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# Regular congruences on Croisot-Teissier and Baer-Levi semigroups

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Clifford and Preston have defined a class of Croisot-Teissier semigroups of type $(p, q)$ which are simple with a minimal right ideal when $p=q$ ([**1**] 8.2). In this paper, we define a modified class of Croisot-Teissier semigroups (1.3) which are simple with a minimal right ideal for all $q \leqq p$ (1.9). The generalized Baer-Levi semigroups ([**1**] 8.1) are shown to be right simple generalized Croisot-Teissier semigroups under the new definition (1.5).

In the concluding sections, we investigate group and band congruences on these semigroups. We find a set, $E$, which is contained in the kernel of every group congruence (2.3), and also find necessary and sufficient conditions for $E$ to be the kernel of such a congruence (2.11). Using this result, we show that a Baer-Levi semigroup has a non-trivial group congruence if and only if $p>q$ (2.14).

Finally, we relate band congruences on simple semigroups with a minimal right ideal to the ordering of the $\mathcal{L}$-classes under the usual ordering (3.3), and after investigating this structure in Baer-Levi semigroups (3.5), we show they have only trivial band congruences. This is sufficient to show that the only regular congruences on Baer-Levi semigroups are group congruences (3.6).

The terminology and notation will be that of Clifford and Preston [**1**].

## § 1. Generalized Croisot-Teissier semigroups.

In this section we discuss a class of simple semigroups with a minimal right ideal, which are generalized Baer-Levi semigroups ([**1**] 8.1). Clifford and Preston ([**1**] 8.2) have defined Croisot-Teissier semigroups of type $(p, q)$ which are simple with a minimal right ideal in the case $p=q$. We will modify their definition to obtain a class of generalized Groisot-Teissier semigroups of type $(p, q)$ $(p \geqq q)$, each member of which is simple with a minimal right ideal.

(1.1) DEFINITION ([**1**] vol. II, p. 86). Let $p$ and $q$ be infinite cardinals with $p \geqq q$, and let $X$ be a set with $|X| \geqq p$. Suppose $\mathcal{E} = \{\mathcal{E}_i : i \in I\}$ is a set of distinct equivalences on $X$ such that each quotient set $X/\mathcal{E}_i$, $i \in I$, is of cardinal $p$. A subset $B \subseteqq X$ will be said to be *well-separated by* $\mathcal{E}$ if i) $|B| = p$, and ii) $\mathcal{E}_i \cap (B \times B) = \Delta_B$, the identity relation on $B$, for all $i \in I$.

(1.2) DEFINITION. Let $p$, $q$, $X$ and $\mathcal{E}$ be as in (1.1). Let $B$ be a subset of $X$ which is well-separated by $\mathcal{E}$. Then $B$ is said to be *$q$-well-separated by* $\mathcal{E}$ if $C_i$, the collection of all $\mathcal{E}_i$-classes of $X$ which do not intersect $B$, has cardinal less than or equal to $q$, for each $i \in I$.

When $\mathcal{E}$ is clear from the context, we will simply say that $B$ is $q$-well-separated.

(1.3) DEFINITION. a. Let $p$, $q$, $X$ and $\mathcal{E}$ be as in (1.1). For each $i \in I$, let $T_i^*$ denote the collection of all maps $t_i$ of $X$ into $X$ for which i) $t_i \circ t_i^{-1} = \mathcal{E}_i$, and ii) there exists a subset $B$ $(= B(t_i)$, in general depending upon $t_i)$ of $X$ with $B$ $q$-well-separated by $\mathcal{E}$, and $Xt_i \subseteqq B$ with $|B \backslash Xt_i| = q$.

b. If $X$ contains a $q$-well-separated subset $B$, then $T_i^* \neq \square$ for any $i \in I$. In this case, we denote by $CT^*(X, \mathcal{E}, p, q)$ the union of the sets $T_i^*$ for all $i \in I$.

A set $X$, with a collection of equivalences $\mathcal{E}$, may have a subset well-separated by $\mathcal{E}$ which is not $q$-well-separated by $\mathcal{E}$ when $p > q$. For example, we let $X$ be a set of infinite cardinal $p$ and $A \subseteqq X$ with $B = X \backslash A$, for which $|A| = |B| = p$. Let $\mathcal{E}_1 = \Delta_X$, the identity relation on $X$, and let $\mathcal{E}_2 = (A \times A) \cup \Delta_B$, where $\Delta_B$ is the identity relation on $B$. Then for $q < p$, $X$ has no subset $q$-well-separated by $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2\}$. It is clear, however, that $B$ is well-separated by $\mathcal{E}$.

The proof of Lemma (1.4) is almost identical to that of [**1**] Lemma 8.9.

(1.4) LEMMA. *Any set of mappings $CT^*(X, \mathcal{E}, p, q)$ forms, under composition, an idempotent free semigroup in which each $T_i^*$ is a right ideal.*

Using (1.3) and (1.4), one easily checks the following:

(1.5) COROLLARY. *Let $S = CT^*(X, \mathcal{E}, p, q)$. Then if $\mathcal{E}$ consists of exactly one equivalence relation, $S$ is right simple. Furthermore, if $\mathcal{E} = \{\Delta_X\}$, where $\Delta_X$ is the identity relation on $X$, then $S$ is a Baer-Levi semigroup of type $(p, q)$.*

(1.6) DEFINITION. The semigroup $CT^*(X, \mathcal{E}, p, q)$ will be known as a *generalized Croisot-Teissier semigroup of type* $(p, q)$, or simply a *Croisot-Teissier semigroup*.

We note the following for the reader's convenience.

(1.7) NOTE. Clifford and Preston ([**1**] 8.2) define the Croisot-Teissier semigroup of type $(p, q)$, denoted $CT(X, \mathcal{E}, p, q)$, as follows:

a. Let $p$, $q$, $X$ and $\mathcal{E}$ be as in (1.1). For each $i \in I$, let $T_i$ denote the collection of all maps $t_i$ of $X$ into $X$ for which i) $t_i \circ t_i^{-1} = \mathcal{E}_i$, and ii) there exists a subset $B$ $(= B(t_i)$, in general depending upon $t_i)$ of $X$ with $B$ well-separated by $\mathcal{E}$, and $Xt_i \subseteqq B$ with $|B \backslash Xt_i| = q$.

b. If $X$ contains a well-separated subset $B$, then $T_i \neq \square$ for any $i \in I$. In this case, $CT(X, \mathcal{E}, p, q)$ is the union of the sets $T_i$ for all $i \in I$.

It is easily checked that $CT^*(X, \mathcal{E}, p, q) \subset CT(X, \mathcal{E}, p, q)$ ($\subset$ indicates proper containment), for $p \neq q$, and that $CT^*(X, \mathcal{E}, p, p) = CT(X, \mathcal{E}, p, p)$.

The following is an example of $CT^*(X, \mathcal{E}, p, q)$ with $\mathcal{E}$ having more than one element.

(1.8) EXAMPLE. [6] Let $X$ be a set, $|X| = p$, where $p$ is an infinite cardinal. Let $x, y \in X$, $x \neq y$, and define $\mathcal{E}_1 = \{(x, y), (y, x)\} \cup \Delta_X$. Let $\mathcal{E}_2 = \Delta_X$. If $B = X \backslash \{x\}$, then for any infinite $q \leqq p$, $B$ is $q$-well-separated by $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2\}$. Thus $CT^*(X, \mathcal{E}, p, q)$ exists for all infinite $q \leqq p$.

The next theorem is the main result of this section, and it is proven in much the same way as [1] Theorem 8.11.

(1.9) THEOREM. *Each semigroup $CT^*(X, \mathcal{E}, p, q)$ is a simple idempotent free semigroup which is the union of its minimal right ideals $T_i^*$, $i \in I$.*

## § 2. Group congruences.

In this section we will give a necessary condition for $CT^*(X, \mathcal{E}, p, q)$ to have a group congruence (2.5). This condition will also be necessary and sufficient in the case $CT^*(X, \mathcal{E}, p, q)$ is right simple (2.12). A sufficient condition for $CT^*(X, \mathcal{E}, p, q)$ to have a non-trivial group congruence will also be given (2.11).

We first quote a theorem which will be the basis for our investigation.

(2.1) THEOREM ([4] Theorem (1.9)). *Let $S$ be a simple semigroup with a minimal right ideal. Then if $S$ has a group congruence $\rho$, the kernel $E$ of $\rho$ is a unitary subsemigroup such that*

i) *$E$ is a right cross section ($E \cap aS \neq \square$, for all $a \in S$),*

ii) *For $a \in S$, $a = ae$ for some $e \in E$,*

iii) *If $xEy \cap E \neq \square$ for $x, y \in S$, then $xEy \subseteqq E$.*

*Conversely, if $E$ is a unitary subsemigroup of $S$ satisfying* i)–iii), *then there exists a group congruence $\rho$ with $E$ as its kernel.*

(2.2) DEFINITION. Let $S = CT^*(X, \mathcal{E}, p, q)$, and let $a \in S$. Then we say a subset $Y_a$ of $X$ is *fixed by a*, if $Y_a$ is $q$-well-separated by $\mathcal{E}$ and $a$ restricted to $Y_a$ is the identity map. We say that *a has a fixed set* if there exists a set $Y_a$ fixed by $a$.

We note the following relation between $E = \{e \in S : e \text{ has a fixed set}\}$ and the kernel of any group congruence on $CT^*(X, \mathcal{E}, p, q)$.

(2.3) PROPOSITION. *Let $S = CT^*(X, \mathcal{E}, p, q)$ and $U$ be the kernel of any group congruence on $S$. Then $E \subseteqq U$.*

PROOF. Let $\sigma$ be a group congruence on $S$ with the kernel $U$. Let $e \in E$, then $e$ fixes some $q$-well-separated subset $Y$ of $X$. Clearly, there exists $a \in S$ such that $Xa = Y$, so that $ae = a$, and $(ae, a) \in \sigma$. But $S/\sigma$ is a group, therefore $e \in U$ and $E \subseteq U$.

We will now find necessary and sufficient conditions for $E$ to be the kernel of a group congruence. Combining (2.1) and (2.3) we have:

(2.4) LEMMA. *If $S$, $E$, and $U$ are as in* (2.3), *and if $\langle E \rangle$ is the subsemigroup of $S$ generated by $E$, then $\langle E \rangle \subseteq U$.*

(2.5) PROPOSITION. *Let $S = CT^*(X, \mathcal{E}, p, q)$. If $S$ has a non-trivial group congruence, then $p > q$.*

PROOF. Suppose $p = q$, and $S$ has a non-trivial group congruence $\rho$ with kernel $U$. Let $E = \{e \in S : e$ has a fixed set$\}$, then the subsemigroup $\langle E \rangle$ of $S$ generated by $E$ is contained in $U$. We will show that $\langle E \rangle = S$, and hence $\rho$ will be a trivial congruence contrary to our assumption.

Let $a \in S$, we will show that there exists $e_1, e_2 \in E$ such that $a = e_1 e_2 \in \langle E \rangle$. Clearly $a \in T_i^*$ for some $i \in I$, so that there exists $A$, a $p$-well-separated subset of $X$, such that $Xa \subseteq A$ and $|A \backslash Xa| = p$. Let $W \subseteq A \backslash Xa$ with $|W| = |(A \backslash Xa) \backslash W| = p$. If $\mathcal{C}_i$ is the collection of $\mathcal{E}_i$-classes of $X$ which do not intersect $W$, then clearly $|\mathcal{C}_i| = p$. Therefore, since $p$ is an infinite cardinal, there exists a one-to-one map $d$ of $\mathcal{C}_i$ into $(A \backslash Xa) \backslash W$, such that

(*) $|[(A \backslash Xa) \backslash W] \backslash \mathcal{C}_i d| = p$.

Define $e_1$ on any $x$ in an $\mathcal{E}_i$-class contained in $\mathcal{C}_i$ by letting $xe_i$ be the image of the $\mathcal{E}_i$-class of $x$ under $d$. Let $we_1 = w$ for all $w \in W$. Since $W$ is a well-separated subset of $X$, every $\mathcal{E}_i$-class intersects $W$ in exactly one element. Therefore if $(w, w') \in \mathcal{E}_i$ for some $w \in W$, and we let $w'e_1 = w$, then $e_1$ is a well-defined map on all of the $\mathcal{E}_i$-classes which intersect $W$. It follows that $Xe_1 = \mathcal{C}_i d \cup W$, and hence $|A \backslash Xe_1| = p$. It is also true that $e_1 \circ e_1^{-1} = \mathcal{E}_i$, so that $e_1 \in T_i^*$. Clearly $W$ is a set fixed by $e_1$, and thus $e_1 \in E$. Since $e_1 \circ e_1^{-1} = \mathcal{E}_i = a \circ a^{-1}$, if we define $e_2$ on $Xe_1$ by $xe_1 e_2 = xa$, then $e_2$ is well-defined on $Xe_1$, in fact, $e_2$ is a one-to-one map of $Xe_1$ onto $Xa$. For each $xe_1 \in Xe_1$, we extend $e_2$ to the $\mathcal{E}_i$-class $\overline{xe}_1$ of $xe_1$ by $(\overline{xe}_1)e_2 = xa$. Since $Xe_1$ is well-separated by $\mathcal{E}$, this extension is well defined. From (*) it is clear that $|A \backslash (Xe_1 \cup Xa)| = p$, and therefore there exists $Y \subseteq A \backslash (Xe_1 \cup Xa) = F$, with $|Y| = |F \backslash Y| = |F| = p$. If $\mathcal{G}$ is the collection of all $\mathcal{E}_i$-classes of $X$ which do not intersect $Xe_1 \cup Y$, clearly $|\mathcal{G}| = p$, and since $p$ is an infinite cardinal, there exists $d'$, a one-to-one map of $\mathcal{G}$ into $F \backslash Y$, with $|(F \backslash Y) \backslash \mathcal{G}d'| = p$. If $x$ is an $\mathcal{E}_i$-class in $\mathcal{G}$, let $xe_2$ be the image of the $\mathcal{E}_i$-class of $x$ under $d'$. Finally, for $y \in Y$, let $ye_2 = y$, and if $(y, y') \in \mathcal{E}_i$ for $y \in Y$, let $y'e_2 = y$. Then we have $|A \backslash Xe_2| = |((F \backslash Y) \backslash \mathcal{G}d') \cup Xe_1| = p$, and $e_2 \circ e_2^{-1} = \mathcal{E}_i$, so that $e_2 \in T_i^* \subseteq S$. Clearly $e_2 \in E$ with fixed set $Y$. Thus combining all of the above, we have $a = e_1 e_2 \in \langle E \rangle$, and hence every element of $S$ is in $\langle E\rangle \subseteqq U$ and $\rho$ is a trivial congruence.

(2.6) DEFINITION. Let $S=CT^*(X, \mathcal{E}, p, q)$. If $C$ and $D$ are $q$-well-separated subsets of $X$, and $i, j\in I$, then the set

$$\mathcal{M}(C, D, i, j)=\{c\in C\colon \text{there is } d\in D \text{ with } (c, d)\in \mathcal{E}_i\cap\mathcal{E}_j\}$$

is said to be the *$(i, j)$-mesh of $C$ with $D$.*

(2.7) CONDITION G. Let $S=CT^*(X, \mathcal{E}, p, q)$. Then $S$ is said to satisfy condition $G$ if and only if $p>q$ and, for any $i, j\in I$ and for any subsets $C$ and $D$ of $X$ which are $q$-well-separated by $\mathcal{E}$, the set $\mathcal{M}(C, D, i, j)$ is $q$-well-separated by $\mathcal{E}$.

An example of a Croisot-Teissier semigroup that satisfies condition $G$ is given in (1.8).

We will now use a series of lemmas to prove the main result.

(2.8) LEMMA. *If $S=CT^*(X, \mathcal{E}, p, q)$ satisfies condition $G$, then $E=\{e\in S$: $e$ has a fixed set$\}$ is a subsemigroup of $S$.*

PROOF. Let $a, b\in E$ with sets $Y_a$ and $Y_b$ fixed by $a$ and $b$ respectively. Assume further that $a\in T_i{}^*$ and $b\in T_j{}^*$ for some $i, j\in I$. Then, since $Y_a$ and $Y_b$ are $q$-well-separated, $Y_{ab}=\mathcal{M}(Y_b, Y_a, i, j)$ is $q$-well-separated (2.7). We now show that $Y_{ab}$ is a fixed set for $ab$. Let $y\in Y_{ab}$, then there exists a $y'\in Y_a$ with $(y, y')\in\mathcal{E}_i\cap\mathcal{E}_j$. Clearly, since $a\in T_i{}^*$ and $y'\in Y_a$, a fixed set for $a$, $ya=y'a=y'$. Similarly, since $b\in T_j{}^*$ and $y\in Y_{ab}\subseteqq Y_b$, a fixed set for $b$, we have $y'b=yb=y$. But then, combining these equations, we have $yab=y'b=y$. Thus we have shown that $Y_{ab}$ is a fixed set for $ab$, and hence, $ab\in E$, and $E$ is a subsemigroup of $S$.

(2.9) LEMMA. *If $S$, $E$ are as in (2.8), then $E$ is unitary in $S$.*

PROOF. First we show that $E$ is left unitary. Suppose $a\in E$ and $ab\in E$, and suppose further that $a\in T_i{}^*$ and $b\in T_j{}^*$. Since $T_i{}^*$ is a right ideal, $ab\in T_i{}^*$. Then there are sets $Y_a$ and $Y_{ab}$ fixed by $a$ and $ab$ respectively. As in the proof of (2.8), we see that $Y_b=\mathcal{M}(Y_{ab}, Y_a, i, j)$ is a $q$-well-separated subset of $X$. We will show that $b\in E$ by showing that $Y_b$ is a fixed set for $b$. Let $y\in Y_b$, then there exists $y'\in Y_a$ with $(y, y')\in\mathcal{E}_i\cap\mathcal{E}_j$. We now proceed as we did in (2.8) to get $ya=y'a=y'$, and $yb=y'b$. But then, since $Y_{ab}$ is a fixed set for $ab$, we have $y=yab=y'b=yb$. Thus $Y_b$ is a fixed set for $b$. Therefore, if $a\in E$ and $ab\in E$ for any $b\in S$, we have $b\in E$ and $E$ is left unitary.

Finally we show $E$ is right unitary. Let $ab\in E$ and $b\in E$, and suppose $a\in T_i{}^*$ and $b\in T_j{}^*$. Then as $T_i{}^*$ is a right ideal, we have $ab\in T_i{}^*$. Since $ab\in E$ and $b\in E$, there exist $Y_{ab}$ and $Y_b$, subsets of $X$ fixed by $ab$ and $b$ respectively. It is easily checked that if $A$ and $B$ are any $q$-well-separated subsets of $X$ which are both contained in the same $q$-well-separated subset of $X$, then $A\cap B$ is a $q$-well-separated subset of $X$. It follows that $Y=$

$Y_{ab} \cap Y_b$ is a $q$-well-separated subset of $X$, since $Y_{ab} \subseteqq Xab \subseteqq Xb$. We also know that $Xa$ is a $q$-well-separated subset of $X$, so that $Y_a = \mathcal{M}(Xa, Y, i, j)$ is $q$-well-separated by (2.7). We now show that $Y_a$ is a fixed set for $a$. If $y \in Y_a$, then there exists $y' \in Y$ such that $(y, y') \in \mathcal{E}_i \cap \mathcal{E}_j$. We proceed as in (2.8) to get $ya = y'a$, $yab = y'ab = y'$, and $yb = y'b = y'$. Now since $y \in Xa$, there exists $y'' \in X$ such that $y = y''a$, and combining all of these equations, we have $y'ab = y' = y'b = yb = y''ab$. But since $ab \in T_i^*$, it follows that $(ab) \circ (ab)^{-1} = \mathcal{E}_i$, and therefore $(y', y'') \in \mathcal{E}_i$. We also have $a \in T_i^*$ and hence $ya = y'a = y''a = y$. Thus $Y_a$ is a fixed set for $a$. We have shown that if $ab \in E$ and $b \in E$ for some $a \in S$, then $a \in E$, and $E$ is right unitary. Our result follows.

We recall that in (2.1) we showed that a unitary subsemigroup of a simple semigroup $S$ with a minimal right ideal was the kernel of a group congruence on $S$ if and only if it satisfied conditions i)–iii) in the following lemma.

(2.10) LEMMA. *If $S$ and $E$ are as in* (2.8), *then $E$ satisfies:*

i) *$E$ is a right cross-section ($E \cap T_i^* \neq \square$, for all $i \in I$),*

ii) *For $a \in S$, $a = ae$ for some $e \in E$,*

iii) *If $aEb \cap E \neq \square$ for $a$, $b \in S$, then $aEb \subseteqq E$.*

PROOF. Conditions i) and ii) are easily checked. To show condition iii), we let $a \in T_i^*$ and $b \in T_j^*$ for which $aEb \cap E \neq \square$. We will show $aEb \subseteqq E$. Let $e_1$, $e_2 \in E$ such that $e_2 = ae_1b \in aEb \cap E$. Clearly $e_2 \in T_i^*$, and we may assume that $e_1 \in T_k^*$ for some $k \in I$. If $e \in E \cap T_m^*$ for any $m \in I$, then we show that $aeb \in E$. Let $Y$, $Y_1$, and $Y_2$ be subsets of $X$ fixed by $e$, $e_1$, and $e_2$ respectively. If $(Y_2)_i^* = \{\bar{x} \in X/\mathcal{E}_i : \bar{x} \cap Y_2 = \square\}$, then it is clear that since $a \in T_i^*$ and $Y_2$ is $q$-well-separated, $|Xa \backslash Y_2a| = |(Y_2)_i^*| \leqq q$. But $Y_2a \subseteqq Xa$, which is $q$-well-separated, and hence it is easily checked that $Y_2a$ is $q$-well-separated. Let $Z_1' = \mathcal{M}(Y_2a, Y_1, j, k)$, then by condition $G$, $|Y_2a \backslash Z_1'| \leqq q$. It is also clear that if $Z_1 = \{y \in Y_2 : ya \in Z_1'\}$, then $|Y_2 \backslash Z_1| = |Y_2a \backslash Z_1'| \leqq q$, and hence $Z_1$ is $q$-well-separated. Let $y \in Z_1$, then there is $y' \in Y_1$ for which $(ya, y') \in \mathcal{E}_j \cap \mathcal{E}_k$. Then, since $e_1 \in T_k^*$ and $b \in T_j^*$, we have $yae_1 = y'e_1$ and $yab = y'b$. We combine these equations to get $y = ye_2 = yae_1b = y'e_1b = y'b = yab$, so that $Z_1$ is a fixed set for $ab$. Now let $Z_2' = \mathcal{M}(Z_1', Y, j, m)$, then $Z_2'$ is $q$-well-separated, and as above, if $Z_2 = \{y \in Z_1 : ya \in Z_2'\}$, then $Z_2$ is $q$-well-separated. But if $y \in Z_2$, then there exists $y' \in Y$ with $(ya, y') \in \mathcal{E}_j \cap \mathcal{E}_m$. It follows that $y = yab$ since $y \in Z_1$, a fixed set for $ab$, and that $y' = y'e$ since $y' \in Y$, a fixed set for $e$. We combine these equations, and $y = yab = y'b = y'eb = yaeb$ follows since $(ya, y') \in \mathcal{E}_j \cap \mathcal{E}_m$. Therefore we have shown that $Z_2$ is a fixed set for $aeb$, so that $aeb \in E$, and our result follows.

(2.11) THEOREM. *Let $S = CT^*(X, \mathcal{E}, p, q)$, then $E = \{e \in S : e$ has a fixed set$\}$ is the kernel of a group congruence on $S$ if and only if $S$ satisfies condition*

$G$. *Moreover, in this case, $E$ is the kernel of the minimum group congruence on $S$.*

PROOF. Combining (1.9), (2.1), and (2.8)-(2.10), we see that $E$ is the kernel of some group congruence on $S$, if $S$ satisfies condition $G$. In [4] Theorem (2.1) it is shown that if $\rho$ is a group congruence with kernel $E'$, then for any $a\in S$, $a\rho=E'aE'$. Let $E_1$ and $E_2$ be the kernels for group congruences $\rho_1$, and $\rho_2$ on $S$ respectively with $E_1\subseteqq E_2$. If $(a, b)\in\rho_1$, then $E_1aE_1=E_1bE_1$, so that $a=e_1be_2$ where $e_1, e_2\in E_1$. But then $e_1, e_2\in E_2$ and $a\in E_2bE_2=b\rho_2$, thus $\rho_1\subseteqq\rho_2$. By (2.3) we have $E$ is contained in the kernel of every group congruence on $S$, therefore it is the kernel of the minimum group congruence on $S$.

Suppose $E$ is the kernel of a group congruence on $S$. Let $Y$ and $Z$ be $q$-well-separated subsets of $X$. We will show that for all $i, j\in I$, $\mathcal{M}(Z, Y, i, j)$ is $q$-well-separated, i.e., $S$ satisfies condition $G$. Let $Y_1\subset Y$ and $Z_1\subset Z$ such that $|Y\backslash Y_1|=|Z\backslash Z_1|=q$. Clearly $Y_1$ and $Z_1$ are $q$-well-separated, and hence there exists $e_i\in T_i^*\cap E$ with fixed set $Z_1$ and $e_j\in T_j^*\cap E$ with fixed set $Y_1$. Since $E$ is a subsemigroup (2.1), $e_je_i\in E$, hence $e_je_i$ has a fixed set $Y^*$, and $Y^*=Y^*e_je_i\subseteqq Xe_i$. We also have $Z_1\subseteqq Xe_i$, thus $Y^*$ and $Z_1$ are $q$-well-separated subsets of the same $q$-well-separated set $Xe_i$, and therefore $Y'=Y^*\cap Z_1$ is a $q$-well-separated set. Let $Y_1'=\{y\in Y' : ye_j\in Y_1\}$. We have $Y'e_j\subseteqq Xe_j$ is $q$-well-separated and $Y_1\subseteqq Xe_j$ is $q$-well-separated, hence $Y_1'e_j=Y'e_j\cap Y_1$ is $q$-well-separated. But $Y_1'\subseteqq Y^*$, therefore $Y_1'e_je_i=(Y_1'e_j)e_i=Y_1'$, and $Y_1'$ is $q$-well-separated. Let $y\in Y_1'$, then $y\in Y^*$, a fixed set for $e_je_i$, and $y\in Z_1$, a fixed set for $e_i$, so that $(ye_j)e_i=y=ye_i$, and $(y, ye_j)\in\mathcal{E}_i$ since $e_i\in T_i^*$. It is also clear from the definition of $Y_1'$ that $ye_j\in Y_1$, a fixed set for $e_j$, so that $ye_j=(ye_j)e_j$. But $e_j\in T_j^*$, therefore $(y, ye_j)\in\mathcal{E}_j$. It follows that $(y, ye_j)\in\mathcal{E}_i\cap\mathcal{E}_j$, and since $y\in Z$ and $ye_j\in Y$, we have $Y_1'\subseteqq\mathcal{M}(Z, Y, i, j)$. Finally, since $Y_1'$ is $q$-well-separated, $\mathcal{M}(Z, Y, i, j)$ must be $q$-well-separated. By (2.5) $p>q$, and we have our result.

(2.12) LEMMA. *Let $S=CT^*(X, \mathcal{E}, p, q)$ be right simple, then $S$ satisfies condition $G$ if and only if $p>q$.*

PROOF. Assume $p>q$. Since $S$ is right simple, we can write $\mathcal{E}=\{E_1\}$. Then for all $C$, $D$, $q$-well-separated subsets of $X$, it is clear that $\mathcal{M}(C, D, 1, 1)$ is $q$-well-separated, since $p>q$. Thus $S$ satisfies condition $G$.

If $S$ satisfies condition $G$, then $p>q$.

(2.13) THEOREM. *If $S=CT^*(X, \mathcal{E}, p, q)$ be right simple, then $S$ has a non-trivial group congruence if and only if $p>q$.*

And

(2.14) COROLLARY. *A Baer-Levi semigroup of type $(p, q)$ has a non-trivial group congruence if and only if $p>q$.*

We recall the following:

(2.15) THEOREM ([**6**]). *Let $S$ be a right simple idempotent-free semigroup and let $E \subseteqq S$. Then $E$ is the kernel of some group congruence $\rho$ on $S$ if and only if $E$ is a subsemigroup of $S$ which is unitary in $S$ and satisfies the condition $EaE \subseteqq aE$ for every $a \in S$. Moreover, for all $a \in S$, $aE$ is the $\rho$-class of $a$.*

Combining (2.11) and (2.15), we get:

(2.16) COROLLARY. *If $S$ is a Baer-Levi semigroup of type $(p, q)$ where $p > q$, and $\rho$ is any congruence on $S$ with $\rho \subseteqq \gamma$, the minimum group congruence on $S$, then for $(a, b) \in \rho$, $D = \{x \in X : xa \neq xb\}$ has cardinal less than or equal to $q$.*

PROOF. We know that the kernel of $\gamma$ is $E = \{e \in S : e$ has a fixed set$\}$ from (2.11). If $(a, b) \in \rho$, then $(a, b) \in \gamma$. But then $aE = bE$, by (2.15), also, $|X \backslash (Xa \cap Xb)| = q$, and $|Xa \cap Xb| = p$. Since $S$ is right simple, there exists $e \in S$ such that $ae = a$. Then $e \in E$ and so $a = ae \in aE = bE$. Hence there exists $e_1 \in E$ such that $a = be_1$. Therefore there exists $Y_1 \subseteqq X$ such that $Y_1$ is fixed by $e_1$. We put $\{x \in X : xb \in Y_1\} = Y_2$. Then we can prove that $|X \backslash Y_2| \leqq q$ and $|Y_2| = p$. Also we can check that $xa = xb$ for every $x \in Y_2$. Hence $D \subseteqq X \backslash Y_2$ and so $|D| \leqq |X \backslash Y_2| \leqq q$.

We conclude this section with an example and the following proposition which can easily be verified (c. f., proof of [**1**] Theorem 8.11).

(2.17) PROPOSITION. *Let $S = CT^*(X, \mathcal{E}, p, q)$ with $Y$ and $Z$ subsets of $X$, $q$-well-separated by $\mathcal{E}$. If $a$ is a one-to-one map of $Y$ into $Z$ for which $|Z \backslash Ya| = q$, then for any $i \in I$, $a$ can be extended to $a_i \in T_i^*$.*

The following is an example of a Croisot-Teissier semigroup which does not satisfy condition $G$, and has no non-trivial group congruence.

(2.18) EXAMPLE ([**1**] vol. II, p. 87). Let $X$ be the Cartesian product $T \times T$ where $T$ is a set of infinite cardinal $p$. We may write $X = \{(t_1, t_2) : t_1, t_2 \in T\}$. If $(t_1, t_2), (t_1', t_2') \in X$, we say $(t_1, t_2)\mathcal{E}_i(t_1', t_2')$ if and only if $t_i = t_i'$ where $i = 1, 2$. The set $Y = \{(t, t) : t \in T\}$ is a subset of $X$ which is $q$-well-separated by $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2\}$, so that $S = CT^*(X, \mathcal{E}, p, q)$ exists for all $q \leqq p$. One may easily find a $q$-well-separated subset $Z$ of $X$ for which $\mathcal{M}(Y, Z, 1, 2) = \square$, and thus $S$ does not satisfy condition $G$. We will show that the subsemigroup $\langle E \rangle$, generated by $E = \{e \in S : e$ has a fixed set$\}$, is all of $S$, i.e., $S = \langle E \rangle$, and by (2.11), it is clear that $S$ not only does not satisfy condition $G$, but by (2.4), $S$ has no non-trivial group congruence.

Let $a \in S$, then we will show $a = e_1 e_2$ where $e_1, e_2 \in E$ (c. f., (2.5)). We may assume without loss in generality that $a \in T_1^*$. We may define $a$ as follows: $(t_1, t_2)a = ((t_1)y_1, (t_1)y_2)$ where $y_i$ is a one-to-one map of $T$ into $T$ with $|T \backslash Ty_i| = q$ for $i = 1, 2$. Note that the image of $(t_1, t_2)$ under $a$ does not depend on $t_2$. Let $T' \subseteqq T$ for which $|T \backslash T'| = q$, and define $(t_1, t_2)e_1 = (t_1, (t_1)y_2)$ for all $t_1 \in T'$. We extend $e_1$ to all of $X$ by letting $T'' \subseteqq T \backslash T'$ with $|T''| =$

$|(T\backslash T')\backslash T''|=q$ and defining $z_1$ to be a one-to-one map of $T\backslash T'$ onto $T''$, so that if $t_1\in T\backslash T'$, we let $(t_1, t_2)e_1=((t_1)z_1, (t_1)y_2)$. Clearly $e_1\in T_1^*$, and for $t_1\in T'$, $(t_1, (t_1)y_2)e_1=(t_1, (t_1)y_2)$, i. e., $e_1$ has a fixed set. Note that we may extend $z_1$ to be a one-to-one map of $T$ onto $T'\cup T''$ by defining $(t)z_1=t$ for $t\in T'$. Under this definition, we have $(t_1, t_2)e_1=((t_1)z_1, (t_1)y_2)$ for all $(t_1, t_2)\in X$. Define $e_2^*$ on $Xe_1$ by $((t_1)z_1, (t_1)y_2)e_2^*=((t_1)y_1, (t_1)y_2)$, clearly $e_2^*$ is a one-to-one map of $q$-well-separated set $Xe_1$ onto $q$-well-separated set $Xa$. By definition of $S=CT^*(X, \mathcal{E}, p, q)$, there exists a $q$-well-separated set $A$ such that $Xa\subseteqq A$ and $|A\backslash Xa|=q$, thus by (2.17), $e_2^*$ may be extended to an element $e_2$ of $T_2^*$. By the definition it is also clear that $e_2$ fixes $Xa$ and that $e_1e_2=a$. This is the desired result.

## § 3. Band congruences.

In this section, we will concentrate mainly on the minimum band congruence on a right simple semigroup. (In particular, on the minimum band congruence on $CT^*(X, \mathcal{E}, p, q)$ when $\mathcal{E}$ consists of exactly one equivalence.) This study is motivated by the following theorem.

(3.1) THEOREM ([4] (2.8) Theorem). *Let $S$ be a simple semigroup with a minimal right ideal. For any $a\in S$, let $\gamma_a$ be the minimum group congruence on $aS$ ($a$ right simple subsemigroup of $S$), and $\beta_a$ be the minimum band congruence on $aS$. If $\pi$ is the congruence generated by $\bigcup_{a\in S}(\gamma_a\cap\beta_a)$, then $\pi$ is the minimum completely simple congruence on $S$. Moreover, if $\rho$ is a regular congruence on $S$, then $\pi\subseteqq\rho$ and $\rho/\pi$ is a congruence on $S/\pi$. Let $\theta$ be the map of $C$, the lattice of regular congruences on $S$, to $C'$, the lattice of congruences on $S/\pi$ defined by $\rho\theta=\rho/\pi$. Then $\theta$ is a lattice isomorphism of $C$ onto $C'$.*

First we characterize the minimum band congruence $\beta$ on simple semigroups with a minimal right ideal. In such semigroups, $\beta\subseteqq\mathcal{R}$ ([1] vol. II, pp. 93–4, ex. 1).

(3.2) LEMMA. *Let $\rho$ be a band congruence on $S$, a simple semigroup with a minimal right ideal. If $a, b\in S$ such that $S^1a\supseteqq S^1b$ and $a\mathcal{R}b$, then $a\rho^\natural=b\rho^\natural$, where $\rho^\natural$ is the natural homomorphism of $S$ onto $S/\rho$ induced by $\rho$.*

PROOF. Since $S/\rho$ is simple with minimal right ideal and regular, $S/\rho$ is completely simple by ([1] Theorem 8.14). Then by ([1] Corollary 2.49), $S/\rho$ is the union of its minimal left ideals. These ideals are of the form $S^1\rho^\natural a\rho^\natural=(S^1a)\rho^\natural=L_{a\rho^\natural}$ for all $a\in S$. If $a$, $b\in S$ such that $S^1a\supseteqq S^1b$, then $L_{a\rho^\natural}=(S^1a)\rho^\natural\supseteqq(S^1b)\rho^\natural=L_{b\rho^\natural}$. Therefore $L_{a\rho^\natural}=L_{b\rho^\natural}$ since $\mathcal{L}$ is an equivalence relation. If, in addition, $a\mathcal{R}b$, then clearly $R_{a\rho^\natural}=R_{b\rho^\natural}$, and $a\rho^\natural=L_{a\rho^\natural}\cap R_{a\rho^\natural}=L_{b\rho^\natural}\cap R_{b\rho^\natural}=b\rho^\natural$. The last equality holds because every element of $S/\rho$ is an idempotent and each $\mathcal{H}$-class contains at most one idempotent ([1] Lemma

2.15).

(3.3) PROPOSITION. *Let $S$ be a simple semigroup with a minimal right ideal, and let $\rho$ be the congruence on $S$ generated by the relation $\alpha=\{(a, b)\in S\times S: a\mathcal{R}b$ and there is $c\in R_a=R_b$ for which $S^1c\cap S^1a\neq\square$ and $S^1c\cap S^1b\neq\square\}$, then $\rho=\beta$.*

REMARK. $\alpha$ is clearly a reflexive and symmetric relation.

PROOF. Let $(a, b)\in\alpha$, then there exists $c\in R_a$ such that $S^1c\cap S^1a\neq\square$ and $S^1c\cap S^1b\neq\square$. Let $x\in S^1c\cap S^1a$ and $y\in S^1c\cap S^1b$. Clearly $ax, ay\in aS=R_a=R_b=R_c$. We also have $S^1ax\subseteqq S^1x\subseteqq S^1c\cap S^1a$ and $S^1ay\subseteqq S^1y\subseteqq S^1c\cap S^1b$. It follows by (3.2) that $a\beta^{\natural}=(ax)\beta^{\natural}=c\beta^{\natural}=(ay)\beta^{\natural}=b\beta^{\natural}$ and $(a, b)\in\beta$, hence $\rho\subseteqq\beta$.

In order to show $\beta\subseteqq\rho$, we show $S/\rho$ is a band. Let $a\in S$, then $(a, a^2)\in\alpha$, since $a^2\in aS=R_a$ by [1] Lemma 8.13, and $S^1a^2\cap S^1a\neq\square$. Hence $(a, a^2)\in\rho$. Thus, every element of $S/\rho$ is idempotent, $\rho$ is a band congruence and $\beta\subseteqq\rho$. Whence $\beta=\rho$.

Since Baer-Levi semigroups are right simple, thus simple with a minimal right ideal, we need only know how their $\mathcal{L}$-classes are ordered, under the usual ordering, $L_a\leqq L_b$ if and only if $S^1a\subseteqq S^1b$, to describe $\beta$ using (3.3).

We know that for every right simple, idempotent-free semigroup $S$; $a, b\in S$, $L_a=L_b$ if and only if $a=b$ ([1] vol. II, p. 85, ex. 1). We now give necessary and sufficient conditions for $L_a<L_b$ on a Baer-Levi semigroup.

(3.4) LEMMA. *Let $S$ be a Baer-Levi semigroup of type $(p, q)$ on a set $X$. If $L_b>L_a$ for $a, b\in S$, then $Xb\supset Xa$ and $|Xb\backslash Xa|=q$.*

PROOF. Let $L_b>L_a$. There is thus a $c\in S$ such that $cb=a$ and we have $Xb\supseteqq Xa$. Since $b$ is one-to-one, it follows directly that $(X\backslash Xc)b=Xb\backslash Xcb=Xb\backslash Xa$. Thus $|Xb\backslash Xa|=|X\backslash Xc|=q$.

(3.5) THEOREM. *Let $a, b\in S$, a Baer-Levi semigroup of type $(p, q)$ on a set $X$. Then $L_b>L_a$ if and only if $Xb\supset Xa$ and $|Xb\backslash Xa|=q$.*

PROOF. The necessity follows from (3.4).

Conversely, suppose $Xb\supset Xa$ and $|Xb\backslash Xa|=q$. Since $b$ is a one-to-one function of $X$ onto $Xb\supset Xa$, one can define an inverse function, denoted by $b^{-1}$, from $Xb$ onto $X$ in the obvious fashion: $(xb)b^{-1}=x$. The restriction of $b^{-1}$ to $Xa$ is a one-to-one function of $Xa$ into $X$. Therefore $ab^{-1}$ is a one-to-one function of $X$ into itself. But $ab^{-1}$ is in $S$ since $|X\backslash Xab^{-1}|=|(X\backslash Xab^{-1})b|=|Xb\backslash Xa|=q$. It now follows that $a=(ab^{-1})b\in Sb$, and hence $L_a<L_b$.

(3.6) THEOREM. *If $S$ is a Baer-Levi semigroup, then $S$ has no non-trivial band congruence.*

PROOF. Let $S$ be a Baer-Levi semigroup of type $(p, q)$ on a set $X$. Let $a, b$ be arbitrary elements of $S$. We will show that $a$ and $b$ are related under any band congruence on $S$.

CASE 1. If $p>q$, then since $|X\backslash Xa|=|X\backslash Xb|=q$, $q\leqq|X\backslash(Xa\cap Xb)|\leqq q+q=q$ and $|Xa\cap Xb|=p$. Let $Y\subset Xa\cap Xb$ with $|Y|=p$ and $|(Xa\cap Xb)\backslash Y|=q$. It is clear that $|X\backslash Y|=|Xa\backslash Y|=|Xb\backslash Y|=q$, so that there exists $c\in S$ such that $Xc=Y$. By (3.5), $L_c<L_a$ and $L_c<L_b$. Then by the definition of $\alpha$ in (3.3), $(a, b)\in\alpha\subseteqq\beta$, the minimum band congruence on $S$. Thus $(a, b)$ are related by every band congruence on $S$.

CASE 2. If $p=q$, then we may have $|Xa\cap Xb|=p$, in which case the proof in case 1 still holds if $q$ is replaced by $p$. On the other hand, we may have $|Xa\cap Xb|<p$. In this case, we let $Y_1, Y_2\subseteqq X$ be such that $Xa\cap Xb\subseteqq Y_1\subseteqq Xa$ and $Xa\cap Xb\subseteqq Y_2\subseteqq Xb$ with $|Xa\backslash Y_1|=|Xb\backslash Y_2|=|Y_1|=|Y_2|=p$. One can easily check the existence of $c, d_1, d_2\in S$ for which $Xc=Y_1\cup Y_2$, $Xd_1=Y_1$, and $Xd_2=Y_2$. Then $L_{d_1}<La$, $L_{d_1}<L_c$ and $L_{d_2}<L_b$, $L_{d_2}<L_c$. It is now clear from (3.3) that $(a, c)\in\alpha$ and $(b, c)\in\alpha$. Therefore $(a, c)\in\beta$ and $(b, c)\in\beta$, but $\beta$ is an equivalence relation, hence $(a, b)\in\beta$. Thus $a$ and $b$ are related under every band congruence on $S$.

Since $a$ and $b$ are arbitrary in either case, all elements of $S$ are related under any band congruence on $S$. Thus the only band congruence on $S$ is the universal relation, and we have the theorem.

We recall the following:

(3.7) THEOREM ([4] (2.5) Theorem). *Let $S$ be a right simple semigroup. If $\tau$ is a group congruence on $S$, and if $\sigma$ is a band congruence on $S$, then $S/(\tau\cap\sigma)$ is regular. Moreover, if $\rho$ is a regular congruence on $S$, then $\rho=\tau\cap\sigma$ where $\tau$ is a group congruence on $S$ and $\sigma$ is a band congruence on $S$. In this case, $\tau$ and $\sigma$ are uniquely determined by $\rho$.*

(3.8) THEOREM. *Let $S$ be a Baer-Levi semigroup of type $(p, q)$, then $S$ has a non-trivial regular congruence $\rho$, if and only if $p>q$, in which case $\rho$ is a group congruence.*

PROOF. This theorem follows immediately from (2.14), (3.6) and (3.7).

We have characterized the minimum band congruence $\beta$ (3.3), on a simple, idempotent free semigroup $S$, with a minimal right ideal. The following is an elaboration of this construction.

We recall:

(3.9) DEFINITION ([1] vol. I, p. 18). If $\rho$ is any relation on a set $S$, which is reflexive and symmetric, then $\rho T$, *the transitive closure of* $\rho$ is the collection of all pairs $(a, b)$ for which there exists a finite sequence $a=x_0, x_1, \cdots, x_n=b$ with $(x_{i-1}, x_i)\in\rho$ for $i=1, 2, \cdots, n$.

(3.10) LEMMA. *Let $S$ be a simple semigroup with a minimal right ideal. Then $\beta=\alpha C^{**}T$, where $\alpha=\{(a, b)\in S\times S: (a, b)\in\mathcal{R}$, and there exists $c\in R_a=R_b$ for which $S^1c\cap S^1a\neq\square$ and $S^1c\cap S^1b\neq\square\}$ and $\alpha C^{**}=\{(a, b)\in S\times S: a=su, b=sv$ for some $(u, v)\in\alpha$ and $s\in S^1\}$.*

PROOF. We recall that by (3.3), $\beta$ is the smallest congruence containing $\alpha$. It is also true that if $\alpha C^*=\{(a, b)\in S\times S: a=sut$ and $b=svt$ for some $(u, v)\in\alpha$ and $s, t\in S^1\}$, then $\alpha C^*T$ is the smallest congruence containing $\alpha$ ([1] Lemma 10.3). We will show that for a simple, idempotent free semigroup with a minimal right ideal, $\alpha C^*=\alpha C^{**}$. Clearly it is true that $\alpha C^{**}\subseteqq\alpha C^*$. Let $(a, b)\in\alpha C^*$, then $a=sut$, and $b=svt$, where $(u, v)\in\alpha$ and $s, t\in S^1$. Thus there exists $c\in R_u=R_v$ for which $S^1c\cap S^1u\neq\square$ and $S^1c\cap S^1v=\square$. We recall that by [1] Lemma 8.13, we have $R_u=uS$, and it follows that $ct\in R_{ut}=R_{vt}=R_u$. It is also clear that $S^1ct\cap S^1ut\neq\square$, and $S^1ct\cap S^1vt\neq\square$, hence $(ut, vt)\in\alpha$. But then $a=su'$ and $b=sv'$, where $u'=ut$ and $v'=vt$, and therefore $(u', v')\in\alpha$. We now have $(a, b)\in\alpha C^{**}$, so that $\alpha C^*\subseteqq\alpha C^{**}$ and the result follows.

We note the following lemma which is easily proven.

(3.11) LEMMA. *Let $S=CT^*(X, \mathcal{E}, p, q)$. If $a, b\in S$ with $L_a<L_b$, then $Xa\subset Xb$ and $|Xb\backslash Xa|=q$.* (c. f., (3.5))

We recall that Baer-Levi semigroups have no non-trivial band congruences. The following is an example of a right simple, idempotent free semigroup with a non-trivial band congruence.

(3.12) EXAMPLE. Let $S=CT^*(X, \mathcal{E}, p, q)$ where $p>q$ and $\mathcal{E}=\{\mathcal{E}_1\}$, where $\mathcal{E}_1$ is defined by the following: let $Y\subseteqq X$ with $|Y|=|X\backslash Y|=p$, and let $\theta: Y\rightarrow X\backslash Y=Z$ be a one-to-one map of $Y$ onto $Z$. Let $\mathcal{E}_1=\{(y, \theta(y))\}_{y\in Y}\cup\{(\theta(y), y)\}_{y\in Y}\cup\Delta_X$. Note that both $Y$ and $Z$ are $q$-well-separated by $\mathcal{E}$, thus $S=CT^*(X, \mathcal{E}, p, q)$ exists. We will now proceed to verify that $\beta$ is non-trivial by checking the following two claims.

a) Let $\alpha$ be as defined in (3.10). If $(a, b)\in\alpha$ and $|Xa\cap Z|\leqq q$, then $|Xb\cap Z|\leqq q$. To show this we let $(a, b)\in\alpha$ with $|Xa\cap Z|\leqq q$. Then there exists $c\in S$ such that $S^1c\cap S^1a\neq\square$ and $S^1c\cap S^1b\neq\square$, and thus there exists $s_i\in S^1$, for $i=1, 2, 3, 4$, such that $s_1c=s_2a$ and $s_3c=s_4b$. Since $s_1c=s_2a$, $L_{s_1c}\leqq L_a$, and hence by (3.11), we have $Xs_1c\subseteqq Xa$. Also by (3.11), we get $|Xc\backslash Xs_1c|\leqq q$, and as a consequence of this, we see that $|Xa\cap Z|\leqq q$ implies that $|Xc\cap Z|\leqq q$. We apply this argument again to see that $|Xc\cap Z|\leqq q$ implies $|Xb\cap Z|\leqq q$, the desired result.

b) Let $\alpha$ be as defined in (3.10). If $(a, b)\in\alpha C^{**}T$ with $|Xa\cap Z|\leqq q$, then $|Xb\cap Z|\leqq q$. This follows by an argument similar to that used in a), and by finite induction.

We conclude that if $a, b\in S$ with $Xa\subseteqq Y$ and $Xb\subseteqq Z$, then we have $|Z\cap Xa|=0<q$ and $|Z\cap Xb|=|Xb|=p$, so that by b), we have $(a, b)\notin\beta$. Thus $\beta$ is non-trivial.

We note that $S$ is right simple, therefore has a non-trivial minimum group congruence $\gamma$. Let $T=\Lambda\times S$ be the direct product of $S$ and a non-trivial, left zero semigroup $\Lambda$ (c. f., [4] (1.10) Example). $T$ is simple with a minimal right ideal. If $(\lambda, a)$, $(\mu, b) \in T$, let $(\lambda, a)\beta'(\mu, b)$ if and only if $\lambda = \mu$ and $a\beta b$. Clearly $\beta'$ is a congruence on $T$, and if $(\lambda, a) \in T$, $\lambda^2 = \lambda$ and $a\beta a^2$, therefore $(\lambda, a)^2 = (\lambda^2, a^2)\beta'(\lambda, a)$, so that $\beta'$ is a band congruence on $T$. Similarly, for $(\lambda, a)$, $(\mu, b) \in T$, let $(\lambda, a)\gamma'(\mu, b)$ if and only if $a\gamma b$, then $\gamma'$ is a group congruence on $T$ (c. f., [4] (1.10) Example). It is clear that $\pi = \gamma' \cap \beta'$ is the minimum completely simple congruence on $T$ (3.1). We also note that since $\Lambda$ is non-trivial, $T/\pi$ is not right simple, and that since $\beta \subset \mathcal{R}$, $T/\pi$ is not left simple.

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