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Non-normal functions $f(z)$ with $\iint_{|z|<1} |f'(z)| dxdy < \infty$

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1. Let $f(z)$ be a function holomorphic in the open unit disk D. The spherical derivative of $f(z)$ is given by

$$\rho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

The function $f(z)$ is said to be *normal* in D (see [5]) if there exists a constant $K>0$ such that

$$\rho(f(z)) \leqq \frac{K}{1-|z|^2}$$

for each $z \in D$; and $f(z)$ is said to be *uniformly normal* in D (see [2]) if there exists a constant $K>0$ such that

$$|f'(z)| \leqq \frac{K}{1-|z|^2}$$

for each $z \in D$.

Using the notations

$$\mathcal{D}(f) = \iint_D |f'(z)|^2 dxdy$$

and

$$\mathcal{S}(f) = \iint_D |f'(z)| dxdy,$$

we state the following questions:

(1) Does $\mathcal{D}(f) < \infty$ imply $f(z)$ is uniformly normal?

(2) Does $f(z)$ uniformly normal imply $\mathcal{D}(f) < \infty$?

(3) Does $\mathcal{S}(f) < \infty$ imply $f(z)$ is uniformly normal?

(4) Does $f(z)$ uniformly normal imply $\mathcal{S}(f) < \infty$?

Mathews [6] has answered question (1) in the affirmative; and questions (2) and (4) have been answered in the negative by Mergeljan [7] who has proved the existence of a bounded holomorphic function $g(z)$ for which

$$\iint_D |g'(z)| dxdy = \infty.$$

Here we answer question (3) in the negative; even more we show that $\mathcal{S}(f) < \infty$ need not imply $f(z)$ is normal.

2. Throughout we let

$$B(z)=\prod_{n=1}^{\infty}\frac{a_n-z}{1-a_nz}$$

and

$$f_\alpha(z)=\frac{B(z)}{(1-z)^\alpha},$$

where $a_n=1-\frac{1}{e^n}$ and $\alpha>0$.

LEMMA 1. *The function $f_\alpha(z)$ is non-normal for all $\alpha>0$.*

PROOF. Bagemihl and Seidel [**1**, Example 4, p. 11] have shown: (1) there exists a constant $K>0$ such that $\rho(a_n, a_{n+1})<K$ for all n, where $\rho(a_n, a_{n+1})$ is the non-Euclidean hyperbolic distance between a_n and a_{n+1}; and (2) there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $a_n<x_n<a_{n+1}$ and

$$\liminf_{n\to\infty}|B(x_{2n})|>0.$$

From (2) it is clear that $f_\alpha(x_{2n})\to\infty$ as $n\to\infty$ while $f_\alpha(a_{2n})\to 0$ as $n\to\infty$ for all $\alpha>0$. Also by (1) we have $\rho(a_{2n}, x_{2n})<K$ for all n. It now follows from a result of Lappan [4, Lemma 3, p. 188] that $f_\alpha(z)$ is non-normal for all $\alpha>0$.

LEMMA 2. $\iint_D \frac{|B'(z)|}{|1-z|^\alpha}dxdy<\infty$ *for* $\alpha<\frac{1}{2}$.

PROOF. Using logarithmic differentiation, we get

$$B'(z)=\sum_{k=1}^{\infty}B_k(z)\left\{\frac{a_k^2-1}{(1-a_kz)^2}\right\},$$

where

$$B_k(z)=\prod_{\substack{n=1\\ n\neq k}}^{\infty}\frac{a_n-z}{1-a_nz}.$$

Hence

$$|B'(z)|\leqq\sum_{n=1}^{\infty}\frac{1-a_n^2}{|1-a_nz|^2};$$

and we have

$$\int_0^{2\pi}\frac{|B'(re^{i\theta})|}{|1-re^{i\theta}|^\alpha}d\theta\leqq\frac{1}{(1-r)^\alpha}\sum_{n=1}^{\infty}(1-a_n^2)\int_0^{2\pi}\frac{d\theta}{|1-a_nre^{i\theta}|^2}$$

$$=\frac{2\pi}{(1-r)^\alpha}\sum_{n=1}^{\infty}\frac{1-a_n^2}{1-(a_nr)^2}$$

$$\leqq\frac{4\pi}{(1-r)^\alpha}\sum_{n=1}^{\infty}\frac{1-a_n}{1-a_nr}.$$

Now for $\alpha<\frac{1}{2}$

$$\left\{\int_0^1 \frac{dr}{(1-r)^\alpha(1-a_n r)}\right\}^2 \leqq \int_0^1 \frac{dr}{(1-r)^{2\alpha}}\int_0^1 \frac{dr}{(1-a_n r)^2}$$
$$=\left(\frac{1}{1-2\alpha}\right)\left(\frac{1}{1-a_n}\right).$$

It follows that for $\alpha<\frac{1}{2}$

$$\iint \frac{|B'(z)|}{|1-z|^\alpha}\, dxdy \leqq \frac{4\pi}{\sqrt{1-2\alpha}} \sum_{n=1}^\infty \sqrt{1-a_n}$$
$$= \frac{4\pi}{\sqrt{1-2\alpha}} \sum_{n=1}^\infty e^{-\frac{n}{2}} < \infty,$$

and the lemma is proved.

THEOREM 1. *For each* $\alpha\in\left(0, \frac{1}{2}\right)$ *the function* $f_\alpha(z)$ *is non-normal and*

$$\iint_D |f'_\alpha(z)|\, dxdy < \infty .$$

PROOF. Since

$$f'_\alpha(z)= \frac{B'(z)}{(1-z)^\alpha} + \frac{\alpha B(z)}{(1-z)^{1+\alpha}}$$

and

$$\iint_D \frac{|B(z)|}{|1-z|^{1+\alpha}}\, dxdy \leqq \iint_D \frac{dxdy}{|1-z|^{1+\alpha}} < \infty$$

for $\alpha<1$, it follows from Lemma 2 that

$$\iint_D |f'_\alpha(z)|\, dxdy < \infty$$

for $\alpha<\frac{1}{2}$. In view of Lemma 1, the theorem is proved.

3. We now consider real-valued harmonic functions in D. Such a function $u(z)$ is said to be *normal* in D (see [3]) if there exists a constant $K>0$ such that

$$\frac{\sqrt{(u_x(z))^2+(u_y(z))^2}}{1+|u(z)|^2} \leqq \frac{K}{1-|z|^2}$$

for each $z\in D$. The *surface area* of $u(z)$ is equal to

$$\iint_D \sqrt{1+(u_x(z))^2+(u_y(z))^2}\, dxdy .$$

THEOREM 2. *For each* $\alpha\in\left(0, \frac{1}{2}\right)$ *the function* $u(z)=\mathrm{Re}\,(f_\alpha(z))$ *is a non-normal harmonic function with finite surface area.*

PROOF. According to Lappan [3, Theorem 5, p. 158] $u(z)$ is non-normal since $f_\alpha(z)$ is non-normal. That $u(z)$ has finite surface area follows from Theorem 1 and the fact that

$$\sqrt{1+(u_x(z))^2+(u_y(z))^2} \leqq 1+|f'_\alpha(z)|.$$

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