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On the K-theoretic characteristic numbers of weakly almost complex manifolds with involution

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§ 0. Introduction.

In [7], tom Dieck has defined the equivariant unitary cobordism ring U_G for any compact Lie group G. U_G-theory seems to be a strong tool in the theory of the differentiable transformation group.

We are concerned only with the case of $G=Z_2$, the cyclic group of order 2, and throughout in this paper, the letter G stands for Z_2. Let $\mathcal{O}_*^U(G)$ be the bordism ring of U-manifolds with involution. T. tom Dieck has shown that elements of $\mathcal{O}_*^U(G)$ are detected by G-equivariant characteristic numbers. More precisely we construct a ring homomorphism

$$\Phi:\ U_G^* \longrightarrow \text{Inv. Lim. } R(G)[[t_1, \cdots, t_s]]$$

and its localization

$$\Phi_L:\ U_G^* \longrightarrow \text{Inv. Lim. } Q[[t_1, \cdots, t_s]]\,.$$

Then the restriction of Φ on U_G^n is injective. We shall recapitulate this fact in (1.1) for the sake of completeness, and we give the explicit form of Φ_L in (3.1) and its relation to Φ in (3.2).

As corollaries of (1.1) and (3.2), the following results will be proved in § 4.

THEOREM (0.1). *Let $[M, T]\in \mathcal{O}_*^U(G)$. The normal bundle ν_F of a connected component of the fixed point set F in M naturally has a complex structure. Assume the following two conditions:*

(i) *For each connected component F, ν_F is trivial,*

(ii) *$\dim_C \nu_F$ is independent of F and equals a constant n.*

Then $\sum[F]\in 2^n U$ and there are two elements of U_*, $[N]$ and $[L]$ such that

$$[M, T]=[CP(1), \tau]^n[N]+[G, \sigma][L] \qquad \textit{in } \mathcal{O}_*^U(G)$$

where $[CP(1), \tau]\in \mathcal{O}_*^U(G)$ is the class of $CP(1)$ with the involution $[z_1, z_2]\rightarrow [z_1, -z_2]$ and $[G, \sigma]\in \mathcal{O}_*^U(G)$ is the class of G with the natural involution $1\rightarrow -1$.

THEOREM (0.2). *Let $[M, T]\in \mathcal{O}_*^U(G)$. If M is a Kähler manifold, and T*

preserves the given metric, then

$$\sum_i (-1)^i H^{0,i}(-1) = \sum_F (2)^{-|\nu_F|} \langle \operatorname{ch} \sum_k b_k(\bar{\nu}_F)(2)^{-k} \operatorname{td}(F), [F] \rangle$$

where $H^{0,i}$ *is the vector space over* $\boldsymbol{C}$ *spanned by the harmonic forms of type* $(0, i)$ *which is considered to be an element of* $R(G)$, *and* $H^{0,i}(-1)$ *is the value of its trace on* (-1), b_k *is the dual K-theory characteristic class,* $|\nu_F| = \dim_C \nu_F$, *and* $\bar{\nu}_F$ *is the conjugate bundle of* ν_F.

Theorem (0.2) is conjectured by Conner in [**6**], p. 115, and is proved implicitly by Atiyah-Singer-Segal in [**2**] by using the localization theorem in K_G-theory. In a special case of (0.1), we get the following theorem of Conner-Floyd [**5**].

THEOREM (Conner-Floyd). *Let* M *be a U-manifold of* $\dim 2n$ *and let* T *be an involution of* M *which is compatible with the U-structure. If* T *has only isolated fixed points, then the number of the fixed points is a form of* $2^n k$ *and* $[M] = k[CP(1)]^n$ *in* $U_*/2U_*$.

This theorem is an immediate corollary of Theorem (0.1) and also of Theorem (0.2) if M is a Kähler manifold and T preserves the given metric, $k = \sum_i (-1)^i H^{0,i}(-1)$.

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§ 1. Equivariant characteristic numbers.

Let $\mathcal{E}_G(X)$ be the semi-ring of G-vector bundles over a G-space X. We consider an equivariant analogy of Atiyah's γ-operation [**1**]. In $\mathcal{E}_G(X)$, we have the exterior power operations λ^i. Thus if ξ is a G-vector bundle of dim k, we have the G-vector bundles $\lambda^i(\xi)$, $i = 0, 1, \cdots$. These have the following formal properties (in $\mathcal{E}_G(X)$)

1) $\lambda^0(x) = 1$, 2) $\lambda^1(x) = x$, 3) $\lambda^i(x+y) = \sum \lambda^j(x)\lambda^{i-j}(y)$,

4) $\lambda^i(x) = 0$ for $i > \dim x$.

Introducing an indeterminate t, we put $\lambda_t(x) = \sum \lambda^i(x) t^i$. Let $A_G(X)$ denote the multiplicative unit group of formal power series in t with coefficients in $K_G(X)$ and leading term 1. Then 1) and 3) assert that λ_t defines a ring homomorphism $\mathcal{E}_G(X) \to A_G(X)$. Hence we get a homomorphism $\lambda_t : K_G(X) \to A_G(X)$. Taking the coefficients of λ_t, this define the operations $\lambda^i : K_G(X) \to K_G(X)$. Now we introduce the operations $a_i^G : K_G(X) \to K_G(X)$ by

$$a_i^G(x) t^i = \lambda_{t/1-t}(x - \varepsilon(x))$$

where $\varepsilon(x)$ is the dimension of x. Moreover we define the dual operations $b_i^G : K_G(X) \to K_G(X)$ by

$$(1+a_1^Gt+a_2^Gt^2+\cdots)^{-1}=1+b_1^Gt+b_2^Gt^2+\cdots.$$

Now let $\mathcal{O}_*^U(G)$ be the geometric bordism ring of U-manifolds with involution which is compatible with the given U-structure [6]. From now on, we represent an element of $\mathcal{O}_*^U(G)$ in a form $[M, T]$ where M is a U-manifold and T is a compatible involution on M. As M is a U-manifold, we have the K-theory Gysin homomorphism $p_!: K_G(M)\to K_G(pt)=R(G)$, induced by the collapsing map to a point, $p: M\to pt$. Let τ_M be the stable tangent bundle of M with the given complex structure, we call $\{p_!(b_{i_1}^G(\tau_M)\cdots b_{i_j}^G(\tau_M))\}$ equivariant K-theory characteristic numbers of $[M, T]$. For brevity we write $b_{i_1}^G(x)b_{i_2}^G(x)\cdots b_{i_j}^G(x)$ in the form $b_{i_1}^G\cdots b_{i_j}^G(x)$.

Before defining Φ, we comment on the structure of U_G. Let V_1 be the 1-dimensional complex vector space with the G-action: $z\to -z$. The equivariant cobordism euler class of the G-vector bundle η_1: $V_1\to pt$ which we denote ζ, belongs to U_G^2. Let i: $\mathcal{O}_*^U(G)\to U_G^*$ be the map obtained by the Pontrjagin-Thom construction, then i is injective and U_G^* is generated by ζ over $i(\mathcal{O}_*^U(G))$ and U_G^{odd} is zero (tom Dieck [7]).

NOTATION. We use the symbol $\omega(i_1,\cdots,i_j)$ in the following sense. Let $t_1, t_2,\cdots$ and $x_1, x_2,\cdots$ be two infinite sequences of indeterminates, and put $A_s=\prod_{i=1}^{s}(1+x_1t_i+x_2t_i+\cdots)\in R[[t_1,\cdots,t_s]]$ where R is the polynomial ring over Z with variables $x_1, x_2,\cdots$. Substituting $t_s=0$, we obtain $f_s: R[[t_1,\cdots,t_s]]\to R[[t_1,\cdots,t_{s-1}]]$. Since $f_s(A_s)=A_{s-1}$, the family A_s determines an element A in Inv. Lim. $R[[t_1,\cdots,t_s]]$. Then $A=\sum x_{i_1}\cdots x_{i_j}\ \omega(i_1,\cdots,i_j)$, where $(i_1,\cdots,i_j)$ is a series of non-negative integers, and $\omega(i_1,\cdots,i_j)$ is the inverse limit of the smallest symmetric polynomials which involve the term $t_1^{i_1}\cdots t_j^{i_j}$.

Now we are prepared to define Φ.

DEFINITION. We define a map Φ: $U_G^*\to$ Inv. Lim. $R(G)[[t_1,\cdots,t_s]]$ as follows. For $[M, T]\in\mathcal{O}_*^U(G)$,

$$\Phi([M, T])=\sum p_!(b_{i_1},\cdots,b_{i_j}(\tau_M))\omega(i_1,\cdots,i_j)$$

where τ_M is the stable tangent bundle of M, and for $\zeta\in U_G^2$,

$$\Phi(\zeta)=(1-\eta)W$$

where η is the generator of $R(G)$ and W is a power series of the form

$$\sum(-2)^i\omega(\overbrace{1,\cdots,1}^{i}).$$

THEOREM (1.1). *Φ is well defined. It is a ring homomorphism, and for each integer n, the restriction of Φ on U_G^n is injective.*

Theorem (1.1) is essentially due to tom Dieck [7]. For the sake of completeness we shall give its proof in § 4.

REMARK (1.2). If M is a Kähler manifold and T is an involution which preserves the given metric, then the first term of Φ ($[M, T]$) equals $\sum(-1)^i H^{0,i}$, where $H^{0,i}$ is the vector space over C spanned by the harmonic forms of type $(0, i)$. For the proof, we recall the first term of Φ ($[M, T]$) being $p_!(1)$. Consider the following commutative diagram

$$\begin{array}{ccc} K_G(M) & \xrightarrow{\;p_!\;} & R(G) \\ \phi\downarrow & & \| \\ K_G(\tau_M) & \xrightarrow{\;\tilde{p}_!\;} & R(G) \end{array}$$

where ϕ is the K-theory Thom isomorphism. We see that $p_!(1)=\tilde{p}_!(\Lambda_{-1}\tau_M)$ ($\Lambda_{-1}\tau_M$ is the K-theory Thom class of τ_M), and this is the topological index of $\bar{\partial}$-operator [2]. By the equivariant index theorem of Atiyah-Singer, this is equal to the analytical index of $\bar{\partial}$-operator: $\sum(-1)^i H^{0,i}$.

§2. Calculation of Φ in two cases.

PROPOSITION (2.1). i) *For* $[M, T]\in \mathcal{O}_*^U(G)$ *where* T *is a trivial* G-*action*,

$$\Phi([M, T])=\sum\langle \operatorname{ch} b_{i_1}\cdots b_{i_j}(\tau_M)\operatorname{td}(M), [M]\rangle\omega(i_1, \cdots, i_j)$$

where $\operatorname{td}(M)$ *is the todd class of* M *and* $[M]$ *is the homology fundamental class of* M.

ii) *For* $[M, T]\in \mathcal{O}_*^U(G)$ *where* T *is a free* G-*action*,

$$\Phi([M, T])=(1+\eta)\sum\langle \operatorname{ch} b_{i_1}\cdots b_{i_j}(\tau_{M/T})\operatorname{td}(M/T), [M/T]\rangle\omega(i_1, \cdots, i_j)$$

where M/T *is the orbit space of* $[M, T]$.

PROOF. i) The assertion follows immediately from the definition considering the fact that $b_i^G=b_i$ in the trivial case (b_i denote the ordinary dual K-theory class), and for any $x\in K(M)$, $p_!(x)=\langle \operatorname{ch} x \operatorname{td}(M), [M]\rangle$.

ii) At first we recall the bordism ring of U-manifolds with free involution: $U_*(BG)$. There is a spectral sequence such that $E_2=U_*\otimes H_*(BG)$ and it converges to $U_*(BG)$. This spectral sequence collapses and hence $E_2=E_\infty$. Therefore $U_*(BG)$ is generated over U_* by $[G, \sigma]$ and $[S^{2n-1}, a]$, $n=1, 2, \cdots$, where S^{2n-1} is the $(2n-1)$-dim. sphere and a is the antipodal involution. Consider the natural map $k: U_*(BG)\to \mathcal{O}_*^U(G)$. Obviously $k([S^{2n-1}, a])=0$. Hence any element $[M, T]$ in $\mathcal{O}_*^U(G)$ with T free involution is equal to an element which has the form $[N][G, \sigma]$, where N has trivial G-action and σ operates on G by $1\to -1$. Therefore we can assume $[M, T]=[N][G, \sigma]$, and hence $[M]=2[N]$ in U_*. Since the characteristic numbers of M are all twice of those of M/T, $[M]=2[M/T]$ in U_*, hence $[M/T]=[N]$. So that $\Phi([M, T])$ $=\Phi([N])\Phi([G, \sigma])=\Phi([M/T])\Phi([G, \sigma])$ and we have only to calculate $\Phi([G,$

$\sigma]$). $\Phi([G, \sigma])$ equals $p_!(1)$ and we accomplish the proof by the following lemma.

LEMMA. *Let $1 \in K_G(G)$ be the element represented by the vector bundle $\boldsymbol{C} \times G \to G$, where $\boldsymbol{C}$ is the complex plane with trivial G-action, then $p_!(1) = 1 + \eta \in R(G)$.*

PROOF. Let V_1 be the 1-dim. complex vector space with the G-action $z \to -z$. Imbed G into V_1 equivariantly, $G \to \{1, -1\} \subset V_1$, and take an equivariant neighborhood. Let S^2 be the one-point compactification of V_1 and $q: S^2 \to S^2 \vee S^2$ the map collapsing the outside of the above neighborhood. G operate $S^2 \vee S^2$ by permutation of the factors, and q is equivariant. Considering the following commutative diagram, we obtain the result.

$$\begin{array}{ccc} K_G(S^2 \vee S^2) & \xrightarrow{\ q^*\ } & K_G(S^2) \\ \uparrow & & \uparrow \\ K_G(G) & \xrightarrow{\ p_!\ } & K_G(pt) \end{array}$$

where the vertical maps are the Thom isomorphisms.

§ 3. The localization of Φ and some application.

DEFINITION (3.1). We define a map $\Phi_L : U_G^* \to \mathrm{Inv.\,Lim.}\, Q[[t_1, \cdots, t_s]]$ where $\boldsymbol{Q}$ is the rational number field, as follows.

For $\zeta \in U_G^2$, $\Phi_L(\zeta) = 2W \qquad (W = \sum (-2)^i \omega(\overbrace{1, \cdots, 1}^{i}))$.
For $[M, T] \in \mathcal{O}_*^U(G)$,

$$(*) \qquad \Phi_L([M, T]) =$$
$$\sum_F (2W)^{-|\nu_F|} \Big\langle \mathrm{ch} \prod_s \sum_i b_i(\tau_F) t_s^i \sum_j b_j(\nu_F) \Big(\frac{-t_s}{1-2t_s}\Big)^j \sum_k b_k(\bar{\nu}_F) \Big(\frac{1}{2}\Big)^k \mathrm{td}(F), [F] \Big\rangle$$

where $|\nu_F|$ is $\dim_C \nu_F$, τ_F is the stable tangent bundle of F with the given complex structure, and the evaluation is applied after expanding the right hand side into the power series of $t_1, t_2, \cdots$.

NOTE. Let $c(\nu_F)$, $c(\tau_F)$ be the total Chern classes of ν_F, τ_F and $c(\nu_F) = \prod_j (1 + y_j)$, $c(\tau_F) = \prod_i (1 + x_i)$ formally, then the right hand side of (*) can be written as follows.

$$\Phi_L([M, T]) = \sum_F \Big\langle \prod_s \prod_i \frac{1}{1 + t_s(e^{x_i} - 1)} \cdot \frac{1}{1 - e^{-x_i}} \prod_j \frac{1}{1 - t_s(e^{y_j} + 1)} \cdot \frac{1}{1 + e^{-y_j}}, [F] \Big\rangle.$$

THEOREM (3.2). *The map Φ_L is well defined. It is a ring homomorphism from U_G^* to* Inv. Lim. $Q[[t_1, \cdots, t_s]]$ *and the following diagram commutes.*

$$\begin{array}{ccc} U_G^* & \xrightarrow{\Phi} & \text{Inv. Lim. } R(G)[[t_1, \cdots, t_s]] \\ & \searrow^{\Phi_L} & \downarrow \eta = -1 \\ & & \text{Inv. Lim. } Q[[t_1, \cdots, t_s]] \end{array}$$

The proof of the above theorem will be given in § 4. The rest of this section is devoted to the proof of the Theorems (0.1) and (0.2).

PROOF OF THEOREM (0.1). From the assumption and the definition of Φ_L,

$$\Phi_L([M, T]) = (2W)^{-n} \sum \langle \mathrm{ch}\, b_{i_1} \cdots b_{i_j}(\tau_F)\, \mathrm{td}\,(F), [F] \rangle\, \omega(i_1, \cdots, i_j)\,.$$

But by the Theorem (3.2), $\Phi_L([M, T]) \in \text{Inv. Lim. } Z[[t_1, \cdots, t_s]]$, and so for all $(i_1, \cdots, i_j)$, $\sum_F \langle \mathrm{ch}\, b_{i_1} \cdots b_{i_j}(\tau_F)\, \mathrm{td}(F), [F] \rangle \in 2^n Z$. Hence all of the dual K-theory characteristic numbers of $\sum [F]$ are divisible by 2^n. From the theorem of Hattori-Stong [8], there is an element $[N] \in U_*$ such that $\sum [F] = 2^n [N]$ in U. Now,

$$\begin{aligned} \Phi_L([CP(1), \tau]^n [N]) &= \sum (W)^{-n} \langle \mathrm{ch}\, b_{i_1} \cdots b_{i_j}(\tau_N)\, \mathrm{td}\,(N), [N] \rangle \omega(i_1, \cdots, i_j) \\ &= \sum (2W)^{-n} \sum_F \langle \mathrm{ch}\, b_{i_1} \cdots b_{i_j}(\tau_F) \mathrm{td}(F), [F] \rangle \omega(i_1, \cdots, i_j) \\ &= \Phi_L([M, T])\,. \end{aligned}$$

Hence $\Phi([M, T] - [CP(1), \tau]^n [N])$ is in the ideal generated by $(1+\eta)$, therefore the K-theory characteristic numbers (ordinary) of $[M] - [CP(1)]^n [N]$ are all divisible by 2. Again by the theorem of Hattori-Stong, we see that there is an element $[L] \in U_*$ such that $[M] - [CP(1)]^n [N] = 2[L]$, and from the Proposition (2.1) ii), we see $\Phi([M, T] - [CP(1), \tau]^n [N]) = \Phi([G, \sigma][L])$. Thus the theorem follows.

PROOF OF THEOREM (0.2). According to Remark (1.2), the first term of $\Phi([M, T])$ equals $\sum (-1)^i H^{0,i} \in R(G)$, and from Theorem (3.2), its evaluation on (-1) is equal to the first term of $\Phi_L([M, T])$ which is

$$\sum_F (2)^{-|\nu_F|} \langle \mathrm{ch} \sum_k b_k(\bar{\nu}_F)(1/2)^k\, \mathrm{td}\,(F), [F] \rangle\,. \qquad \text{Q. E. D.}$$

NOTE (3.3). Let

$$0 \longrightarrow U_n \longrightarrow \mathcal{O}_n^U(G) \longrightarrow \mathcal{M}_n \longrightarrow U_{n-1}(BG) \longrightarrow 0$$

be the exact sequence mentioned in [6], p. 63. Now U_2 is generated by $[CP(1)]$, $\mathcal{M}_2$ by $[D^2 \to pt]$ and $[CP(1)]$, and $U_1(BG)$ by $[S^1, a]$ which is the class of the 1-dim. sphere with the antipodal involution. Hence $\mathcal{O}_2^U(G)$ is generated over Z by $[CP(1)]$, $[CP(1)][G, \sigma]$ and $[CP(1), \tau]$, that is for any element x in $\mathcal{O}_2^U(G)$, x is represented as $A[CP(1)] + B[CP(1)][G, \sigma] + C[CP(1), \tau]$ where A, B, C are integers. Now by a simple calculation we see that

$$\Phi([CP(1)])=1-2\omega(1),$$

$$\Phi([CP(1)][G, \sigma])=(1+\eta)(1-2\omega(1)),$$

$$\Phi([CP(1), \tau])=\text{Inv. Lim.}\prod_s(1-2\eta t_s+\sum_{j=2}(1-\eta)^j t_s^j).$$

Hence B is decided by the first term of $\Phi(x)$, and A and C are decided by $\Phi_L(x)$. Let S_n (n is an even integer $\geqq 4$) be the algebraic curve in $CP(2)$ defined by the equation $z_1^n+z_2^n+z_3^n=0$ and let T be the involution $[z_1, z_2, z_3]\to[z_1, z_2, -z_3]$ on S_n. Then $[S_n, T]\in\mathcal{O}_2^U(G)$ and we have

$$\Phi_L([S_n, T])=n/2+n\omega(1)+\text{higher order terms},$$

and the first term of $\Phi([S_n, T])$ equals

$$H^{0,0}-H^{0,1}=(4n-n^2)/4+(n^2-2n)/4 \qquad ([\mathbf{6}]).$$

Then, comparing the coefficients, we have the following equation

$$[S_n, T]=(n/2)[CP(1), \tau]+(n(n-2)/4)[G, \sigma][CP(1)]$$

in $\mathcal{O}_2^U(G)$.

REMARK. Prof. N. Shimada informed to me that Theorem (0.1) is obtained from the structure of $U_*(BG)$. (See [**9**].)

§4. Proof of (1.1) and (3.2).

In this section we prove the Theorems (1.1) and (3.2). Let $\{a_i\}$ be the K-theory characteristic classes (Atiyah's γ-operation [**1**]). Then $K(BU)=Z[a_1, a_2, \cdots]$. A map $\chi: U^*(X)\to\text{Hom}(K(BU), K^*(X))$ is defined as follows. For $x\in U^k(X)$ which is represented by a map $f: S^{2n-k}\wedge X^+\to MU_n$, $\chi(x)$ is the following composed map.

$$K(BU)\xrightarrow{j}K(BU_n)\xrightarrow{\phi}\tilde{K}(MU_n)\xrightarrow{f^*}\tilde{K}(S^{2n-k}\wedge X^+)=K^k(X)$$

where j is the inclusion $BU_n\subset BU$, ϕ is the K-theory Thom isomorphism. By the coalgebra structure of $K(BU)$, that is, $\varDelta: K(BU)\to K(BU)\otimes K(BU)$ with $\varDelta(a_k)=\sum_{i+j=k}a_i\otimes a_j$, $\text{Hom}(K(BU), K^*(X))$ forms a ring, and χ is a ring homomorphism. When X is BG, the classifying space of G, χ is injective (tom Dieck [**7**]). Let α be the bundling map: $U_G^*\to U^*(BG)$, it is injective ([**7**]). Now for any element x of U_G^*, $\chi(\alpha(x))(a_{i_1}\cdots a_{i_j})\in K^*(BG)$, and by the fact $U_G^{\text{odd}}=0$ and Bott periodicity, it is considered an element of $K^0(BG)=\widehat{R(G)}$, where $\widehat{R(G)}$ is the completion of $R(G)$ in $I(G)$-adic topology ($I(G)$ denotes the augmentation ideal of $R(G)$). We define a map $\Phi': U_G^*\to\text{Inv. Lim.}\widehat{R(G)}[[t_1, \cdots, t_s]]$ by the following equation,

$$\Phi'(x)=\sum\chi(\alpha(x))(a_{i_1}\cdots a_{i_j})\omega(i_1, \cdots, i_j).$$

Since χ and α are both injective, the restriction of Φ' on U_G^n for each integer n is injective. The natural inclusion $R(G) \subset \widehat{R(G)}$ induces the inclusion j: Inv. Lim. $R(G)[[t_1, \cdots, t_s]] \to$ Inv. Lim. $\widehat{R(G)}[[t_1, \cdots, t_s]]$. The following lemma gives the proof of Theorem (1.1).

LEMMA (4.1). *$j \circ \Phi$ and Φ' coincide.*

PROOF. i) For $\zeta \in U_G^2$. Since ζ is the equivariant euler class of the bundle $\eta_1 \colon V_1 \to pt$, $\alpha(\zeta)$ is the 1-dim. Conner-Floyd class of the bundle η: $V_1 \times_G EG \to BG$, where $EG \to BG$ is the universal G-bundle. Consider the commutative diagram,

$$\begin{array}{ccccc} K^*(BG) & \xleftarrow{s^*} & \tilde{K}^*(V^c \wedge EG^+/G) & \xleftarrow{\tilde{f}^*} & \tilde{K}^*(MU) \\ & \nwarrow^{\times(1-\eta)} & \uparrow & & \uparrow \\ & & K^*(BG) & \xleftarrow{f^*} & K^*(BU) \end{array}$$

(V^c is the one-point compactification of V) where f is the classifying map of the bundle η, s is the zero-cross-section, and the vertical maps are the Thom isomorphisms. As η is a line bundle, $a_i(\eta) = 0$ for $i > 1$, and $a_1(\eta) = \eta - 1$. Since $\eta^2 = 1$ and $(1-\eta)^2 = 2(1-\eta)$, we can calculate as follows,

$$\begin{aligned} \Phi'(\zeta) &= (1-\eta) \sum a_{i_1} \cdots a_{i_j}(\eta) \omega(i_1, \cdots, i_j) \\ &= (1-\eta) \sum (a_1(\eta))^i \omega(\overbrace{1, \cdots, 1}^{i}) \\ &= (1-\eta) \sum (\eta-1)^i \omega(\overbrace{1, \cdots, 1}^{i}) \\ &= (1-\eta) W . \end{aligned}$$

ii) For $[M, T] \in \mathcal{O}_*^U(G)$. Fix an equivariant embedding of M into a G-module (over $\boldsymbol{C}$) V, and let ν be the equivariant normal bundle. We have the following two vector bundles

$$\nu \times_G EG \longrightarrow M \times_G EG ,$$

$$V \times_G EG \longrightarrow BG .$$

Let $T(\nu)$ be the Thom space of ν, $q \colon V^c \to T(\nu)$ the map collapsing the outside of the disk bundle of ν, and $q \wedge 1 \colon V^c \wedge EG^+/G \to T(\nu) \wedge EG^+/G$. Now as in § 1, we put $A = \sum a_{i_1} \cdots a_{i_j} \omega(i_1, \cdots, i_j)$ and for a vector bundle ξ, $A(\xi) = \sum a_{i_1} \cdots a_{i_j}(\xi) \omega(i_1, \cdots, i_j)$ and similarly $B(\xi) = \sum b_{i_1} \cdots b_{i_j}(\xi) \omega(i_1, \cdots, i_j)$. Then, $A(\xi) B(\xi) = 1$ and $A(\xi + \eta) = A(\xi) A(\eta)$, $B(\xi + \eta) = B(\xi) B(\eta)$. Moreover for an element $x \in U(?)$, we have $\chi(x)(a_{i_1} \cdots a_{i_j}) \in K(?)$ and we put $A(x) = \sum \chi(x)(a_{i_1} \cdots a_{i_j}) \omega(i_1, \cdots, i_j)$. Let

$$\begin{aligned}
\phi_\nu &: U^*(M\times_G EG) \longrightarrow \tilde{U}^*(T(\nu)\wedge EG^+/G)\\
\phi_V &: U^*(BG) \longrightarrow \tilde{U}^*(V^c\wedge_G EG^+)\\
\phi_\nu &: K^*(M\times_G EG) \longrightarrow \tilde{K}^*(T(\nu)\wedge EG^+/G)\\
\phi_V &: K^*(BG) \longrightarrow \tilde{K}^*(V^c\wedge_G EG^+)
\end{aligned}$$

be the Thom isomorphisms, then we have

$$A(x)A(V\times_G EG)=\phi_V^{-1}A(\phi_V(x)) \qquad \text{for } x\in U^*(BG)\,,$$

$$A(x)A(\nu\times_G EG)=\phi_\nu^{-1}A(\phi_\nu(x)) \qquad \text{for } x\in U^*(M\times_G EG)$$

(Thom isomorphisms are applied on each coefficients of $\omega(i_1, \cdots, i_j)$). Now by the definition of α, we see easily that $\alpha\circ i$ ($[M, T]$) equals $\phi_V^{-1}\circ q^*\wedge 1\circ\phi_\nu(1)$ $\in U^*(BG)$, where $1\in U^0(M\times_G EG)$ is the unit. Then

$$\begin{aligned}
\Phi'([M, T])A(V\times_G EG) &= A(\phi_V^{-1}\circ q^*\wedge 1\circ\phi_\nu(1))A(V\times_G EG)\\
&=\phi_V^{-1}A(q^*\wedge 1\circ\phi_\nu(1))\\
&=\phi_V^{-1}\circ q^*\wedge 1\circ A(\phi_\nu(1))\\
&=\phi_V^{-1}\circ q^*\wedge 1\circ \phi_\nu A(\nu\times_G EG)\\
&=\phi_V^{-1}\circ q^*\wedge 1\circ \phi_\nu(B(\tau_M\times_G EG)A((M\times V)\times_G EG))\\
&=\phi_V^{-1}\circ q^*\wedge 1\circ \phi_\nu(B(\tau_M\times_G EG)\circ p^*\times 1\circ A(V\times_G EG))\\
&\qquad\text{(where } p\times 1: M\times_G EG\to BG,\ p: M\to pt,\\
&\qquad\text{the collapsing map)}\\
&=\phi_V^{-1}\circ q^*\wedge 1\circ \phi_\nu(B(\tau_M\times_G EG))A(V\times_G EG)\,.
\end{aligned}$$

Hence

$$\Phi'([M, T])=\phi_V^{-1}\circ q^*\wedge 1\circ\phi_\nu B(\tau_M\times_G EG)\,.$$

But consider the following commutative diagram

$$\begin{array}{ccccccc}
K(M\times_G EG) & \longrightarrow & \tilde{K}(T(\nu)\wedge EG^+/G) & \xrightarrow{q^*\wedge 1} & \tilde{K}(V^c\wedge EG^+/G) & \longleftarrow & K(BG)\\
C\uparrow & & C\uparrow & & C\uparrow & & C\uparrow\\
\widehat{K_G(M)} & \longrightarrow & \widehat{\tilde{K}_G(T(\nu))} & \longrightarrow & \widehat{\tilde{K}_G(V^c)} & \longleftarrow & \widehat{R(G)}
\end{array}$$

where the lower sequence represents the K_G-theory Gysin homomorphism, and C is the isomorphism of Atiyah and Segal [4]. Since $C(b_i^G(\xi))=b_i(\xi\times_G EG)$, we obtain

$$\Phi'([M, T])=\sum p_!\,(b_{i_1}^G\cdots b_{i_j}^G(\tau_M))\omega(i_1, \cdots, i_j)\,.$$

Q. E. D.

Next we go on to the proof of Theorem (3.2). We consider the localized ring

$U^*(BG)[C^{-1}]$, where C is the euler class of the canonical line bundle η over BG, which is equal to $\alpha(\zeta)$. Let Λ: $U^*(BG)\rightarrow U^*(BG)[C^{-1}]$ be the natural map. For $[M, T]\in \mathcal{O}_*^U(G)$, $\Lambda\circ\alpha\circ i([M, T])\in U^*(BG)[C^{-1}]$ can be written by the term of the fixed point set and its normal bundle in M as follows (tom Dieck [7]),

$$\Lambda\circ\alpha\circ i([M, T])=\sum_F e(\nu_F\otimes\eta)^{-1}/Z_F$$

where F ranges over the components of the fixed point set, ν_F is the normal bundle of F in M, $\nu_F\otimes\eta$ is the tensor bundle over $F\times BG$, $e(\nu_F\otimes\eta)\in U^*(F\times BG)$ is its euler class, and $Z_F\in U_*(F)$ is the fundamental class, and finally $\cdots/Z_F$ represents the evaluation on Z_F.

Now we consider the localized ring Inv. Lim. $\widehat{R(G)}[[t_1, \cdots, t_s]][\Phi(\zeta)^{-1}]$. Since $\Phi(\zeta)=(1-\eta)W$ and W is invertible, it is equal to the ring localized at $(1-\eta)$. Since $\widehat{R(G)}=Z+$Inv. Lim. $Z/2^nZ$, and in $\widehat{R(G)}[(1-\eta)^{-1}]$, $1+\eta=0$, we have $\widehat{R(G)}[(1-\eta)^{-1}]=Q_2$ where Q_2 is the 2-adic completed rational number field. Let Φ'_L be the composed map: $U_G^*\xrightarrow{\alpha}U^*(BG)\xrightarrow{\Lambda}U^*(BG)[C^{-1}]\longrightarrow$ Inv. Lim. $Q_2[[t_1, \cdots, t_s]]$, then it is a ring homomorphism and the following diagram commutes.

$$\begin{array}{ccc} U_G^* & \xrightarrow{\Phi'} & \text{Inv. Lim. } \widehat{R(G)}[[t_1, \cdots, t_s]] \\ & {\scriptstyle \Phi'_L}\searrow & \big\downarrow \eta=-1 \\ & & \text{Inv. Lim. } Q_2[[t_1, \cdots, t_s]] \end{array}$$

The proof of Theorem (3.2) is contained in the following lemma.

LEMMA (4.2). *Let $j: Q\subset Q_2$ be the inclusion, then Φ'_L and $j\circ\Phi_L$ coincide.*

PROOF. Let $V=V_0^{k_0}\oplus V_1^{k_1}$ be a G-module (over $\boldsymbol{C}$), where V_0, V_1 are the trivial and non-trivial part respectively. For $[M, T]\in\mathcal{O}_*^U(G)$, assume that M is embedded in V. Let F be a connected component of the fixed point set, then $F\subset V_0^{k_0}$. We put as follows

ν: the normal bundle of M in V,

ν_0: the normal bundle of F in $V_0^{k_0}$,

ν_F: the normal bundle of F in M.

If we write $\nu|_F=\nu_0+\rho_F$, we have $\rho_F+\nu_F=F\times V_1^{k_1}$. Thus

$$e(\nu_F\otimes\eta)^{-1}=C^{-k_1}e(\rho_F\otimes\eta)$$

in $U^*(F\times BG)[C^{-1}]$.

The element $e(\rho_F\otimes\eta)/z_F$ of $U^*(BG)$ is represented by the following composed map,

$$V^c \wedge BG^+ \longrightarrow T(\nu_0)\wedge F^+ \wedge BG^+ \longrightarrow T(\nu_0)\wedge T(\rho_F\otimes\eta)$$
$$\xrightarrow{\tilde f\wedge\tilde g} MU\wedge MU \longrightarrow MU$$

where $q: V^c \to T(\nu_0)\wedge F^+$ is a product of the usual collapsing map onto the Thom space and the projection onto F, $s: F\times BG \to T(\rho_F\otimes\eta)$ is the zero cross-section and f, g are the classifying maps of the bundles ν_0, $\rho_F\otimes\eta$ respectively. Now let $p: F\to pt$ be the collapsing map and $d: F\to F\times F$ the diagonal, and finally $V^c \xrightarrow{q_1} T(\nu_0) \xrightarrow{q_2} T(\nu_0)\wedge F^+$ be the decomposition of q. Consider the following diagram (the two left squares commute and the vertical maps are all Thom isomorphisms).

$$\begin{array}{ccccccc}
K^*(BG) & \xleftarrow{p_!\otimes 1} & K^*(F\times BG) & \xleftarrow{d^*\times 1} & K^*(F\times F\times BG) & & K^*(F\times F\times BG) \\
\downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_2} & & \downarrow{\scriptstyle\phi_3} & & \downarrow{\scriptstyle\phi_4} \\
\tilde K^*(V^c\wedge BG^+) & \xleftarrow{q_1^*} & \tilde K^*(T(\nu_0)\wedge BG^+) & \xleftarrow{q_2^*} & \tilde K^*(T(\nu_0)\wedge F^+\wedge BG^+) & \xleftarrow{1\wedge s^*} & \tilde K^*(T(\nu_0\oplus\rho_F\otimes\eta))
\end{array}$$

In the above diagram, since $K^*(F\times BG)=K^*(F)\otimes K^*(BG)$, $p_!\otimes 1$ is defined. For $x\in K^*(F\times F\times BG)$, we have

$$\begin{aligned}\phi_1^{-1}\circ q_1^*\circ q_2^*\circ 1\wedge s^*\circ\phi_4 x &= p_!\otimes 1\circ d^*\times 1\circ\phi_3^{-1}\circ 1\wedge s^*\circ\phi_4 x\\ &= p_!\otimes 1(d^*\times 1(x)\lambda(\nu_0\oplus\rho_F))\end{aligned}$$

where $\lambda(?)$ denotes the K-theory euler class.

Now we use the notations $A(\ )$, $B(\ )$ as in the proof of (1.1). Then by the definition of α, we have,

$$\begin{aligned}\Phi'_L([M,T]) &= \sum_F (\Phi(\zeta))^{-k_1}\phi_1^{-1}\circ q_1^*\circ q_2^*\circ 1\wedge s^*\circ\phi_4 A(\nu_0+\rho_F\otimes\eta)\\
&= \sum((1-\eta)W)^{-k_1}p_!\otimes 1\ (A(\nu_0+\rho_F\otimes\eta)\lambda(\rho_F\otimes\eta))\\
&= \sum((1-\eta)W)^{-k_1}p_!\otimes 1\ (B(\tau_M+\nu_F\otimes\eta)\lambda(\rho_F\otimes\eta))A(V_1^{k_1}\times_G EG)\\
&= \sum((1-\eta)W)^{-k_1}p_!\otimes 1\ (B(\tau_M+\nu_F\otimes\eta)\lambda(\rho_F\otimes\eta))W^{k_1}\\
&= \sum(1-\eta)^{-k_1}p_!\otimes 1\ (B(\tau_M+\nu_F\otimes\eta)\lambda(\rho_F\otimes\eta))\,.\end{aligned}$$

Now formally,

$$B(\tau_F)=\prod_s\sum_i b_i(\tau_F)t_s^i\,,$$
$$B(\nu_F\otimes\eta)=\prod_s\sum_i b_i(\nu_F)\Big(\frac{\eta t_s}{1+(\eta-1)t_s}\Big)^i\prod_s\Big(\frac{1}{1+(\eta-1)t_s}\Big)^{\dim F},$$
$$\lambda(\rho_F\otimes\eta)=(1-\eta)^{\dim\rho_F}\sum_i b_i(\bar\nu_F)\Big(\frac{\eta}{1-\eta}\Big)^i$$

and we put $\eta=-1$, then the lemma follows. Q. E. D.

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