J. Math. Soc. Japan
Vol. 24, No. 2, 1972

On rank 3 groups with a multiply transitive constituent

By Eiichi BANNAI*)

(Received Sept. 1, 1971)

§ 1. Introduction.

We say that a permutation group $(\mathfrak{G}, \Omega)$ is a primitive extension of rank 3 of a permutation group (G, Δ) if the following conditions are satisfied: (i) $\mathfrak{G}$ is primitive and of rank 3 on the set Ω, and (ii) there exists an orbit $\Delta(a)$ of the stabilizer $\mathfrak{G}_a$ $(a \in \Omega)$ such that the action of $\mathfrak{G}_a$ on $\Delta(a)$ is faithful and that $(\mathfrak{G}_a, \Delta(a))$ and (G, Δ) are isomorphic as permutation groups.

The purpose of this note is to prove the following theorem:

THEOREM 1. *Let* (G, Δ) *be a* 4*-ply transitive permutation group. If* (G, Δ) *has a primitive extension of rank* 3, *then one of the following cases holds:*

(I) $|\Delta|=5$, $G=S_5$,

(II) $|\Delta|=7$, $G=S_7$ *or* A_7,

(III)[1] $|\Delta|=57$ *and* $G \neq S_{57}, A_{57}$,

where S_n *and* A_n *denote the symmetric and alternating groups on* Δ $(|\Delta|=n)$ *respectively.*

Theorem 1 is regarded as a sort of generalization of the results in T. Tsuzuku [6] and S. Iwasaki [3] where primitive extensions of rank 3 of symmetric and alternating groups are determined.

The author wishes to express his hearty thanks to Professor Ryuzaburo Noda for his kind criticisms, which have improved Theorem 1 to this general form.[2] The author also thanks Mr. Hikoe Enomoto for the valuable discussions we had.

§ 2. Proof of Theorem 1.

LEMMA 1. *Let* $\mathfrak{G}$ *be a primitive rank* 3 *permutation group on* Ω, *and let* $\mathfrak{G}_a$ *be doubly transitive on one of its orbits* $\Delta(a)$. *Let* $\Gamma(a)$ *be another orbit*

*) Supported in part by the Fujukai Foundation.

1) Professor Noboru Ito has kindly shown the author the proof of the non-existence of non-trivial 4-ply transitive permutation group of degree 57 in a letter dated on Aug. 18, 1971. Therefore the case (III) of Theorem 1 does not occur.

2) In the original manuscript Theorem 1 is proved with the additional hypothesis that the case (B) in the proof of Theorem 1 holds.

($\neq \{a\}$, $\Delta(a)$) of $\mathfrak{G}_a$, and let us set $|\Delta(a)|=k$, $|\Gamma(a)|=l$ *and* $|\Delta(a)\cap\Delta(b)|=\mu$ $(b\in\Gamma(a))$. *Then*

(i) $\mu l=k(k-1)$ *and* $0<\mu<k-1$,

(ii) *if* $b, c\in\Delta(a)$, $b\neq c$, *then there exist a point* $d\in\Gamma(a)$ *and an automorphism* σ *of the group* $\mathfrak{G}_{a,b}$ *such that* $(\mathfrak{G}_{a,b,c})^{\sigma}\leqq\mathfrak{G}_{a,d}$.

PROOF OF (i). This is essentially due to Manning [4]. For an ingeneous proof of the full statement of (i), see P. J. Cameron: Proofs of some theorems of W. A. Manning, Bull. London Math. Soc., Vol. 1 (1969), 349-352.

PROOF OF (ii). Since the orbit $\Delta(a)$ is self-paired, there exists an element x of $\mathfrak{G}$ which interchanges a and b. Let σ be the automorphism of $\mathfrak{G}_{a,b}$ induced by the conjugation by x, then we easily have the assertion, since c^x (let us set $=d$) $\in\Gamma(a)$.

REMARK. More strengthened form of Lemma 1 is stated in S. Montague [5] as Theorem 3.1 (page 509). However Theorem 3.1 (iii) is incorrect. For example, $U_3(5)$ (which is a primitive extension of rank 3 of A_7 with subdegrees 1, 7, 42) and Higman-Sims's simple group of order 44,352,000 (which is a primitive extension of rank 3 of M_{22} with subdegrees 1, 22, 77) give a contradiction to Theorem 3.1 (iii) in [5].

PROOF OF THEOREM 1. Let $(\mathfrak{G}, \Omega)$ be a primitive extension of (G, Δ) and let $k=|\Delta(a)|\geqq 4$, $l=|\Gamma(a)|$ and $\mu=|\Delta(a)\cap\Delta(b)|$ $(b\in\Gamma(a))$. Let σ (an automorphism of $\mathfrak{G}_{a,b}$) and d (a point in $\Gamma(a)$) be as in the statement of Lemma 1 (ii). Then $(\mathfrak{G}_{a,b,c})^{\sigma}$ $(b, c\in\Delta(a), b\neq c)$ is a subgroup of index $|\Delta|-1$ of the 3-ply transitive permutation group $(\mathfrak{G}_{a,b}, \Delta(a)-\{b\})$. Thus by Satz 3 in N. Ito [2], either

(A) $(\mathfrak{G}_{a,b,c})^{\sigma}$ is transitive on $\Delta(a)-\{b\}$ or

(B) $(\mathfrak{G}_{a,b,c})^{\sigma}=\mathfrak{G}_{a,b,e}$ for some $e\in\Delta(a)-\{b\}$.

Let us assume that the case (A) holds. Then the orbits in $\Delta(a)$ by the action of the group G_d $(=\mathfrak{G}_{a,d}\geqq(\mathfrak{G}_{a,b,c})^{\sigma})$ are either $\Delta(a)$ itself, or $\{b\}$ and $\Delta(a)-\{b\}$. Therefore either $\mu=1$, $\mu=k-1$, $\mu=k$ or $\mu=0$. However by Lemma 1 (i) the last three cases are impossible (i. e., contradict the primitivity of $\mathfrak{G}$), therefore $\mu=1$. Next let us assume that the case (B) holds. From the 3-ply transitivity of G, the structure of the orbits of the group G_d $(=\mathfrak{G}_{a,d}\geqq\mathfrak{G}_{a,b,e})$ on Δ is one of the following: (i) Δ, (ii) $\{b\}$, $\Delta-\{b\}$, (iii) $\{e\}$, $\Delta-\{e\}$, (iv) $\{b, e\}$, $\Delta-\{b, e\}$, (v) $\{b\}$, $\{e\}$, $\Delta-\{b, e\}$. Therefore either $\mu=1$, $\mu=2$, $\mu=k-2$, $\mu=k$, $\mu=k-1$ or $\mu=0$. The last three cases are impossible, and if $\mu=k-2$ then we have $\mu=2$ $(k=4)$ by the relation $\mu l=k(k-1)$. Therefore we have $\mu=1$ or 2 in both cases (A) and (B). Firstly let us assume that $\mu=1$. Then from D. G. Higman [1] and 4-ply transitivity of G, we have either $k=7$ or 57. If $k=7$, then G is either A_7 or S_7, and they have a unique primitive extension of rank 3 of type $\mu=1$. On the other hand, A_{57} and S_{57} have not, and so we have the assertion in this case. (Cf. [1'], [3] and [6].) Secondly let us assume that $\mu=2$. We may assume that $k\neq 4$, since there exists no primitive group of rank 3 with subdegrees 1, 4, 6. Then $(G, \Gamma(a))\cong(G, G/G_d)\cong(G, G/G_{(b,e)})$ as a permutation group, and is of rank 3 by the 4-ply transitivity of G on Δ. The lengths of orbits of G_d ($d\in\Gamma(a)$) on $\Gamma(a)$ are 1, $2(k-2)$ and $\frac{1}{2}(k-2)(k-3)$. Now, G_d is transitive on $\Delta(a)\cap\Gamma(d)$. Thus G_d must have an orbit $\Delta(d)\cap\Gamma(a)$ on $\Gamma(a)$ since there exists an element of $\mathfrak{G}$ interchanging a and d[3]. If $k\neq 5$, then $|\Delta(d)\cap\Gamma(a)|=k-2\neq 2(k-2)$ and $\neq\frac{1}{2}(k-2)(k-3)$, and this is impossible. If $k=5$, then $G=S_6$, and S_6 has a unique such extension. (Cf. [1'] and [6].) Thus we have completed the proof of Theorem 1.

University of Tokyo

References

[1] D. G. Higman, Finite permutation groups of rank 3, Math. Z., **86** (1964), 145–156.

[1'] D. G. Higman, Primitive rank 3 groups with a prime subdegree, Math. Z., **91** (1966), 70–86.

[2] N. Ito, Über die Gruppen $PSL_n(q)$, die eine Untergruppe von Primzahlindex enthalten, Acta Sci. Math. Szeged., 21 (1960), 206–217.

[3] S. Iwasaki, A note on primitive extensions of rank 3 of alternating groups, J. Fac. Sci. Hokkaido Univ., **21** (1970), 125–128.

[4] W. A. Manning, A theorem on simply transitive groups, Bull. Amer. Math. Soc., **35** (1929), 330–332.

[5] S. Montague, On rank 3 groups with a multiply transitive constituent, J. Algebra, **14** (1970), 506–522.

[6] T. Tsuzuku, On primitive extensions of rank 3 of symmetric groups, Nagoya Math. J., **27** (1966), 171–177.

[7] D. Wales, Uniqueness of the graph of a rank three group, Pacific J. Math., **30** (1969), 271–276.

Added in proof: (1) The non-existence of rank 3 groups with $k=57$ and $\mu=1$ has just been proved by M. Aschbacher: The non-existence of rank three permutation groups of degree 3250 and subdegree 57, J. Algebra, 19 (1971), 538–540.

(2) The assumption that $\mathfrak{G}_a$ is faithful on $\Delta(a)$ is removable in Theorem 1. (Cf. Theorem 1 of P. J. Cameron (cited in page 253), D. G. Higman [1'] and M. Aschbacher (ibid).)

3) The author has found this argument in D. Wales [7], Theorem 1.