y) &= \sum_\lambda (X_\lambda a|_Y)\omega^\lambda \wedge \omega^{p+1} \wedge \cdots \wedge \omega^m \\ &\quad + \sum a|_Y d(\omega^{p+1} \wedge \cdots \wedge \omega^m)|_Y \\ &= \sum_\lambda X_\lambda(a|_Y)\omega^\lambda \wedge \omega^{p+1} \wedge \cdots \wedge \omega^m + O(|y|). \end{aligned}$$

Hence

$$*'d'*'(\alpha|_Y)(x_0) = \sum_\lambda X_\lambda(a|_Y) *'(\omega^\lambda \wedge \omega^{p+1} \wedge \cdots \wedge \omega^m).$$

Since $\delta_0\alpha = (-1)^{(m+n+1)(p-1)+p} * d * \alpha$ and $\delta'\alpha = (-1)^{(m+1)(p-1)+p} *'d'*'(\alpha|_Y)$, we have proved $\delta_0\alpha(x_0) = (\delta'\alpha_Y)(x_0)$.

§ 5. Closed operator d and its adjoint d^*.

The exterior differential operator d_0 restricted to $\Omega^p(X, Y)$ is a closable operator in the Hilbert space $W_a^p(X)$. Let us denote its smallest extension and its adjoint by d and d^* respectively. In order to obtain informations about the domain $D(d)$ and $D(d^*)$ of d and d^*, we shall make some preparation.

Let T be any p-current on Y, then we denote by $T \otimes \delta_Y$ the following p-current on X:

$$\int_X \alpha \wedge *(T \otimes \delta_Y) = \int_Y \alpha|_Y \wedge *'T \tag{5.1}$$

for any α in $\Omega^p(X)$. If $T \equiv 1$ on Y, we shall denote $1 \otimes \delta_Y$ by δ_Y briefly. We have the following formulae:

$$d_0(T\otimes\delta_Y)=(d'T)\otimes\delta_Y+(-1)^pT\otimes d_0\delta_Y \tag{5.2}$$

$$\delta_0(T\otimes\delta_Y)=(\delta'T)\otimes\delta_Y\,, \tag{5.3}$$

where T is any p-current on Y. Equality (5.3) is a consequence of Proposition 4.1 and the fact that the tangential component of $d_0\delta_Y$ vanishes.

Now we come back to the discussion of $D(d^*)$.

THEOREM 5.1. *A current α in $W_a^{p+1}(X)$ belongs to $D(d^*)$ if and only if there are γ in $W_a^p(X)$ and T in $W_{-a-1+n/2}^p(Y)$ such that*

$$\delta_0\alpha-\gamma=(1+\Delta_0)^{-a}(T\otimes\delta_Y)\,. \tag{5.4}$$

If this holds, $d^*\alpha$ is equal to γ and $\delta'T$ belongs to $W_{-a-1+n/2}^p(Y)$.

PROOF. $\alpha\in W_a^{p+1}(X)$ belongs to $D(d^*)$ if and only if there is an element γ in $W_a^p(X)$ such that

$$(\gamma,\beta)_a=(\alpha,d\beta)_a \qquad \text{for any } \beta \text{ in } \Omega^p(X,Y)\,. \tag{5.5}$$

And $\gamma=d^*\alpha$ if (5.5) holds. Assume that α belongs to $D(d^*)$. Then (5.5) holds for any β in $\Omega_0^p(X-Y)$. This means that

$$(1+\Delta_0)^a(\delta_0\alpha-\gamma)=0 \qquad \text{in } X-Y\,. \tag{5.6}$$

Let x_0 be an arbitrary point of Y. Take a small open set U of X containing x_0. We assume that U is so small that we have coordinate expression introduced in §4. We take a C^∞-function φ with support in U. Then $\sigma=(1+\Delta_0)^a(\delta_0\alpha-\gamma)$ has expression

$$\varphi(1+\Delta_0)^a(\delta_0\alpha-\gamma)=\sum\beta_{i_1\cdots i_p}^{r_1\cdots r_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\otimes X_{r_1}\cdots X_{r_k}\delta_Y\,. \tag{5.7}$$

The summation ranges over all indices $1\leqq i_1\leqq\cdots\leqq i_p\leqq m+n$, $m+1\leqq r_1\leqq\cdots\leqq r_k\leqq m+n$, where k runs over all non-negative integers. The coefficients $\beta_{i_1\cdots i_p}^{r_1\cdots r_k}$ are scalar distributions on Y. The equality (5.7) is a consequence of (5.6) and Schwartz' theorem. (cf. [7] page 102.)

Since α is an element of $W_a^{p+1}(X)$, σ must belong to $W_{-a-1}^p(X)$. Thus for any fixed index $i_1\cdots i_p$, the 0-current

$$\sum_k\sum_{r_1\cdots r_k}\beta_{i_1\cdots i_p}^{r_1\cdots r_k}X_{r_1}\cdots X_{r_k}\delta_Y$$

belongs to $W_{-a-1}(X)$. Applying Lemma 2.3, we have

$$\beta_{i_1\cdots i_p}^{r_1\cdots r_k}=0\,, \qquad \text{if } k\geqq 1\,.$$

Therefore

$$\varphi(1+\Delta_0)^a(\delta_0\alpha-\gamma)=\sum_{i_1\cdots i_p}\beta_{i_1\cdots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\otimes\delta_Y\,.$$

Each of the scalar distributions $\beta_{i_1\cdots i_p}$ is defined only on Y. However, we may assume that U is diffeomorphic to the direct product $(Y\cap U)\times V$, where V is an open set in $\boldsymbol{R}^n$. If we denote by 1_v the function on V which is constantly one, then $\beta_{i_1\cdots i_p}\otimes 1_V$ is a distribution on U. Thus we see that

$$\beta=\sum_{i_1\cdots i_p}\beta_{i_1\cdots i_p}\otimes 1_V\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

is a p-current on U. We have

$$\varphi(1+\Delta_0)^a(\delta_0\alpha-\gamma)=\delta_Y\cdot\beta\,,$$

where $\delta_Y\beta$ is a product of current δ_Y and β. Operating δ_0 to both sides of this, we have (cf. [2])

$$-(d_0\delta_Y)\lrcorner\beta+\delta_Y\cdot(\delta_0\beta)=-\varphi(1+\Delta_0)^a\delta_0\gamma-(d_0\varphi)\lrcorner(1+\Delta_0)^a(\delta_0\alpha-\gamma)\,. \tag{5.8}$$

Since the right side of (5.8) belongs to $W^{p-1}_{-a-1}(X)$ and $(\delta_0\beta)\delta_Y$ belongs to $W^{p-1}_{b,-a-b-2}(X)$, $-(d_0\delta_Y)\lrcorner\beta$ must belong to $W^{p-1}_{b,-a-b-2}(X)$. The coordinate expression of $(d_0\delta_Y)\lrcorner\beta$ is equal to

$$\sum_{r=m+1}^{m+n}(\beta_{r\,i_1\cdots i_{p-1}}\otimes 1_V)\frac{\partial}{\partial y_r}\delta_Y=\sum_{r=m+1}^{m+n}\beta_{r\,i_1\cdots i_{p-1}}\frac{\partial}{\partial y_r}\delta_Y\,.$$

By Lemma 2.3, we know that $\beta_{ri_1\cdots i_p}=0$ if $r\geqq m+1$. This is equivalent to the fact that $\beta=\beta'\otimes 1_Y$, where β' is a current on Y (i. e. β' is tangential to Y). Since $d_0\delta_Y\lrcorner\beta=0$, $\delta_Y(\delta_0\beta)\in W^{p-1}_{-a-1}(Y)$. This again implies that $\delta_0\beta'$ belongs to $W^{p-1}_{-a-1+n/2}(Y)$. Collecting this result by partition of unity, we have proved that

$$(1+\Delta_0)^a(\delta_0\alpha-\gamma)=T\otimes\delta_Y\,, \tag{5.9}$$

and $T\in W^p_{-a-1+n/2}(Y)$ and $\delta'T$ belongs to $W^{p-1}_{-a-1+n/2}(Y)$.

Conversely, if (5.9) holds with a current T on Y, then for any β in $\Omega^p(X)$ we have

$$\begin{aligned}(5.10)\qquad &\int_X(1+\Delta_0)^{a/2}\alpha\wedge*(1+\Delta_0)^{a/2}d_0\beta\\ &=\int_X(1+\Delta_0)^a\gamma\wedge*\beta+\int_X T\otimes\delta_Y\wedge*\beta\\ &=\int_X(1+\Delta_0)^{a/2}\gamma\wedge*(1+\Delta_0)^{a/2}\beta+\int_Y T\wedge*'\beta|_Y\,.\end{aligned}$$

Since $\beta|_Y=0$ for any β in $\Omega^p(X, Y)$, we have $(\alpha, d_0\beta)_a=(\gamma, \beta)_a$. This proved our theorem.

REMARK 5.2. The currents $\gamma\in W^p_a(X)$ and $T\in W^p_{-a-1+n/2}(Y)$ are uniquely determined by α. In fact Lemma 2.3 implies that $T=0$ if $(1+\Delta_0)^{-a}(T\otimes\delta_Y)$ belongs to $W^p_a(X)$.

We know that $W^{p+1}_{a+1}(X)$ is contained in $D(d^*)$. But $W^{p+1}_{a+1}(X)$ is strictly smaller than $D(d^*)$. In fact, we have

PROPOSITION 5.3. *If* $T\in W^p_{-a-1+n/2}(Y)$ *with* $\delta'T\in W^{p-1}_{-a-1+n/2}(Y)$, *then*

$$\beta=d_0(1+\Delta_0)^{-1-a}(T\otimes\delta_Y) \tag{5.11}$$

belongs to $D(d^*)$ *and*

$$d^*\beta=-(1+\varDelta_0)^{-a-1}(T\otimes\delta_Y)-d_0(1+\varDelta_0)^{-a-1}(\delta' T\otimes\delta_Y)\,.$$

PROOF. We have

$$\begin{aligned}\delta_0\beta&=\delta_0 d_0(1+\varDelta_0)^{-a-1}(T\otimes\delta_Y)\\&=(1+\varDelta_0)^{-a}(T\otimes\delta_Y)-(1+d_0\delta_0)^{-a-1}(T\otimes\delta_Y)\,.\end{aligned}$$

Hence

$$\delta_0\beta-\gamma=(1+\varDelta_0)^{-a}(T\otimes\delta_Y)\,,$$

where

$$\gamma=-(1+\varDelta_0)^{-a-1}(T\otimes\delta_Y)-d_0(1+\varDelta_0)^{-a-1}(\delta' T\otimes\delta_Y)$$

belongs to $W^p_{2a+b+2,-a-b-1}(X)+W^p_{2a+b+1,-a-b-1}(X)\subset W^p_a(X)$. This proves proposition.

Furthermore we can prove

PROPOSITION 5.4. *Any* $\alpha\in D(d^*)$ *can be written as*

$$\alpha=\beta+\sigma\,,$$

where β *is as in* (5.11) *and* $\sigma\in W^{p+1}_a(X)$ *with* $\delta_0\sigma\in W^p_a(X)$ *and* $d_0\sigma=d_0\alpha$.

PROOF. If α belongs to $D(d^*)$, we have

$$\delta_0\alpha=d^*\alpha+(1+\varDelta_0)^{-a}(T\otimes\delta_Y)$$

with T in $W^p_{-a-1+n/2}(Y)$ and $\delta' T\in W^{p-1}_{-a-1+n/2}(Y)$. Setting $\beta=d_0(1+\varDelta_0)^{-a-1}(T\otimes\delta_Y)$, we have $\delta_0\beta=d^*\beta+(1+\varDelta_0)^{-a}(T\otimes\delta_Y)$. Thus $\sigma=\alpha-\beta$ belongs to $W^{p+1}_a(X)$ and satisfies

$$\delta_0\sigma=d^*\alpha-d^*\beta\in W^p_a(X)\quad\text{and}\quad d_0\sigma=d_0\alpha\,.$$

Little is known about the domain of d. A trivial fact is

PROPOSITION 5.5. *If* $\alpha\in D(d)$, *then* $\alpha\in W^p_a(X)$ *and* $d\alpha=d_0\alpha\in W^{p+1}_a(X)$.

In order to characterize $D(d)\cap D(d^*)$, we begin with the following

LEMMA 5.6. *If a p-current* α *belongs to* $D(d)\cap D(d^*)$, *then the p-current* σ *in Proposition* 5.4 *belongs to* $W^p_{a+1}(X)$.

PROOF. If $\alpha\in D(d)\cap D(d^*)$, then $d_0\alpha=d\alpha$ belongs to $W^{p+1}_a(X)$. Therefore $\delta_0\sigma$ belongs to $W^{p-1}_a(X)$ and $d_0\sigma$ is contained in $W^{p+1}_a(X)$. Hence σ belongs to $W^p_{a+1}(X)$ by well known fact.

LEMMA 5.7. *Let* $\beta=d_0(1+\varDelta_0)^{-a-1}(T\otimes\delta_Y)$ *with* T *in* $W^{p-1}_{-a-1+n/2}(Y)$. *Then the tangential component* $\beta_{p,0}$ *of* β *coincides with a current belonging to* $W^p_{2a+b+2,-a-b-2}(X)$, $b<-n/2$, *in a neighbourhood of* Y. *Furthermore, if* $d'T$ *is in* $W^p_{-a-1+n/2}(Y)$, *the tangential component* $\beta_{p,0}$ *coincides with a current belonging to* $W^p_{2a+b+2,-a-b-1}(X)$ *in a neighbourhood of* Y.

PROOF. We have

$$\beta=(1+\varDelta_0)^{-a-1}\{(d' T\otimes\delta_Y)+(-1)^{p-1}(T\otimes d_0\delta_Y)\}\,. \tag{5.12}$$

The tangential component of $T\otimes d_0(\delta_Y)$ vanishes in a neighbourhood of Y. Let ξ be a cotangent vector $\in T^*(X)$. Then the principal symbol of $(1+\Delta_0)^{-a-1}$ is $|\xi|^{-2a-2}$ and the second symbol is of degree $-2a-4$, where $|\xi|$ is the length of the vector ξ. This implies that the tangential component of $(1+\Delta_0)^{-a-1}(T\otimes d_0\delta_Y)$ belongs to $W^p_{2a+b+3,-a-b-1}(X)$ in a neighbourhood of Y. On the other hand, $(1+\Delta_0)^{-a-1}(d'T\otimes\delta_Y)$ belongs to $W^p_{2a+b+2,-a-b-2}(X)$. Thus we have proved the first part of the Lemma. If $d'T\in W^p_{-a-1+n/2}(Y)$, then $(1+\Delta_0)^{-a-1}(d'T\otimes\delta_Y)$ belongs to $W^p_{2a+b+2,-a-b-1}(X)$. This completes the proof. Here we used the fact that $(1+\Delta_0)^{-a-1}$ is a pseudo-differential operator (cf. [**3**] or [**4**]).

The next lemma plays an important role in the following. (cf. [**9**].)

LEMMA 5.8. *Let $\lambda>0$ and P_λ be the transformation of currents on Y defined by*

$$P_\lambda\colon T\to[(\lambda+\Delta_0)^{-1}(1+\Delta_0)^{-a}(T\otimes\delta_Y)]|_Y\,. \tag{5.13}$$

Then P_λ is an elliptic pseudo-differential operator of order $-2a-2+n$. Furthermore P_λ is an invertible non-negative essentially self-adjoint operator in $W^p_0(Y)$.

PROOF. Let $x_0\in Y$, ξ' be in $T^*_{x_0}(Y)$ and η be in $N^*_{x_0}(Y)$. The principal symbol of $(\lambda+\Delta_0)^{-1}(1+\Delta_0)^{-a}$ is $(|\xi'|^2+|\eta|^2)^{-a-1}$. Hence the principal symbol of P_λ is

$$\frac{1}{(2\pi)^n}\int_{R^n}\frac{d\eta}{(|\xi'|^2+|\eta|^2)^{a+1}}=C_n|\xi'|^{-2a-2+n} \tag{5.14}$$

where

$$C_n=\frac{1}{(2\pi)^n}\int_{R^n}\frac{d\eta}{(1+|\eta|^2)^{a+1}}\,. \tag{5.15}$$

Thus P_λ is an elliptic operator of order $-2a-2+n$. Let φ and ϕ be p-currents belonging to $W^p_0(X)$. Then

$$\begin{aligned}\Phi_\lambda&=(\lambda+\Delta_0)^{-1/2}(1+\Delta_0)^{-a/2}(\varphi\otimes\delta_Y)\\ \Psi_\lambda&=(\lambda+\Delta_0)^{-1/2}(1+\Delta_0)^{-a/2}(\phi\otimes\delta_Y)\end{aligned} \tag{5.16}$$

belong to $W^p_{a+b+1,-n/2-b}(X)\subset W^p_0(X)$. And we have

$$\begin{aligned}\int_Y P_\lambda\varphi\wedge*'\phi&=\int_Y[(\lambda+\Delta_0)^{-1}(1+\Delta_0)^{-a}(\varphi\otimes\delta_Y)]|_Y\wedge*'\phi\\ &=\int_X(\lambda+\Delta_0)^{-1}(1+\Delta_0)^{-a}(\varphi\otimes\delta_Y)\wedge*(\phi\otimes\delta_Y)\\ &=\int_X\Phi_\lambda\wedge*\Psi_\lambda\,.\end{aligned} \tag{5.17}$$

Here we applied equality (5.1). Thus,

$$\int_Y P_\lambda\varphi\wedge*'\varphi=\int_X\Phi_\lambda\wedge*\Phi_\lambda\geqq 0 \tag{5.18}$$

and equality holds if and only if $\Phi_\lambda=0$. This is equivalent to saying that $\varphi=0$.

By the way we have proved

COROLLARY 5.9. *For any ϕ and φ in $W_0^p(Y)$,*

$$\int_Y P_\lambda\varphi\wedge *'\phi=\int_X \Phi_\lambda\wedge *\Psi_\lambda \tag{5.19}$$

where

$$\Phi_\lambda=(\lambda+\Delta_0)^{-1/2}(1+\Delta_0)^{-a/2}(\varphi\otimes\delta_Y) \tag{5.20}$$

$$\Psi_\lambda=(\lambda+\Delta_0)^{-1/2}(1+\Delta_0)^{-a/2}(\phi\otimes\delta_Y).$$

Similar discussion proves

PROPOSITION 5.10. *Let $\lambda>0$ and Q_λ denote the transformation of currents on Y defined by*

$$Q_\lambda:\ T\to[(\lambda+\Delta_0)^{-1}(1+\Delta_0)^{-a}(T\otimes d_0\delta_Y)]|_Y. \tag{5.21}$$

Then Q_λ is an pseudo-differential operator of order $-2a-3+n$.

We denote by V^p the space of $(1+\Delta_0)^{-a-1}(T\otimes\delta_Y)$, where T satisfies conditions in Proposition 5.3.

Now we can prove

THEOREM 5.11. *$D(d)\cap D(d^*)$ is contained in $W^p_{2a+b+1,-a-b}(X)$. A current α in $W^p_{a+1}(X)+dV^{p-1}$ belongs to $D(d)\cap D(d^*)$ if and only if the tangential part $\alpha_{p,0}$ vanishes on Y.*

PROOF. Let a p-current α be in $D(d)\cap D(d^*)$. We have the decomposition $\alpha=\beta+\sigma$

$$\sigma\in W^p_{a+1}(X),\ \beta=d_0(1+\Delta_0)^{-a-1}(T\otimes\delta_Y)\quad\text{and}\quad T\in W^{p-1}_{-a-1+n/2}(Y) \tag{5.21}$$

with $\delta' T\in W^{p-2}_{-a-1+n/2}(Y)$. Thus, we have the restriction $\alpha|_Y$ by Lemma 5.7. This $\alpha|_Y$ belongs to $W^p_{a-n/2}(X)$. Let us prove that $\alpha|_Y=0$. Take an arbitrary μ in $\Omega^p(Y)$ and set

$$\nu=d_0(1+\Delta_0)^{-1-a}(\mu\otimes\delta_Y). \tag{5.22}$$

Then ν belongs to $D(d^*)$ by Proposition 5.3. So we have

$$(d\alpha,\nu)_a=(\alpha,d^*\nu)_a. \tag{5.23}$$

On the other hand, we can prove the identity

$$(\alpha,d^*\nu)_a=(d_0\alpha,\nu)_a-\int_Y\alpha|_Y\wedge *'\mu. \tag{5.24}$$

Once we admit this, we have

$$\int_Y\alpha|_Y\wedge *'\mu=0\qquad\text{for any }\mu\text{ in }\Omega^p(Y).$$

Hence $\alpha|_Y=0$.

Now we prove (5.24). We can find a sequence $\{\varphi_k\}_{k=1}^{\infty}$ of C^{∞}-functions on X satisfying the following conditions;

(i) Support of φ_k is contained in the tubular neighbourhood B of Y.

(ii) $\lim_{k\to\infty}\varphi_k=\delta_Y$ in $W_b^0(X)$ with some $b<-n/2$.

(iii) $d\varphi_k(x)\in N_x^*(Y)$ if $x\in Y$.

Set $\nu_k=d_0(1+\Delta_0)^{-a-1}(\mu\otimes\varphi_k)$, then $\nu_k\in D(d^*)$ and converges to ν in $W_a^{p+1}(X)$. $\{d^*\nu_k\}$ does not converge to $d^*\nu$. However, $d^*\nu_k-(1+\Delta_0)^{-a}(\mu\otimes\varphi_k)$ converges to $d^*\nu$. In fact, $\delta_0\nu_k=d^*\nu_k$ and $\delta_0\nu_k-(1+\Delta_0)^{-a}(\mu\otimes\varphi_k)=-(1+\Delta_0)^{-a-1}(\mu\otimes\varphi_k)$ $-d_0(1+\Delta_0)^{-a-1}\delta_0(\mu\otimes\varphi_k)=-(1+\Delta_0)^{-a-1}(\mu\otimes\varphi_k)-d_0(1+\Delta_0)^{-a-1}(\delta'\mu\otimes\varphi_k)$, because $d\varphi_k\in N^*(Y)$ implies that $d\varphi_k \lrcorner \mu_k=0$. $\mu\otimes\varphi_k$ and $\delta'\mu\otimes\varphi_k$ converges to $\mu\otimes\delta_Y$ and $\delta'\mu\otimes\delta_Y$ in $W_b^p(X)$ and $W_b^{p-1}(X)$ respectively. Hence $\{\delta_0\nu_k-(1+\Delta_0)^{-a}(\mu\otimes\varphi_k)\}$ converges to $\delta_0\nu-(1+\Delta_0)^{-a}(\mu\otimes\delta_Y)=d^*\nu$ in $W_a^p(X)$. Now we have

$$\begin{aligned}(\alpha,\ &\delta_0\nu_k-(1+\Delta_0)^{-a}(\mu\otimes\varphi_k))_a\\ &=\int_X(1+\Delta_0)^{a/2}\alpha\wedge*(1+\Delta_0)^{a/2}(\delta_0\nu_k-(1+\Delta_0)^{-a}(\mu\otimes\varphi_k))\\ &=\int_X(1+\Delta_0)^{a/2}d_0\alpha\wedge*(1+\Delta_0)^{a/2}\nu_k-\int_X\alpha\wedge*(\mu\otimes\varphi_k)\,.\end{aligned}$$

Letting k go to infinity, we have (5.24).

Now we make use of the fact $\alpha|_Y=0$. This implies that $\beta|_Y=-\sigma|_Y$ belongs to $W_{a+1-n/2}^p(Y)$. On the other hand we have $\beta|_Y=d'P_1T$. Thus, $P_1d'T=\beta|_Y-[d',P_1]T$ belongs to $W_{a+1-n/2}^p(Y)$, because the commutator $[d',P_1]$ $=d'P_1-P_1d'$ of d' and P_1 is of order $-2a-2+n$. We know from Lemma 5.8 that $d'T$ belongs to $W_{-a-1+n/2}^p(Y)$. Thus, combining this with the fact that $\delta'T\in W_{-a-1+n/2}^{p-2}(Y)$, we have proved that T is contained by $W_{-a+n/2}^{p-1}(Y)$. This implies that β belongs to $W_{2a+b+1,-a-b}^p(X)$.

Conversely, if we assume (5.21) and $\alpha|_Y=0$, then we can prove $T\in W_{-a+n/2}^{p-1}(Y)$ as above. The fact that α belongs to $D(d^*)$ is clear. We have only to prove that α belongs to $D(d)$. Let β be in $D(d^*)$. Set $\delta_0\beta=$ $d^*\beta+(1+\Delta_0)^{-a}(S\otimes\delta_Y)$, $S\in W_{-a-1+n/2}^{p+1}$ and $\delta'S\in W_{-a-1+n/2}^p$. Then

$$\begin{aligned}(\alpha,\ d^*\beta)_a&=\int_X(1+\Delta_0)^{a/2}\alpha\wedge*(1+\Delta_0)^{a/2}d^*\beta\\ &=\int_X(1+\Delta_0)^{a/2}\alpha\wedge*(1+\Delta_0)^{a/2}\delta_0\beta-\int_Y\alpha|_Y\wedge*'S\\ &=(d_0\alpha,\ \beta)_a\,,\end{aligned}$$

because $\alpha|_Y=0$. This completes proof of the theorem.

Summing up, we have

THEOREM 5.12. *The following sequence of vector spaces is exact;*

(5.25) $$0 \longrightarrow D(d) \cap D(d^*) \xrightarrow{\iota} W^p_{\alpha+1}(X) + d_0 V^{p-1} \xrightarrow{\iota'} W^p_{\alpha+1-n/2}(Y) \longrightarrow 0\,,$$

where ι' is the restriction mapping and ι is inclusion.

PROOF. We have only to prove that ι' is an onto mapping. This is clear because ι' maps $W^p_{\alpha+1}(X)$ onto $W^p_{\alpha+1-n/2}$.

§ 6. Relative Hodge-Kodaira decomposition.

We introduce generalized Laplacian operator

(6.1) $$L = dd^* + d^*d\,.$$

The operator L is a non-negative self-adjoint operator in $W^p_\alpha(X)$. Thus $(\lambda+L)^{-1}$ exists if $\lambda>0$. Theorem 5.11 implies that $(\lambda+L)^{-1}$ is a completely continuous operator. As a consequence (cf. [2] and [5]),

THEOREM 6.1. *The spectrum of L consists of eigenvalues of finite multiplicity. The range of L is a closed subspace of $W^p_\alpha(X)$, $p=0, 1, 2, \cdots, m+n$.*

We have commutative relations

PROPOSITION 6.2. *$Ld=dL$, $d^*L=Ld^*$, $d(\lambda+L)^{-1} \supset (\lambda+L)^{-1}d$, $d^*(\lambda+L)^{-1} \supset (\lambda+L)^{-1}d^*$, where $A \supset B$ means that A is defined and coincides with B on the domain of B.*

PROPOSITION 6.3. *A p-current α belongs to* $\ker(L)$ *if and only if $\alpha \in D(d) \cap D(d^*)$ and $d\alpha=0$ and $d^*\alpha=0$. Moreover α is orthogonal to the image of d.*

Proof is ommitted here.

Let H be the orthogonal projection onto the kernel of L. H is given by the formula

(6.2) $$H = \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} (\lambda - L)^{-1} d\lambda\,,$$

where ε is so small that all positive eigenvalues of L lie outside the circle $|\lambda|=\varepsilon$ of the complex plane. We define the Green operator of L as

(6.3) $$G = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} (\lambda - L)^{-1} d\lambda\,,$$

where Γ is a contour enclosing all positive eigenvalues.

PROPOSITION 6.4. *G is a bounded linear mapping which satisfies*

(6.4) $$GH = HG = 0\,, \qquad LG = I - H \supset GL\,,$$

where I is the identity.

THEOREM 6.5. *We have the following decomposition;*

(6.5) $$I = (dd^* + d^*d)G + H\,,$$

(6.6) $$I=dGd^*+d^*Gd+H \qquad on\ D(d)\cap D(d^*).$$

So far are formal consequences of preceding sections.

Our main aim in this section is to prove

THEOREM 6.6. *The kernel of L is canonically isomorphic to the relative de-Rham cohomology group $H^*(X, Y)$.*

In order to prove this, we define following graded vector spaces:

(6.7)
$$U_0^*=\bigoplus_{p=0}^{m+n} U_0^p, \qquad U_0^p=D(d)\cap D(d^*)+d(D(d)\cap D(d^*)),$$
$$U_1^*=\bigoplus_{p=0}^{m+n} U_1^p, \qquad U_1^p=W_{a+1}^{p}(X)+d_0V^{p-1}+d_0W_{a+1}^{p-1}(X),$$
$$U_2^*=\bigoplus_{p=0}^{m+n} U_2^p, \qquad U_2^p=W_{a+1-n/2}^{p}(Y)+d'W_{a+1-n/2}^{p-1}(Y).$$

U_1^* and U_2^* are complexes with the exterior differentiation d_0 and d'. U_0^* is also a complex with operation d. We have following sequence of complexes;

(6.8) $$0 \longrightarrow U_0^* \xrightarrow{\ \iota\ } U_1^* \xrightarrow{\ \iota'\ } U_2^* \longrightarrow 0.$$

PROPOSITION 6.7. *The above sequence (6.8) is exact.*

PROOF. We have only to prove that $\mathrm{Im}\,\iota=\ker\iota'$. Assume that

(6.9) $$\alpha=\sigma+\beta+d_0\gamma,$$

$\sigma\in W_{a+1}^p(X)$, $\beta\in d_0V^{p-1}$ and $\gamma\in W_{a+1}^{p-1}(X)$. The fact $\iota'\alpha=0$ means that

(6.10) $$\sigma|_Y+\beta|_Y+d'(\gamma|_Y)=0.$$

Let G' and H' be the Green operator of $d'\delta'+\delta'd'$ and the projection onto the space of harmonic forms on Y respectively. We have

(6.11) $$\gamma|_Y=H'\gamma|_Y+d'\delta'G'\gamma|_Y+\delta'G'd'\gamma|_Y.$$

Since $\sigma|_Y$ and $\beta|_Y$ belong to $W_{a+1-n/2}^{p-1}(Y)$, equality (6.10) implies that $\delta'G'd'\gamma|_Y$ belongs to the space $W_{a+2-n/2}^{p-1}(Y)$. Setting

(6.12) $$A=P_1^{-1}(H'\gamma|_Y+\delta'G'd'\gamma|_Y) \quad\text{and}\quad B=P_1^{-1}\delta'G'\gamma|_Y,$$

we have that

(6.13) $$A\in W_{-a+n/2}^{p-1}(Y) \quad\text{and}\quad B\in W_{-a+n/2}^{p-2}(Y).$$

Therefore, if we define

(6.14)
$$\mu=(1+\Delta_0)^{-a-1}(A\otimes\delta_Y)$$
$$\nu=(1+\Delta_0)^{-a-1}(B\otimes\delta_Y),$$

then we have equalities

(6.15) $$\mu|_Y = H'\gamma|_Y + \delta' G' d' \gamma|_Y\,, \qquad \nu|_Y = \delta' G' \gamma|_Y \quad \text{and}$$

$$\gamma|_Y = \mu|_Y + (d_0\nu)|_Y\,.$$

Note that

(6.16) $$\mu \in W^{p-1}_{2a+b+2,-a-b}(X) \subset W^{p-1}_{a+1}(X), \quad \nu \in W^{p-2}_{2a+b+2,-a-b}(X) \subset W^{p-2}_{a+1}(X)\,.$$

Since $A \in W^{p-1}_{-a+n/2}(Y)$, we have $d_0\mu \in D(d^*)$. Similarly $d_0\nu \in D(d^*)$. Set

(6.17) $$\gamma - \mu - d_0\nu =$$

Then $\rho \in W^{p-1}_{a+1}(X) + d_0 V^{p-2}$ and $\rho|_Y = 0$. Hence ρ belongs to $D(d) \cap D(d^*)$ by Theorem 5.9. Replacing γ in (6.9) by $\mu + d_0\nu + \rho$, we have $\alpha = \sigma + \beta + d_0\mu + d_0\rho$. We know that $\rho \in D(d) \cap D(d^*)$ and $\sigma + \beta + d_0\mu \in W^{p}_{a+1}(X) + d_0 V^{p-1}$. Moreover we have $\sigma + \beta + d_0\mu|_Y = \alpha|_Y = 0$, because $\rho|_Y = 0$. This implies that $\sigma + \beta + d_0\mu \in D(d) \cap D(d^*)$. Proof is now complete.

PROPOSITION 6.8. *The cohomology group of complexes U_1^* and U_2^* are canonically isomorphic to de Rham cohomology groups $H^*(X)$ and $H^*(Y)$ respectively.*

PROOF. A cochain $\alpha = \sigma + \beta + d_0\gamma$, $\sigma \in W^{p}_{a+1}(X)$, $\beta \in d_0 V^{p-1}$, $\gamma \in W^{p-1}_{a+1}(X)$, is a cocycle if and only if $d_0\sigma = 0$. On the other hand α is a coboundary if and only if there is a $\nu \in W^{p-1}_{a+1}(X)$ such that $\alpha = d_0\nu$. This is equivalent to the fact that $\sigma + \beta + d_0(\gamma - \nu) = 0$. As $\beta = d_0(1+\Delta_0)^{-a-1}(T \otimes \delta_Y)$ with some $T \in W^{p-1}_{-a-1+n/2}(Y)$, we have $(1+\Delta_0)^{-a-1}(T \otimes \delta_Y) \in W^{p-1}_{2a+b+2,-a-b-1}(X) \subset W^{p-1}_{a+1}(X)$. Thus we have proved that α is a coboundary if and only if $\sigma = d_0\tau$ with some τ in $W^{p-1}_{a+1}(X)$. Therefore the cohomology group of the complex U_1^* is isomorphic to $H^*(X)$. Similar argument proves that $H^*(U_2^*) \cong H^*(Y)$.

PROPOSITION 6.8. *The kernel of L is canonically isomorphic to the cohomology of complex U_0^*.*

PROOF. Let $\alpha = \beta_1 + d\beta_2$, $\beta_1, \beta_2 \in D(d) \cap D(d^*)$. This is a cocycle if and only if $d\beta_1 = 0$. If we apply Theorem 6.6 to β_1, we have $\alpha = H\beta_1 + d(Gd^*\beta_1 + \beta_2)$. Thus α and $H\beta_1$ are cohomologous. On the other hand if $\beta \in \operatorname{Ker} L$ then $\beta \in D(d) \cap D(d^*)$ and $d\beta = 0$ by Proposition 6.3. Hence β is a cocycle in U_0^*. If β is a coboundary, β must be zero by virtue of Proposition 6.3. This completes proof.

Now we can prove Theorem 6.6.

Set $\operatorname{Ker}^p L$ the space of p-currents in the kernel of L. Then it follows from Propositions 6.7, 6.8 and 6.9 that the following sequence is exact; $0 \to \operatorname{Ker}^0 L \to H^0(X) \to H^0(Y) \to \cdots \to H^{p-1}(Y) \to \operatorname{Ker}^p L \to H^p(X) \to H^p(Y) \to \cdots$. This and the five lemma prove our theorem.

REMARK. Another more natural proof of Theorem 6.6 will be given in § 8.

§ 7. Boundary conditions.

We shall treat the equation

(7.1) $$(\lambda+L)\alpha=\sigma, \quad \text{with } \lambda\geqq 0 \text{ and } \sigma\in W_{\alpha}^{p}(X).$$

We shall first treat the case $\lambda>0$. $\alpha\in D(d)\cap D(d^{*})$ means that there exists $S\in W_{-\alpha+n/2}^{p-1}(Y)$ such that

(7.2) $$\delta_{0}\alpha-d^{*}\alpha=(1+\Delta_{0})^{-\alpha}(S\otimes\delta_{Y})$$

and

(7.3) $$\alpha|_{Y}=0.$$

The condition $d\alpha\in D(d^{*})\cap D(d)$ is equivalent to the fact that there exists $T\in W_{-\alpha+n/2}^{p}(Y)$ such that

(7.4) $$\delta_{0}d_{0}\alpha-d^{*}d_{0}\alpha=(1+\Delta_{0})^{-\alpha}(T\otimes\delta_{Y}).$$

From (7.2) and (7.4) we have

$$\Delta_{0}\alpha-L\alpha=d_{0}(1+\Delta_{0})^{-\alpha}(S\otimes\delta_{Y})+(1+\Delta_{0})^{-\alpha}(T\otimes\delta_{Y}).$$

Using (7.1) we have

(7.5) $$(\lambda+\Delta_{0})\alpha=\sigma+d_{0}(1+\Delta_{0})^{-\alpha}(S\otimes\delta_{Y})+(1+\Delta_{0})^{-\alpha}(T\otimes\delta_{Y}).$$

Therefore, α is given by

(7.6) $$\alpha=(\lambda+\Delta_{0})^{-1}\sigma+d_{0}(\lambda+\Delta_{0})^{-1}(1+\Delta_{0})^{-\alpha}(S\otimes\delta_{Y}) +(\lambda+\Delta_{0})^{-1}(1+\Delta_{0})^{-\alpha}(T\otimes\delta_{Y}).$$

The condition (7.3) is

(7.7) $$(\lambda+\Delta_{0})^{-1}\sigma|_{Y}+d'P_{\lambda}S+P_{\lambda}T=0.$$

We must check the condition $d^{*}\alpha\in D(d)\cap D(d^{*})$. $d^{*}\alpha\in D(d^{*})$ is automatically satisfied. From (7.2) and (7.6) we have

$$\begin{aligned} d^{*}\alpha &= \delta_{0}\alpha-(1+\Delta_{0})^{-\alpha}(S\otimes\delta_{Y}) \\ &= \delta_{0}(\lambda+\Delta_{0})^{-1}\sigma-\lambda(\lambda+\Delta_{0})^{-1}(1+\Delta_{0})^{-\alpha}(S\otimes\delta_{Y}) \\ &\quad -d_{0}(\lambda+\Delta_{0})^{-1}(1+\Delta_{0})^{-\alpha}(\delta'S\otimes\delta_{Y}) \\ &\quad +(\lambda+\Delta_{0})^{-1}(1+\Delta_{0})^{-\alpha}(\delta'T\otimes\delta_{Y}). \end{aligned}$$

This belongs to $D(d)$ if and only if $d^{*}\alpha|_{Y}=0$, that is,

(7.8) $$(\delta_{0}(\lambda+\Delta_{0})^{-1}\sigma)|_{Y}-\lambda P_{\lambda}S-d'P_{\lambda}\delta'S+P_{\lambda}\delta'T=0.$$

Since P_{λ} is invertible, we can define

(7.9) $$\tilde{\delta}=P_{\lambda}\delta'P_{\lambda}^{-1}.$$

Thus we have proved

THEOREM 7.1. *Equation* $(\lambda+L)\alpha=\sigma$, $\lambda>0$ *and* $\sigma\in W_{a}^{p}(X)$, *is equivalent to the system of equations*

$$\alpha=(\lambda+\Delta_0)^{-1}\sigma+d_0(\lambda+\Delta_0)^{-1}(1+\Delta_0)^{-a}(S\otimes\delta_Y)+(\lambda+\Delta_0)^{-1}(1+\Delta_0)^{-a}(T\otimes\delta_Y), \tag{7.10}$$

$$(\lambda+\Delta_0)^{-1}\sigma|_Y+d'P_\lambda S+P_\lambda T=0, \tag{7.11}$$

$$(\delta_0(\lambda+\Delta_0)^{-1}\sigma)|_Y-\lambda P_\lambda S-d'\tilde{\delta}P_\lambda S+\tilde{\delta}P_\lambda T=0, \tag{7.12}$$

$$\alpha\in W_a^p(X),\quad S\in W^{p-1}_{-a+n/2}(Y),\quad T\in W^{p}_{-a+n/2}(Y).$$

Before going further, we give an interpretation of the meaning of the operator $\tilde{\delta}$. Let $\langle\,,\rangle_\lambda$ be the scalar product in $\Omega^p(Y)$ defined by

$$\langle\varphi,\phi\rangle_\lambda=\int_Y P_\lambda^{-1}\varphi\wedge*'\phi=\int_Y\varphi\wedge*'P_\lambda^{-1}\phi,\qquad\text{for } \varphi,\ \phi \text{ in } \Omega^p(Y). \tag{7.13}$$

Then $\Omega^p(Y)$ is a pre-Hilbert space by virtue of Lemma 5.8. The operators d' and $\tilde{\delta}$ are mutually adjoint with respect to this scalar product. In fact, we have

$$\begin{aligned}\langle d'\varphi,\phi\rangle_\lambda&=\int_Y P_\lambda^{-1}d'\varphi\wedge*'\phi\\&=\int_Y d'P_\lambda P_\lambda^{-1}\varphi\wedge*'P_\lambda^{-1}\phi\\&=\int_Y P_\lambda^{-1}\varphi\wedge*'P_\lambda\delta'P_\lambda^{-1}\phi\\&=\langle\varphi,\tilde{\delta}\phi\rangle_\lambda.\end{aligned} \tag{7.14}$$

This implies that the operator $L_\lambda=d'\tilde{\delta}+\tilde{\delta}d'$ is a non-negative self-adjoint operator with respect to the Hilbert space structure $\langle\,,\rangle_\lambda$. L_λ is an elliptic pseudo-differential operator of order 2.

We come back to equations (7.10), (7.11) and (7.12).

PROPOSITION 7.2. *The system of equation* (7.11) *and* (7.12) *is equivalent to the system of equations*

$$(d'\tilde{\delta}+\tilde{\delta}d')P_\lambda S+\lambda P_\lambda S=-\tilde{\delta}((\lambda+\Delta_0)^{-1}\sigma)|_Y+(\delta_0(\lambda+\Delta_0)^{-1}\sigma)|_Y \tag{7.15}$$

$$d'P_\lambda S+P_\lambda T=-((\lambda+\Delta_0)^{-1}\sigma)|_Y. \tag{7.16}$$

If $\sigma\in W_a^p(X)$ is given, we can find S by (7.15) and T by (7.16). Hence α is given by (7.10).

As a consequence

THEOREM 7.3. *Assume that* $(\lambda+L)\alpha=\sigma$, *with some* $\lambda>0$ *and* σ *in* $W^p_{a+r}(X)$, $r\geqq 0$. *Then* S *and* T *must belong to* $W^{p-1}_{-a+r+1+n/2}(Y)$ *and* $W^{p}_{-a+r+n/2}(Y)$ *respec-*

tively. And α *belongs to* $W^p_{2a+b+1,-a-b+r+1}(X)$, *where b is an arbitrary real number* $< -n/2$.

PROOF. The fact that $\sigma \in W^p_{a+r}(X)$ implies that $(\delta_0(\lambda+\Delta_0)^{-1}\sigma)|_Y$ and $\tilde{\delta}((\lambda+\Delta_0)^{-1}\sigma)|_Y$ belong to $W^p_{a+r+1-n/2}(Y)$. Hence $S \in W^p_{-a+r+1+n/2}(Y)$. Similarly $T \in W^p_{-a+r+n/2}(Y)$. This proves theorem.

COROLLARY 7.4. *The domain* $D(L)$ *of* $L = dd^* + d^*d$ *is contained in* $W^p_{2a+b+1,-a-b+1}(X)$.

Now we treat the case $\lambda = 0$. Equation (7.5) holds also in this case. Let us recall classical Hodge-Kodaira decomposition of currents on X. We shall denote by G_0 the Green operator of the Laplacian $\Delta_0 = d_0\delta_0 + \delta_0 d_0$ and by H_0 the projection operator onto the space of harmonic forms on X (cf. [2]). Then the equation (7.5) with $\lambda = 0$ is equivalent to

$$H_0\sigma + H_0(T \otimes \delta_Y) = 0 \tag{7.17}$$

and

$$(1-H_0)\alpha = G_0\sigma + d_0 G_0(1+\Delta_0)^{-a}(S \otimes \delta_Y) + G_0(1+\Delta_0)^{-a}(T \otimes \delta_Y). \tag{7.18}$$

We introduce the following operator P, which will play a similar role as P_λ in the case of $\lambda > 0$.

DEFINITION 7.5. P is an operator which operates on currents on Y as follows:

$$P: T \longrightarrow G_0(1+\Delta_0)^{-a}(T \otimes \delta_Y)|_Y. \tag{7.19}$$

Just as Lemma 5.8, we have

PROPOSITION 7.6. *P is an elliptic pseudo-differential operator of order* $-2a-2+n$. *P is an isomorphism from* $W^p_0(Y)$ *onto* $W^p_{2a+2-n}(Y)$.

PROOF. If $\varphi,\ \phi \in \Omega^p(Y)$, there holds

$$\int_Y P\varphi \wedge *'\phi = \int_X \Psi \wedge *\Phi, \tag{7.20}$$

where $\Psi = G_0^{1/2}(1+\Delta_0)^{-a/2}(\varphi \otimes \delta_Y)$ and

$$\Phi = G_0^{1/2}(1+\Delta_0)^{-a/2}(\phi \otimes \delta_Y).$$

In particular,

$$\int_Y P\varphi \wedge *'\varphi = \int_X \Psi \wedge *\Psi \geqq 0. \tag{7.21}$$

Here equality holds if and only if $\Psi = 0$, that is, $(1+\Delta_0)^{-a/2}(\varphi \otimes \delta_Y)$ is harmonic on X. However this occurs only when $\varphi = 0$.

We need one more operator Q.

DEFINITION 7.7. We define operator Q which operates on currents on Y as follows:

$$Q: S \longrightarrow H_0(S \otimes \delta_Y)|_Y. \tag{7.22}$$

Let $h_1, \cdots, h_k$ be the orthonormal basis of harmonic forms on X. Then

$$QS = \sum_{j=1}^{k} \Big(\int_X h_j \wedge *(S \otimes \delta_Y)\Big) h_j|_Y \tag{7.23}$$

$$= \sum_{j=1}^{k} \Big(\int_Y h_j|_Y \wedge *' S\Big) h_j|_Y .$$

Just as the case $\lambda > 0$, the fact $\alpha \in D(d) \cap D(d^*)$ means that

$$(H_0\alpha + G_0\sigma)|_Y + d'PS + PT = 0. \tag{7.24}$$

Since

$$\delta_0\alpha = \delta_0 G_0\sigma - d_0 G_0(1+\Delta_0)^{-a}(\delta' S \otimes \delta_Y) - H_0(1+\Delta_0)^{-a}(S \otimes \delta_Y)$$
$$+ G_0(1+\Delta_0)^{-a}(\delta' T \otimes \delta_Y) + (1+\Delta_0)^{-a}(S \otimes \delta_Y),$$

we have

$$d^*\alpha = \delta_0 G_0\sigma - d_0 G_0(1+\Delta_0)^{-a}(\delta' S \otimes \delta_Y) \tag{7.25}$$
$$- H_0(1+\Delta_0)^{-a}(S \otimes \delta_Y) + G_0(1+\Delta_0)^{-a}(\delta' T \otimes \delta_Y).$$

This belongs to $D(d)$ if and only if $d^*\alpha|_Y = 0$, i.e.,

$$(\delta_0 G_0 \sigma)|_Y - d' P\delta' S - QS + P\delta' T = 0. \tag{7.26}$$

Just as we did in the case $\lambda > 0$, we define

$$\delta_1' = P\delta' P^{-1}. \tag{7.27}$$

Then we have

THEOREM 7.8. *Equation $L\alpha = \sigma$ is equivalent to system of equations concerning $\alpha \in W_a^p(X)$, $S \in W_{-a+n/2}^{p-1}(Y)$ and $T \in W_{-a+n/2}^{p}(Y)$:*

$$\Delta_0\alpha = \sigma + d_0(1+\Delta_0)^{-a}(S \otimes \delta_Y) + (1+\Delta_0)^{-a}(T \otimes \delta_Y), \tag{7.28}$$

$$d'PS + PT + (H_0\alpha + G_0\sigma)|_Y = 0, \tag{7.29}$$

$$d'\delta_1' PS - \delta_1' PT + \pi PS - (\delta_0 G_0 \sigma)|_Y = 0, \tag{7.30}$$

where the operator π is defined by

$$\pi = QP^{-1}. \tag{7.31}$$

PROOF. Equality (7.28) implies (7.17) and (7.18). Thus (7.29) means that $\alpha \in D(d) \cap D(d^*)$. We have

$$d\alpha = d_0 G_0 \sigma + d_0 G_0 (1+\Delta_0)^{-a}(T \otimes \delta_Y) \tag{7.32}$$

and (7.25). Equality (7.32) means that $d\alpha \in D(d) \cap D(d^*)$ and

(7.33) $$d^*d\alpha=\delta_0 d_0 G_0\sigma-d_0\delta_0 G_0(1+\Delta_0)^{-a}(T\otimes\delta_Y) -H_0(1+\Delta_0)^{-a}(T\otimes\delta_Y)\,,$$

because

$$\delta_0 d\alpha=\delta_0 d_0 G_0\sigma-d_0\delta_0 G_0(1+\Delta_0)^{-a}(T\otimes\delta_Y) -H_0(1+\Delta_0)^{-a}(T\otimes\delta_Y)+(1+\Delta_0)^{-a}(T\otimes\delta_Y)\,.$$

On the other hand (7.30) means that $d^*\alpha\in D(d)\cap D(d^*)$ and

(7.34) $$dd^*\alpha=d_0\delta_0 G_0\sigma+d_0 G_0(1+\Delta_0)^{-a}(\delta' T\otimes\delta_Y)\,.$$

This and (7.33) give that

$$L\alpha=(1-H_0)\sigma-H_0(T\otimes\delta_Y) =\sigma$$

by virtue of (7.17).

In the case $\sigma=0$, we have stronger version of this theorem.

THEOREM 7.9. *The equation* $L\alpha=0$ *is equivalent to the system of equations concerning* $\alpha\in W^p_a(X)$ *and* $S\in W^{p-1}_{-a+n/2}(Y)$ *given by*

(7.35) $$\Delta_0\alpha=d_0(1+\Delta_0)^{-a}(S\otimes\delta_Y)\,,$$

(7.36) $$d'PS+H_0\alpha|_Y=0\,,$$

(7.37) $$d'\delta_1' PS=0\,,$$

(7.38) $$\pi PS=0\,.$$

PROOF. If equality (7.35)~(7.38) hold, then (7.28), (7.29) and (7.30) hold with $T=0$, $\sigma=0$. Hence $L\alpha=0$. Conversely if $L\alpha=0$, then $d\alpha=d^*\alpha=0$. Equality $d\alpha=0$ implies that $T=0$ because of (7.4). The fact that $d^*\alpha=0$ implies that $H_0(1+\Delta_0)^{-a}(S\otimes\delta_Y)=0$ by virtue of (7.25). Hence we have $\pi PS=0$. If we apply these to Theorem 7.8, we prove Theorem 7.9.

§ 8. Boundary conditions and the long exact sequence.

Let us define a new scalar product in $\Omega^p(Y)$, by

(8.1) $$\langle\varphi,\phi\rangle=\int_Y P^{-1}\varphi\wedge *'\phi\,,$$

where P is the operator defined by (7.19). Making use of Proposition 7.6, we know that the scalar product can be extended to $W^p_{a+1-n/2}(Y)$ continuously and $W^p_{a+1-n/2}(Y)$ becomes a Hilbert space with this scalar product. We always consider this Hilbert space structure when we refer to the space $W^p_{a+1-n/2}(Y)$. The exterior differential operator d' restricted to $\Omega^p(Y)$ is closable in this space. We shall denote its smallest closed extension by the same symbol d'.

PROPOSITION 8.1. *The operator* δ_1' *is the adjoint of* d' *in the space* $W^p_{\alpha+1-n/2}(Y)$. *The operator* π *defined by* (7.31) *is a symmetric operator of finite rank.*

PROOF. We have only to prove that π is symmetric.

From (7.23), we have

$$\pi S = \sum_{j=1}^{k} \Big(\int_Y h_j|_Y \wedge *' P^{-1} S\Big) h_j|_Y . \tag{8.2}$$

$$= \sum_{j=1}^{k} \langle S, h_j|_Y \rangle h_j|_Y .$$

This shows that π is symmetric.

Making use of operator d' and δ_1', we can prove an analogue of Hodge-Kodaira theory in the space $W^p_{\alpha+1-n/2}(Y)$.

PROPOSITION 8.2. *The operator* $L' = d'\delta_1' + \delta_1' d'$ *is a non-negative self-adjoint elliptic pseudo-differential operator with only point spectrum of finite multiplicity.*

PROPOSITION 8.3.

$$\mathrm{Ker}\, L' = \{T \text{ in } D(L') \mid d'T = 0 \text{ and } \delta_1' T = 0\} .$$

THEOREM 8.4.

$$I = H' + (d'\delta_1' + \delta_1' d')G' \tag{8.3}$$

$$= H' + d'G'\delta_1' + \delta_1' G' d' ,$$

where H' *is the orthogonal projection onto the* Ker L' *and* G' *is the Green operator of* L'.

THEOREM 8.5. Ker L' *is isomorphic to the de Rham cohomology group* $H^*(Y)$.

Now we give an interpretation of the boundary conditions in §7 and give a new proof of Theorem 6.6. Let us denote by $\mathrm{Ker}^p \Delta_0$, $\mathrm{Ker}^p L'$ and $\mathrm{Ker}^p L$ the space of p-currents belonging to kernels of Δ_0, L' and L respectively.

First we define a mapping ρ by

$$\begin{array}{ccc} \mathrm{Ker}^p \Delta_0 & \xrightarrow{\rho} & \mathrm{Ker}^p L' \\ \Cup & & \Cup \\ \alpha & \longrightarrow & H'(\alpha|_Y) . \end{array} \tag{8.4}$$

Secondly ρ' by

$$\begin{array}{ccc} \mathrm{Ker}^p L & \xrightarrow{\rho'} & \mathrm{Ker}^p \Delta_0 \\ \Cup & & \Cup \\ \alpha & \longrightarrow & H_0 \alpha . \end{array} \tag{8.5}$$

And finally ρ'' by

$$\begin{array}{ccc} \mathrm{Ker}^p L' & \xrightarrow{\rho''} & \mathrm{Ker}^{p+1} L \\ \cup\!\!\!| & & \cup\!\!\!| \\ T & \longrightarrow & d_0 G_0(1+\Delta_0)^{-\alpha}(P^{-1}JT\otimes\delta_Y)\,, \end{array} \tag{8.6}$$

where J is the orthogonal projection onto $\mathrm{Ker}^p L' \ominus H' \,\mathrm{Im}\,\pi$, the orthogonal complement of $H'\,\mathrm{Im}\,\pi$ in the space $\mathrm{Ker}^p L'$. We must prove that image of ρ'' is contained in $\mathrm{Ker}^{p+1}L$. If T is in $H'\,\mathrm{Im}\,\pi$, this is trivial because $JT=0$. Assume that T belongs to $\mathrm{Ker}^p L'\ominus H'\,\mathrm{Im}\,\pi$. Then $0=\langle T, H'\,\mathrm{Im}\,\pi\rangle =\langle H'T, \mathrm{Im}\,\pi\rangle=\langle T, \mathrm{Im}\,\pi\rangle$. This implies that T belongs to $\mathrm{Ker}\,\pi$ because π is symmetric. Hence (7.35), (7.36), (7.37) and (7.38) are satisfied if we set $\alpha=d_0G_0(1+\Delta_0)^{-\alpha}(P^{-1}JT\otimes\delta_Y)$ and $S=P^{-1}T$. Therefore $d_0G_0(1+\Delta_0)^{-\alpha}(P^{-1}JT\otimes\delta_Y)$ belongs to $\mathrm{Ker}^{p+1}L$.

Now we have defined a sequence of homomorphisms

$$\begin{aligned} (8.7)\qquad & 0\longrightarrow \mathrm{Ker}^0 L \xrightarrow{\rho'} \mathrm{Ker}^0\Delta_0 \xrightarrow{\rho} \mathrm{Ker}^0 L' \xrightarrow{\rho''} \\ & \xrightarrow{\rho''} \mathrm{Ker}^1 L \xrightarrow{\rho'} \mathrm{Ker}^1\Delta_0 \xrightarrow{\rho} \mathrm{Ker}^1 L' \xrightarrow{\rho''} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \xrightarrow{\rho''} \mathrm{Ker}^p L \xrightarrow{\rho'} \mathrm{Ker}^p\Delta_0 \xrightarrow{\rho} \mathrm{Ker}^p L' \xrightarrow{\rho''} \\ & \xrightarrow{\rho''} \mathrm{Ker}^{p+1} L \longrightarrow \cdots\cdots \end{aligned}$$

THEOREM 8.6. *The sequence* (8.7) *is exact.*

PROOF. We start by proving

$$\mathrm{Im}\,\rho'=\mathrm{Ker}\,\rho\,. \tag{8.8}$$

Assume that α is in $\mathrm{Ker}^p L$, then Theorem 7.9 implies that

$$\alpha=\rho'\alpha+d_0G_0(1+\Delta_0)^{-\alpha}(S\otimes\delta_Y) \tag{8.9}$$

and

$$\rho'\alpha|_Y+d'PS=0\,. \tag{8.10}$$

Hence $\rho\rho'\alpha=H'(\rho'\alpha|_Y)=-H'(d'PS)=0$. Conversely, if $\alpha\in\mathrm{Ker}^p\Delta_0$ and $\rho\alpha =H'(\alpha|_Y)=0$, then $\alpha|_Y=d'\delta'G'(\alpha|_Y)$, because $d'\alpha|_Y=0$. We define S by $S=-P^{-1}\delta_1'G'(\alpha|_Y)$. We have

$$d'PS+\alpha|_Y=0$$

and

$$d'\delta_1'PS=0\,.$$

Further $\pi PS=0$ because $\pi\delta_1'=0$. Setting $\beta=\alpha+d_0G_0(1+\Delta_0)^{-\alpha}(S\otimes\delta_Y)$, we have

$\rho'\beta=\alpha$. This proves (8.8).

Next we prove

$$\text{Im}\,\rho=\text{Ker}\,\rho''. \tag{8.11}$$

The fact that $\rho''\rho=0$ is trivial. Assume that $\alpha\in\text{Ker}^pL'$ and $\rho''\alpha=0$, i.e., $d_0(P^{-1}JS\otimes\delta_Y)$ is harmonic. This is possible if and only if $P^{-1}JS=0$, that is, S belongs to $H'\,\text{Im}\,\pi=$ image of ρ. (8.11) is proved.

Finally we prove

$$\text{Im}\,\rho''=\text{Ker}\,\rho'. \tag{8.12}$$

We have only to prove $\text{Ker}\,\rho'\subset\text{Im}\,\rho''$. Assume that $\alpha\in\text{Ker}\,\rho'$. Then $H_0\alpha=0$ and $\alpha\in\text{Ker}^pL$. Hence

$$\alpha=d_0G_0(1+\Delta_0)^{-a}(S\otimes\delta_Y) \tag{8.13}$$

with

$$d'PS=0, \tag{8.14}$$

$$d'\delta_1'PS=0, \tag{8.15}$$

and

$$\pi PS=0. \tag{8.16}$$

Equalities (8.14) and (8.15) imply that $PS\in\text{Ker}^pL'$. This proves (8.12). Therefore Theorem 8.5 is proved.

REMARK 8.7. Since Ker^pL' is isomorphic to $H^p(Y)$ and $\text{Ker}^p\Delta_0$ is isomorphic to $H^p(X)$, Theorem 8.5 means that Ker^pL is isomorphic to $H^p(X, Y)$ by virtue of five lemma.

Summing up the above results, we have

THEOREM 8.8. *Every cohomology class in $H^p(X, Y)$ is represented by a current α in $W_a^p(X)$ such that there is an $S\in W^{p-1}_{-a+n/2}(Y)$ satisfying*

$$\Delta_0\alpha=d_0(1+\Delta_0)^{-a}(S\otimes\delta_Y), \tag{8.17}$$

$$d'PS+H_0\alpha|_Y=0, \tag{8.18}$$

$$d'\delta_1'PS=0 \tag{8.19}$$

and

$$\pi PS=0. \tag{8.20}$$

REMARK 8.9. It should be noted that $\alpha|_Y=0$.

THEOREM 8.10. *Every cohomology class $H^p(X, Y)$ is represented by a current γ in $X-Y$ which is harmonic in $X-Y$ and may have singularity at Y majorized as*

(8.21) $$|\gamma(x)|=(\gamma(x) \lrcorner \gamma(x))^{1/2}=O(r^{1-n})$$

for any $x\in X-Y$. *Here* $r=$ *geodesic distance from the point* x *to* Y *and* $n=$ *codimension of* Y.

PROOF. As we noted in Remark 3.4, the current α in Theorem 8.8 can be identified with a current γ in $X-Y$ by the formula (3.6). On the other hand (8.17) gives that

(8.22) $$\Delta_0(1+\Delta_0)^a\alpha=d_0(S\otimes\delta_Y).$$

This means that γ is harmonic in $X-Y$ because support of $d_0(S\otimes\delta_Y)$ is contained in Y. Applying the Green operator G_0 to both sides of (8.22), we have

(8.23) $$(1+\Delta_0)^a\alpha=H_0\alpha+G_0d_0(S\otimes\delta_Y).$$

Since $H_0\alpha$ is smooth, $(1+\Delta_0)^a\alpha$ has the same singularity as $G_0d_0(S\otimes\delta_Y)=d_0G_0(S\otimes\delta_Y)$. This proves our theorem.

§ 9. Addenda.

The aim of this section is to get new representatives of cohomology classes of $H^p(X, Y)$ that is more natural than the one given in the previous sections. In order to do this we introduce a new Hilbert space structure of $W_a^p(X)$ by the following inner product:

(9.1) $$[\alpha, \beta]=(\Delta_0^{a/2}\alpha, \Delta_0^{a/2}\beta)+(H_0\alpha, H_0\beta).$$

Since the topologies of $W_a^p(X)$ are the same, the closed operator d is unchanged. However, the adjoint which we denote again by d^* is different from the one treated in § 5. We have

THEOREM 9.1. *A current* α *in* $W_a^{p+1}(X)$ *belongs to* $D(d^*)$ *if and only if there are* γ *in* $W_a^p(X)$ *and* T *in* $W_{-a-1+n/2}^p(Y)$ *such that*

(9.2) $$\delta_0\alpha-\gamma=G_0^a(T\otimes\delta_Y)+H_0(T\otimes\delta_Y).$$

If this holds, $d^*\alpha=\gamma$ *and* $\delta' T$ *belongs to* $W_{-a-1+n/2}^p(Y)$.

Proof is similar to that of Theorem 5.1.

Introducing the space

(9.3) $$V'^p=\{G_0^{a+1}(T\otimes\delta_Y)\,|\,T\in W_{-a-1+n/2}^p(Y) \text{ and } \delta' T\in W_{-a-1+n/2}^{p-1}(Y)\},$$

we have

THEOREM 9.2. *The following sequence of vector spaces are exact*

(9.4) $$0\longrightarrow D(d)\cap D(d^*)\longrightarrow W_{a+1}^p(X)+d_0V'^{p-1}\longrightarrow W_{a+1-n/2}^p(Y)\longrightarrow 0.$$

Let $\mathrm{Ker}^p L$ be the space of all p-currents belonging to the kernel of the operator

(9.5) $$L=dd^*+d^*d.$$

Then just as Theorem 6.6, we can prove

THEOREM 9.3. *$H^p(X, Y)$ is isomorphic to the space* $\mathrm{Ker}^p L$.

Now we define the operator P' operating on currents of Y given by

$$P' : T \longrightarrow G_0^{a+1}(T \otimes \delta_Y)|_Y . \tag{9.6}$$

PROPOSITION 9.4. *P' is a non-negative self adjoint invertible elliptic pseudo-differential operator of order $-2a-2+n$.*

We define the following operators:

$$\delta_2' = P'\delta' P'^{-1} . \tag{9.7}$$

$$\pi' = QP'^{-1} . \tag{9.8}$$

THEOREM 9.5. *Equation $L\alpha \equiv (dd^*+d^*d)\alpha = f$ is equivalent to system of equations concerning $\alpha \in W_a^p(X)$, $S \in W_{-a+n/2}^{p-1}(Y)$ and T in $W_{-a+n/2}^p(Y)$:*

$$\begin{aligned} f &= \Delta_0\alpha - d_0 G_0^a(S\otimes\delta_Y) - G_0^a(T\otimes\delta_Y) - H_0(T\otimes\delta_Y) \\ (G_0 f)|_Y &= -(H_0\alpha)|_Y - d'P'S - P'T \\ (\delta_0 G_0 f)|_Y &= \pi' P'S + d'\delta_2' P'S - \delta_2' P'T . \end{aligned} \tag{9.9}$$

In particular,

THEOREM 9.6. *Equation $(dd^*+d^*d)\alpha = 0$ is equivalent to the system of equations:*

$$\begin{aligned} \Delta_0\alpha &= d_0 G_0^a(S\otimes\delta_Y) \\ (H_0\alpha)|_Y + d'P'S &= 0 \\ d'\delta_2' P'S &= 0 \\ \pi' P'S &= 0 \end{aligned} \tag{9.10}$$

for α in $W_a^p(X)$ and S in $W_{-a+n/2}^{p-1}(Y)$.

THEOREM 9.7. *Every cohomology class of $H^p(X, Y)$ is uniquely represented by a current α such that there exists a current S in $W_{-a+n/2}^{p-1}(Y)$ satisfying* (9.10).

REMARK 9.8. The current α in Theorem 9.6 satisfies equation

$$\Delta_0^{a+1}\alpha = 0 \quad \text{in } X-Y . \tag{9.11}$$

That is α is poly-harmonic in $X-Y$.

REMARK 9.9. We can identify an arbitrary current α in $W_a^p(X)$ with a current γ on $X-Y$ by the formula

$$\gamma = (\Delta_0^a\alpha + H_0\alpha)|_{X-Y} . \tag{9.12}$$

(cf. Remark 3.4.)

Then

$$\Delta_0\gamma = 0 \quad \text{on } X-Y \tag{9.13}$$

(9.14) $$|\gamma(x)|=O(r^{1-n}).$$

(cf. Theorem 8.10.)

Since the operator P' enjoys the same properties as P, we can define an inner product

(9.15) $$\langle\alpha, \beta\rangle'=\int_Y P'^{-1}\alpha\wedge*'\beta$$

for currents α and β on Y. (cf. § 8.)

The operators d' and δ_2' are mutually adjoint with respect to this Hilbert space structure. We can make Hodge-Kodaira decomposition of currents on Y using this inner product.

THEOREM 9.10. *$H^p(Y)$ is isomorphic to the space of all p-currents T on Y satisfying equation*

(9.16) $$(d'\delta_2'+\delta_2'd')T=0.$$

Thus we can give interpretation of the long exact sequence (1.2) from our new stand point. Discussion is completely parallel to that of § 8 and the detail is ommitted here.

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