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Herbrand uniformity theorems for infinitary languages[1)]

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We begin by recalling several aspects of Herbrand's theorem for $L_{\omega,\omega}$, or more precisely, of several corollaries to Herbrand's original theorem ([**3**], [**6**], [**8**], not in these references but elsewhere in the literature these corollaries are sometimes confused with the theorem itself). $L^{\vee}$ is an extension of L by arbitrarily many function symbols of each number of arguments.

(1) *Semantic versions.*

(a) *Reduction, for validity, to existential sentences.* For every sentence φ of L there is an existential sentence $\check{\varphi}$ of $L^{\vee}$ such that φ is valid if and only if $\check{\varphi}$ is valid.

(b) *Weak Uniformity theorem.* A prenex existential sentence $\theta=\exists x_1 \cdots x_m \phi(x_1, \cdots, x_m)$ is valid if and only if it is valid in all canonical (term) models; i.e., if and only if for each model $\mathfrak{A}$ of θ there are terms $t_1, \cdots, t_m$ such that $\mathfrak{A} \models \phi(t_1, \cdots, t_m)$.

(b)′ *Uniformity theorem.* θ is valid if and only if for some finite set T of terms $\bigvee\!\!\!\bigvee_{t_1,\cdots,t_m \in T} \phi(t_1, \cdots, t_m)$ is valid.

A third aspect of Herbrand's theorem will be considered in (2) (b) below.

There are many possible sentences $\check{\varphi}$ which can be used for a given φ in (1)(a). In the case that φ is in prenex form, the *validity functional form* (often called the Herbrand normal form), which is dual to the Skolem form, always suffices. For example, if $\varphi=\exists y \forall z \varphi_1(y, z)$ with φ_1 quantifier-free, the validity functional form of φ is

$$\text{(i)} \qquad \exists y \varphi_1(y, f(y)) .$$

Following Denton and Dreben [**3**], one can directly associate existential $\check{\varphi}$ with *any* φ; $\check{\varphi}$ is an Herbrand normal form of a prenex form of φ.

To illustrate (1)(a) and (1)(b)′, consider the sentence

1) This paper forms part of the author's Ph. D. thesis, submitted to Stanford University in May, 1970. We would like to thank our thesis adviser, Professor Solomon Feferman, for his advice and direction and unfailing willingness to give of his time and effort throughout our work. We also owe a special debt of thanks to Professor Georg Kreisel for suggesting the initial direction of our research and for many helpful suggestions.

(ii) $$\exists y \forall z(r(y) \vee \sim r(z))$$

which is trivially valid. The formula $\check{\varphi}$ is then

(iii) $$\exists y(r(y) \vee \sim r(f(y))) .$$

If we take a constant symbol c of $L^{\vee}$, we get a valid disjunction over the finite set $T=\{c, f(c)\}$:

(iv) $$(r(c) \vee \sim r(f(c))) \vee (r(f(c)) \vee \sim r(f(f(c)))) .$$

A simple model-theoretic proof of (1)(a), (b) is given in Kreisel-Krivine [8]. (1)(b)′ then follows by compactness.

(2) *Syntactic versions.* Here we distinguish between *quantitative* and *qualitative* refinements of (1), when, naturally, the starting point is not the validity of a sentence φ or θ, but a derivation by some given rules of inference.

(a) *Explicit definability.* There are functionals (in an explicitly defined class) which map a derivation of φ or θ in (1)(a), resp. (1)(b), into a derivation of $\check{\varphi}$, respectively into a set T of terms, and a derivation of $\bigvee\!\!\!\bigvee_{t_1, \cdots, t_m \in T} \psi(t_1, \cdots, t_m)$.

(b) *Herbrand-Gentzen midsequent theorem.* For any derivation $\mathcal{D}$ of a prenex formula φ there is a derivation $\mathcal{D}'$ in which all propositional inferences precede all quantifier inferences. (The midsequent is the lowest quantifier-free sequent in $\mathcal{D}'$.)

The syntactic versions are called "refinements" because, modulo the completeness theorem, the semantic versions follow easily from them, for example, (1)(b)′ from (2)(b). (Apart from quantitative bounds, (2)(b) follows also from (1)(b)′ using a suitable substitution of variables for terms in a derivable disjunction $\bigvee\!\!\!\bigvee_{t_1, \cdots, t_m \in T} \psi(t_1, \cdots, t_m)$. For example, in (ii) above, use the substitution instance

$$(r(x) \vee \sim r(y)) \vee (r(y) \vee \sim r(z))$$

of (iv).) The reader not familiar with the basic proof theory of $L_{\omega,\omega}$ should look at example 3.3 below.

In this paper we consider generalizations of (1) and (2) to $L_{\omega_1,\omega}$ and its admissible sublanguages L_A and also to $L_{\kappa,\kappa}$, where κ is strongly inaccessible. The main results are as follows.

(1)(a) and (1)(b) generalize, by means of the semantic proofs in § 1. The same methods give, as an easy corollary, a new and simpler proof of Takeuti's interpolation theorem for $L_{\kappa,\kappa}$ [12]; this is done in § 4. As is to be expected, (1)(b)′ does not generalize without changing the meaning of "finite." It does generalize, for $A=HC$ and for countable admissible A, if A-finite sets of terms T are used (the terms themselves remain finite in the ordinary sense).

(2)(a) is shown in § 2 to generalize to $L_{\omega_1,\omega}$. (2)(b) turns out *not* to generalize, at least for the natural generalization of the concept of midsequent theorem given in § 3.

It may be remarked that, in the literature on finite languages, obstacles have been noted to extending Herbrand's (original) theorem; e. g., to arithmetic [7]. But these results seem incomparable to ours.

§ 1. Herbrand uniformity theorems: semantic proof.

Unless otherwise specified, all formulas considered are in reduced (negation normal) form; we write φ^* for the reduced form of φ. In the following, the language L is to be $L_{\omega_1,\omega}$ or one of its countable admissible sublanguages L_A, $A \neq HF$.

1.1. DEFINITION. Given a language L, $L^\vee$ is an extension of L by arbitrarily many function symbols of each number of arguments. Constant symbols are regarded as O-ary function symbols. For every formula $\varphi \in L$, the relation, $\breve{\varphi} \in L^\vee$ is a *validity functional form* (v. f. f.) of φ, is defined inductively as follows:

(i) φ is an atomic or a negated atomic formula and $\breve{\varphi} = \varphi$; or

(ii) $\varphi = \exists x \psi(x)$, $\breve{\psi}(x)$ is a v. f. f. of $\psi(x)$ and $\breve{\varphi} = \exists x \breve{\psi}(x)$; or

(iii) $\varphi = \forall x \psi(x)$, $\breve{\psi}(y_1, \cdots, y_n, x)$ (where $y_1, \cdots, y_n$ are the free variables of φ) is a v. f. f. of $\psi(x)$, and $\breve{\varphi} = \breve{\psi}(y_1, \cdots, y_n, f(y_1, \cdots, y_n))$ where f is an n-ary function symbol not occurring in φ; or

(iv) $\varphi = \bigvee \Phi (\varphi = \bigwedge \Phi)$, $\Phi^\vee$ is a set of v. f. f.'s $\breve{\psi}$ of $\psi \in \Phi$ such that $\psi_1 \neq \psi_2$ implies $\breve{\psi}_1$, $\breve{\psi}_2$ have disjoint sets of new function symbols, and $\breve{\varphi} = \bigvee \Phi^\vee (\breve{\varphi} = \bigwedge \Phi^\vee)$.

$\breve{\varphi}$ is uniquely determined up to a renaming of the function symbols. We shall, therefore, regard two v. f. f.'s $\breve{\varphi}_1$, $\breve{\varphi}_2$ of φ as equivalent and speak of *the* validity functional form $\breve{\varphi}$ of φ.

As an example, consider

$$\varphi = \exists w \forall x \exists y \forall z \psi(w, x, y, z),$$

ψ quantifier-free. The validity functional form of φ is

$$\breve{\varphi} = \exists w \exists y \psi(w, f(w), y, g(w, f(w), y)).$$

1.2. DEFINITION. *The satisfiability functional form* $\hat{\varphi}$ of φ is the dual of $\breve{\varphi}$, i. e.

$$\hat{\varphi} = (\sim (\breve{\sim \varphi})^*)^*.$$

It is seen that $\hat{\varphi}$ can be defined inductively just as $\breve{\varphi}$ by interchanging $\exists$ and $\forall$. Note, too, that $\breve{\varphi}$ is existential and $\hat{\varphi}$ is universal.

1.3. DEFINITION. Let $\varphi \in L$, T a countable set of terms of L. $\varphi^{(T)}$ is defined by:

(i) If φ is an atomic or a negated atomic formula, $\varphi^{(T)} = \varphi$.

(ii) If $\varphi = \exists x \psi(x)$, $\varphi^{(T)} = \bigvee\!\!\bigvee_{t \in T} \psi^{(T)}(t)$.

(iii) If $\varphi = \forall x \psi(x)$, $\varphi^{(T)} = \bigwedge\!\!\bigwedge_{t \in T} \psi^{(T)}(t)$.

(iv) If $\varphi = \bigwedge\!\!\bigwedge \Psi$, $\Psi^{(T)} = \{\psi^{(T)} : \psi \in \Psi\}$, then $\varphi^{(T)} = \bigwedge\!\!\bigwedge \Psi^{(T)}$.

(v) If $\varphi = \bigvee\!\!\bigvee \Psi$, $\varphi^{(T)} = \bigvee\!\!\bigvee \Psi^{(T)}$.

1.4. DEFINITION. Let $\varphi \in L$, $\check{\varphi} \in L^{\vee}$, T a countable set of terms of $L^{\vee}$. The *Herbrand form of* φ *relative to* T is $\check{\varphi}^{(T)}$.

In general, $\check{\varphi}^{(T)}$ is a formula of $L_{\omega_1,\omega}$. If $L = L_A$, where A is countable admissible, and T is A-finite, then for each $\varphi \in L$, $\check{\varphi}$, $\check{\varphi}^{(T)} \in L^{\vee}$.

1.5. DEFINITION. A *canonical* L-structure $\mathfrak{A}$ is one where every element of $|\mathfrak{A}|$ is denoted by a closed term in L.

Given a model $\mathfrak{A}$ and additional function symbols $f = (f_i)_{i \in I}$ and assignments $\bar{f} = (\bar{f}_i)_{i \in I}$, we write $\langle \mathfrak{A}, \bar{f} \rangle$ for the extension of $\mathfrak{A}$ by the new functions $(\bar{f}_i)_{i \in I}$.

1.6. LEMMA. *Suppose* φ *is a sentence of* L *and* $\mathfrak{A}$ *is an* L*-structure.*

(i) $\mathfrak{A} \models \varphi \Leftrightarrow \langle \mathfrak{A}, \bar{f} \rangle \models \check{\varphi}$ *for every assignment* $\bar{f}$ *to the new function symbols in* $\check{\varphi}$.

(ii) $\mathfrak{A} \models \varphi \Leftrightarrow \langle \mathfrak{A}, \bar{f} \rangle \models \hat{\varphi}$ *for some assignment* $\bar{f}$ *to the new function symbols in* $\hat{\varphi}$.

PROOF. We first show (i)$\Rightarrow$, (ii)$\Rightarrow$. Then $\Leftrightarrow$ follows in both.

(i) $\Rightarrow$ is easily proved by induction on the formation of $\check{\varphi}$.

(ii) $\Rightarrow$ is also proved by induction on the formation of $\hat{\varphi}$. This is straightforward, using the Axiom of Choice. One step worth noting is for the case where ψ is a subformula of φ of the form $\psi = \bigwedge\!\!\bigwedge \Psi$. Let $\Psi^{\wedge} = \{\hat{\sigma} : \sigma \in \Psi\}$ where different formulas have disjoint sets of new function symbols. By induction, for each $\sigma \in \Psi$ there is an extension $\langle \mathfrak{A}, \bar{f}_\sigma \rangle$ of $\mathfrak{A}$ s.t. $\langle \mathfrak{A}, \bar{f}_\sigma \rangle \models \hat{\sigma}$ (relative to a fixed assignment on the free variables of ψ). But now the $\bar{f}_\sigma$ correspond to disjoint sets of function symbols, so by putting together all the $\bar{f}_\sigma$, we get an extension $\langle \mathfrak{A}, \bar{f} \rangle$ s.t. $\langle \mathfrak{A}, \bar{f} \rangle \models \hat{\psi}$.

1.7. LEMMA. *Let* φ *be a sentence of* L. $\check{\varphi} \in L^{\vee}$ *is valid if and only if* $\check{\varphi}$ *is true in all canonical* $L^{\vee}$*-structures.*

PROOF. The "only if" part is trivial. For the converse, suppose we had a model $\mathfrak{A} \models \sim \check{\varphi}$. $(\sim \check{\varphi})^*$ is universal and so $\sim \check{\varphi}$ is true in the canonical $L^{\vee}$-substructure $\mathfrak{B} \subseteq \mathfrak{A}$, where the domain of $\mathfrak{B}$ is the set of valuations $\bar{t} \in |\mathfrak{A}|$ for closed terms t of $L^{\vee}$. But then $\check{\varphi}$ is false in the canonical $L^{\vee}$-structure $\mathfrak{B}$.

1.8. THEOREM. *Let* φ *be a sentence of* L, $\check{\varphi}$ *in* $L^{\vee}$, T *the set of all closed terms in* $L^{\vee}$. *Then* φ *is valid if and only if* $\check{\varphi}^{(T)}$ *is valid.*

PROOF. By 1.6 (i), $\models \varphi$ if and only if $\models \check{\varphi}$, and by 1.7, $\check{\varphi}$ is valid if and only if $\check{\varphi}$ is true in all canonical $L^{\vee}$-structures. But for any canonical $L^{\vee}$-structure $\mathfrak{A}$,

$$\mathfrak{A} \models \check{\varphi} \Leftrightarrow \check{\varphi}^{(T)} .$$

Moreover, $\check{\varphi}^{(T)}$ is quantifier-free, so it is valid if and only if it is true in all canonical $L^{\vee}$-structures. Thus, $\models \varphi$ if and only if $\models \check{\varphi}^{(T)}$.

By using 1.8 and the compactness theorem for finitary propositional logic, we get as in Kreisel-Krivine [8]:

1.9. THEOREM (Herbrand). *Let φ be a finitary formula in prenex form, $\check{\varphi}=\exists x_1 \cdots x_n \psi(x_1, \cdots, x_n)$, ψ quantifier-free. Then φ is valid if and only if there are terms $t_1^{(i)}, \cdots, t_n^{(i)}$, $1 \leqq i \leqq k$, of $L^{\vee}$ s.t.*

$$\psi(t_1^{(1)}, \cdots, t_n^{(1)}) \vee \cdots \vee \psi(t_1^{(k)}, \cdots, t_n^{(k)})$$

is valid.

REMARKS. (1) For $A \neq HF$, $\varphi \in L_A$, and T the set of closed terms of L_A imply that $\check{\varphi} \in L_A^{\vee}$ and $\check{\varphi}^{(T)} \in L_A^{\vee}$. Thus if we generalize "finite" to "A-finite" then 1.8 is a generalization to L_A, $A \neq HF$, of 1.9.

(2) It happens that the counterexample we shall give to the midsequent theorem in § 3 provides a valid formula $\exists x \forall y \psi(x, y)$ of $L_{\omega_1,\omega}$, ψ quantifier-free such that no finite disjunction $\bigvee\!\!\!\bigvee_{i \leqq k} \psi(t_i, f(t_i))$ is valid. Hence there is no uniformity theorem for $L_{\omega_1,\omega}$ in which the meaning of "finite" is kept fixed.

(3) In the finitary case we can always assume that a formula is in prenex normal form, so that the disjunctions over terms in the Herbrand form are outermost in the formula. This is not possible for $L_{\omega_1,\omega}$, as the following example shows. Let

$$\varphi = \bigwedge\!\!\!\bigwedge_{n<\omega} \exists y_n \forall z_n (r_n(y_n) \vee \sim r_n(z_n))$$

$$\check{\varphi} = \bigwedge\!\!\!\bigwedge_{n<\omega} \exists y_n (r_n(y_n) \vee \sim r_n(f_n(y_n))) .$$

φ and $\check{\varphi}$ are valid. Assume we have one constant symbol c and let T be the closure of $\{c, f_0, f_1, \cdots\}$. Then

$$\check{\varphi}^{(T)} = \bigwedge\!\!\!\bigwedge_{n<\omega} \bigvee\!\!\!\bigvee_{t \in T} (r_n(t) \vee \sim r_n(f_n(t)))$$

is valid. The uncountable disjunction

$$\bigvee\!\!\!\bigvee_{\langle t_0, t_1, \cdots \rangle \in T^{\omega}} \bigwedge\!\!\!\bigwedge_{n<\omega} (r_n(t_n) \vee \sim r_n(f_n(t_n)))$$

is also valid, but an easy check shows that no disjunction over a countable subset of T^{ω} is valid.

§ 2. Herbrand uniformity theorems: syntactic proof.

In this section we show how to obtain from a derivation $\mathscr{D}\vdash\varphi$ a derivation $\mathscr{D}'\vdash\check{\varphi}^{(T)}$ where $\check{\varphi}^{(T)}$ is an Herbrand form for some set of terms T of $L^{\vee}$. $\mathscr{D}'$ is obtained quite simply from $\mathscr{D}$, essentially by recursion on $\prec_{\mathscr{D}}$, the ordering of subderivations of $\mathscr{D}$. Moreover, the length of $\mathscr{D}'$ is equal to or less than the length of $\mathscr{D}$. However, we have not analyzed in explicit functional terms the construction of $\mathscr{D}'$ from $\mathscr{D}$.

Derivations are considered to be in a complete calculus with Gentzen-style rules of inference. We follow Feferman [**4**] for details, except that we omit the rules for negation. As axioms we shall take all sequents $\varphi\supset\varphi$; $\varphi, \sim\varphi\supset$; $\supset\varphi, \sim\varphi$ for φ atomic. We need all these axioms because we have omitted the rules for negation. If we included negation rules we would need as axioms only sequents $\varphi\supset\varphi$ for φ atomic as in Feferman [**4**]. If one wished to include negation rules, the definition of $\check{\varphi}$ would have to be extended to formulas not in reduced form.

2.1. DEFINITION. The length, $\mathrm{od}(\mathscr{D})$, of a derivation $\mathscr{D}$ is defined inductively to be

$$\sup\{\mathrm{od}(\mathscr{D}')+1 : \mathscr{D}' \text{ is a proper subderivation of } \mathscr{D}\}.$$

The terminology in the following definition is from Takeuti [**13**].

2.2. DEFINITION. Let $\dfrac{\Gamma_h\supset\Delta_h (h\in H)}{\Gamma\supset\Delta}$ be an instance of a rule (R). The *immediate successors* of an occurrence of a formula ψ_h in $\Gamma_h(\Delta_h)$ are defined as follows:

(i) If the rule is a structural rule (S) $\dfrac{\Gamma\supset\Delta}{\Gamma'\supset\Delta'}$, the immediate successors of ψ in $\Gamma(\Delta)$ are all occurrences of ψ in $\Gamma'(\Delta')$.

(ii) If the rule is (C) and ψ_h is the cut-formula (in Δ, Γ'), ψ_h has no immediate successor. The immediate successors of other formulas in $\Gamma\supset\Delta$, $\Gamma'\supset\Delta'$ are the same formulas in Γ, $\Gamma'-\varphi\supset\Delta-\varphi$, Δ' at the corresponding occurrences.

(iii) If ψ_h is the *active* formula in $\Gamma_h\supset\Delta_h$, its immediate successor is the active formula in $\Gamma\supset\Delta$.

(iv) If ψ_h is a *passive* formula in $\Gamma_h\supset\Delta_h$ its immediate successor is the same formula in $\Gamma\supset\Delta$ at the corresponding occurrence.

A *fiber* in a derivation $\mathscr{D}$ is a maximal chain of formulas $\psi_1, \cdots, \psi_n$ s.t. each ψ_{k+1} is an *immediate successor* of ψ_k. An occurrence of a formula φ in $\mathscr{D}$ is a *successor* of an occurrence of ψ if φ and ψ are in the same fiber and φ comes after ψ in the chain $\psi_1, \cdots, \psi_n$.

It is clear that every fiber begins with a formula which is either in a beginning sequent (an axiom) or is introduced by weakening, and ends either with a cut-formula or a formula in the end-sequent.

2.3. DEFINITION. A free variable w in a derivation $\mathcal{D}$ occurs as an *eigenvariable* of $\mathcal{D}$ if

$$\frac{\Gamma \supset \Delta,\ \phi(w)}{\Gamma \supset \Delta,\ \forall y\phi(y)}(\supset\forall) \quad \text{or} \quad \frac{\Gamma,\ \phi(w) \supset \Delta}{\Gamma,\ \exists y\phi(y) \supset \Delta}(\exists\supset)$$

occurs in $\mathcal{D}$.

We shall assume throughout this section and the next that eigenvariables of a derivation $\mathcal{D}$ have unique occurrences as variables of instances of $(\supset\forall)$, $(\exists\supset)$. Moreover, an eigenvariable occurs only in sequents which precede the sequent where it occurs as an eigenvariable. This is done without loss of generality.

2.4. LEMMA. *With any derivation $\mathcal{D}$ of a formula φ we can associate a derivation $\mathcal{D}_1$ of $\check{\varphi}$ s.t.* $\mathrm{od}(\mathcal{D}_1) \leqq \mathrm{od}(\mathcal{D})$. *Moreover, if $\mathcal{D}$ is cut-free, so is $\mathcal{D}_1$.*

PROOF. Let $\Gamma \supset \Delta$ be a sequent in $\mathcal{D}$ and let

$$\check{\Delta} = \{\check{\phi} : \phi \in \Delta \text{ and } \varphi \text{ is a successor of } \phi\} \cup$$
$$\{\phi : \phi \in \Delta \text{ and } \varphi \text{ is not a successor of } \phi\}$$

$\check{\Delta}$ is well-defined. For suppose φ is a successor of ϕ. We have a chain $\phi = \phi_1, \cdots, \phi_n = \varphi$, where ϕ_k is a substitution instance of a subformula of φ and ϕ_{k+1} is an immediate successor of ϕ_k. The choice of new function symbols for $\check{\varphi}$ thus gives a well-determined choice for $\check{\phi}$, $\check{\phi}_k$, so that $\check{\phi}$, $\check{\phi}_k$ are the substitution instances of subformulas of $\check{\varphi}$ corresponding to the substitution instances ϕ, ϕ_k. We have no rules for negation, so no formula in Γ is a predecessor of φ. (If we were deriving $\varphi_1 \supset \varphi_2$, we should consider $\hat{\varphi}_1 \supset \check{\varphi}_2$ and define $\hat{\Gamma}$ dually to $\check{\Delta}$.)

By induction on the subderivation relation $\prec_{\mathcal{D}}$, we show that for any $\mathcal{D}' \prec \mathcal{D}$ we get a derivation

$$\begin{array}{c}\downarrow \mathcal{D}_1' \\ \Gamma \supset \check{\Delta}\end{array} \quad \text{from} \quad \begin{array}{c}\downarrow \mathcal{D}', \text{ s.t.} \\ \Gamma \supset \Delta\end{array}$$

$\mathrm{od}(\mathcal{D}_1') \leqq \mathrm{od}(\mathcal{D}')$ and $\mathcal{D}_1'$ is cut-free if $\mathcal{D}'$ is cut-free. The only non-trivial step is for instances of $(\supset\forall)$,

$$\left.\begin{array}{c}\downarrow \mathcal{D}'' \\ \dfrac{\Gamma \supset \Delta,\ \phi(w)}{\Gamma \supset \Delta,\ \forall y\phi(y)}\end{array}\right\}\mathcal{D}'$$

where φ is a successor of $\forall y\phi(y)$. $\forall y\phi(y)$ is a substitution instance of a subformula of φ and we take

$$\forall y\widecheck{\phi(y)} = \check{\phi}(t_1, \cdots, t_n, f(t_1, \cdots, t_n))$$

to be the induced substitution instance of the corresponding subformula of $\check{\varphi}$. By inductive hypothesis we have

$$\begin{array}{c}\downarrow \mathcal{D}_1''(w) \\ \Gamma \supset \check{\Delta},\ \check{\psi}(t_1, \cdots, t_n, w)\,.\end{array}$$

But as w does not occur free in Γ, $\check{\Delta}$, we have

$$\begin{array}{c}\downarrow \mathcal{D}_1''(f(t_1, \cdots, t_n)) \\ \Gamma \supset \check{\Delta},\ \check{\psi}(t_1, \cdots, t_n, f(t_1, \cdots, t_n))\,.\end{array}$$

Then $\mathcal{D}_1' = \mathcal{D}_1''(f(t_1, \cdots, t_n))$ is the desired derivation. It is obvious that od$(\mathcal{D}_1')$ $\leqq$ od$(\mathcal{D}')$ and that no cuts are introduced into $\mathcal{D}_1'$.

In case we begin with a cut-free derivation $\mathcal{D}$ of φ it is very easy to obtain a derivation of an Herbrand form $\check{\varphi}^{(T)}$ of φ; this is done in 2.5 and 2.6.

2.5. LEMMA. *Let $\mathcal{D}$ be a cut-free derivation of φ, $\mathcal{D}_1$ a (cut-free) derivation of $\check{\varphi}$ obtained as in 2.4. Let $\mathcal{D}_2$ be obtained from $\mathcal{D}_1$ by replacing all free variables in $\mathcal{D}_1$ by a constant symbol c. Then $\mathcal{D}_2 \vdash \check{\varphi}$.*

PROOF. There are no instances of $(\supset \forall)$ in $\mathcal{D}_1$ and no left-hand rules at all since $\Gamma = 0$ for all sequents $\Gamma \supset \Delta$ in $\mathcal{D}$. Thus all instances of the rules remain valid.

2.6. THEOREM. *Let $\mathcal{D}$ be a cut-free derivation of φ, $\mathcal{D}_2$ a cut-free derivation of $\check{\varphi}$ obtained from $\mathcal{D}$ as in 2.5. Let T' be the set of closed terms of $\mathcal{D}_2$. Then for any $T \supseteqq T'$, we can obtain a cut-free derivation $\mathcal{D}'$ of $\check{\varphi}^{(T)}$, such that* od$(\mathcal{D}') \leqq$ od$(\mathcal{D})$.

PROOF. Let $\Delta_T = \{\psi^{(T)} : \psi \in \Delta\}$ for each sequent $\supset \Delta$ in $\mathcal{D}_2$. Let $\mathcal{D}'$ be obtained from $\mathcal{D}_2$ by replacing $\supset \Delta$ in $\mathcal{D}_2$ by $\supset \Delta_T$. Then $\mathcal{D}' \vdash \check{\varphi}^{(T)}$. This holds because

$$\frac{\Delta,\ \psi(t)}{\Delta,\ \exists y \psi(y)}(\supset \exists) \text{ goes into } \frac{\Delta_T,\ \psi^{(T)}(t)}{\Delta_T,\ \mathbb{W}_{s \in T}\, \psi^{(T)}(s)}(\supset \mathbb{W})$$

which is valid because $t \in T$.

It is clear that $\mathcal{D}'$ is cut-free and that od$(\mathcal{D}') \leqq$ od$(\mathcal{D}_2)$, so od$(\mathcal{D}') \leqq$ od$(\mathcal{D})$.

REMARK. If $\mathcal{D}$ is A-finite, so are $\mathcal{D}_1$, $\mathcal{D}_2$, T', $\mathcal{D}'$. In particular the finitary Herbrand theorem is a special case of 2.6 (assuming the cut-elimination theorem).

We now turn to derivations with cut. From a derivation $\mathcal{D}$ of φ we can pass by cut-elimination to a cut-free derivation $\mathcal{D}'$ of φ and then proceed to obtain a $\mathcal{D}'' \vdash \check{\varphi}^{(T)}$ as in 2.6. It will be shown in 2.8 that such a $\mathcal{D}''$ can be obtained directly from $\mathcal{D}$ without using cut-elimination. Of course $\mathcal{D}''$ may contain cuts.

2.7. DEFINITION. Let $\mathcal{D}'$ be a derivation of a sequent $\Gamma \supset \Delta$. A set of terms T is *closed for* $\mathscr{D}'$ if

(1) T contains the terms in $\Gamma \supset \Delta$,

(2) whenever $f(\bar{v}) \in T$, where $\bar{v} = \langle v_1, \cdots, v_n \rangle$, and $\bar{t}$ is a sequence of terms in T, then $f(\bar{t}) \in T$, and

(3) T contains no terms with occurrences of eigenvariables of $\mathscr{D}'$.

2.8. THEOREM. *Let* $\mathscr{D}$ *be a derivation of* φ, $\mathscr{D}_1$ *a derivation of* $\check{\varphi}$ *obtained as in* 2.4. *From* $\mathscr{D}_1$ *we can obtain directly a derivation* $\mathscr{D}' \vdash \check{\varphi}^{(T)}$ *for some set of terms* T. *Moreover,* $\mathrm{od}(\mathscr{D}') \leqq \mathrm{od}(\mathscr{D})$.

PROOF. For any set of terms T and sequents $\Gamma \supset \Delta$, let Γ_T, Δ_T be the sequences of formulas $\phi^{(T)}$ corresponding to Γ, Δ. We show by induction on $\prec_{\mathscr{D}_1}$:

Suppose we have a derivation $\overset{\downarrow \mathscr{D}'}{\Gamma \supset \Delta}$. There is a set of terms T' *closed* for $\mathscr{D}'$ s.t. for all *closed* $T \supseteqq T'$ there is a derivation

$$\begin{array}{c}\downarrow \mathscr{D}'_T\\ \Gamma_T \supset \Delta_T\,.\end{array}$$

We shall assume that T' is minimal for $\mathscr{D}'$, in particular no variables should occur in T' which do not occur in $\mathscr{D}'$.

The non-trivial rules to consider are $(\supset \forall)$, $(\exists \supset)$, and (C).

(a) Suppose the last rule is $(\supset \forall)$,

$$\text{(i)}\qquad \left.\begin{array}{c}\downarrow \mathscr{D}''(w)\\ \underline{\Gamma \supset \Delta',\ \phi(w)}\\ \Gamma \supset \Delta',\ \forall y \phi(y)\end{array}\right\}\mathscr{D}'\,.$$

If $T(w)$ is a set of terms with occurrences of the free variable w, and t is a term without occurrences of w, we let $T(t) = \{s(t) : s(w) \in T(w)\}$. By inductive hypothesis we have a set $T''(w)$ closed for $\mathscr{D}''(w)$, such that for all closed $T(w) \supseteqq T''(w)$, there is a derivation

$$\text{(ii)}\qquad \begin{array}{c}\downarrow \mathscr{D}''_{T(w)}(w)\\ \Gamma_{T(w)} \supset \Delta'_{T(w)},\ \phi^{(T(w))}(w)\,.\end{array}$$

But now, if $T(w)$ is closed for $\mathscr{D}''(w)$ and $t, s \in T(w)$ are such that w does not occur in t, s, then $T(s) = T(t)$. Let $T = T(t)$ for any such term t. Then also $T = T(t)$ for any $t \in T$. Now let $T' = T''(t)$ for $t \in T''(w)$, such that w does not occur in t. T' is closed for $\mathscr{D}'$, since w does not occur in the terms of T'. Moreover, any $T \supseteqq T'$ such that T is closed for $\mathscr{D}'$ extends to $T(w) \supseteqq T''(w)$ closed for $\mathscr{D}''(w)$ in such a way that $T = T(t)$ for any $t \in T$. Thus, for any $T \supseteqq T'$, such that T is closed for $\mathscr{D}'$, we obtain from (ii),

$$\text{(iii)} \qquad \begin{array}{c} \downarrow \mathscr{D}''_T(t) \\ \Gamma_T \supset \Delta'_T,\ \phi^{(T)}(t) \end{array} \qquad \text{for all } t \in T$$

and hence

$$\left.\begin{array}{c} \downarrow \mathscr{D}''_T(t) \\ \Gamma_T \supset \Delta'_T,\ \phi^{(T)}(t) \quad \text{for all } t \in T \\ \hline \Gamma_T \supset \Delta'_T,\ \bigwedge\!\!\!\bigwedge_{t \in T} \phi^{(T)}(t) \end{array} (\supset \bigwedge\!\!\!\bigwedge) \right\} \mathscr{D}'_T .$$

The argument for $(\exists \supset)$ is dual to this.

(b) Suppose the last rule is (C),

$$\left.\begin{array}{cc} \downarrow \mathscr{D}_1 & \downarrow \mathscr{D}_2 \\ \Gamma \supset \Delta & \Gamma' \supset \Delta' \\ \hline \multicolumn{2}{c}{\Gamma, \Gamma' - \varphi \supset \Delta - \varphi, \Delta'} \end{array} (C) \right\} \mathscr{D}' .$$

Let T_1, T_2 be the minimal closed sets for $\mathscr{D}_1$, $\mathscr{D}_2$ given by the inductive assumption. Let T' be the closure of $T_1 \cup T_2$ under (2) of 2.7. Then T' is closed for $\mathscr{D}'$, because it certainly contains the terms of Γ, $\Gamma' - \varphi$, $\Delta - \varphi$, Δ' and contains only variables from T_1, T_2. (T_1, T_2 by minimality contain only variables from $\mathscr{D}_1$, $\mathscr{D}_2$ respectively.)

Moreover, any $T \supseteqq T'$ closed for $\mathscr{D}'$ is also closed for $\mathscr{D}_1$, $\mathscr{D}_2$. Thus for each such T, we have by inductive assumption

$$\left.\begin{array}{cc} \downarrow \mathscr{D}_{1T} & \downarrow \mathscr{D}_{2T} \\ \Gamma_T \supset \Delta_T & \Gamma'_T \supset \Delta'_T \\ \hline \multicolumn{2}{c}{\Gamma_T, \Gamma'_T - \varphi_T \supset \Delta_T - \varphi_T, \Delta'_T} \end{array} (C) \right\} \mathscr{D}'_T$$

which gives the desired result for cut.

REMARKS. (1) In the derivations without cut, we could be sure that the disjunctions in the derived Herbrand form $\breve{\varphi}^{(T)}$ were over a subset T of the closure of the terms of $\breve{\varphi}$. The disjunctions in derivations with cut may well be over extraneous terms introduced by cut-formulas.

(2) A finite derivation $\mathscr{D} \vdash \varphi$ with cut may well go into an infinite derivation $\mathscr{D}' \vdash \breve{\varphi}^{(T)}$ because the branching becomes infinite in the substitution for instances of $(\supset \forall)$, $(\exists \supset)$. Thus the finitary Herbrand theorem does not follow from 2.8.

(3) For $A \neq HF$, the set of terms T' in the inductive hypothesis can be kept A-finite and so for $\varphi \in L_A$, $A \neq HF$, $\mathscr{D} \vdash \varphi$, $\mathscr{D} \in \mathrm{Der}_A$, we get $\mathscr{D}' \vdash \breve{\varphi}^{(T)}$ where $\mathscr{D}'$, $\breve{\varphi}^{(T)}$ are A-finite.

§ 3. The Midsequent Theorem.

In this section, all derivations are assumed to be cut-free.

3.1. DEFINITION. $\mathscr{D}$ is a derivation in *normal form* if every propositional inference in $\mathscr{D}$ precedes every quantifier inference. The *midsequent* of a derivation in normal form is the lowest sequent prior to the quantifier inferences.

A proof of the following theorem may be found in Kleene [**6**] or Smullyan [**10**].

3.2. MIDSEQUENT THEOREM FOR $L_{\omega,\omega}$. *If $\Gamma \supset \Delta$ consists of prenex formulas of $L_{\omega,\omega}$ and $\Gamma \supset \Delta$ has a derivation, then $\Gamma \supset \Delta$ has a derivation in normal form.*

Suppose we have a normal form derivation of a prenex formula φ. As in section 2 we easily obtain a normal form derivation of $\breve{\varphi}$. The disjunction of the formulas in the midsequent of the second derivation is then a derivable Herbrand disjunct for φ. Thus the Herbrand theorem for finitary logic follows easily from 3.2.

3.3. EXAMPLE. The following is a variant of an example in Kleene [**6**], p. 343. It will be useful as an illustration of some aspects of the theorems below. κ is a constant symbol, α a function symbol, a, b, c, d, e free variables.

$\downarrow$ propositional rules

$$\phi(\alpha(b), d, \kappa, e),\ \phi(a, b, b, c) \qquad (\supset \forall)$$

$$\forall z\phi(\alpha(b), d, \kappa, z),\ \forall z\phi(a, b, b, z) \qquad (\supset \exists)$$

$$\exists y\forall z\phi(\alpha(b), d, y, z),\ \exists y\forall z\phi(a, b, y, z) \qquad (\supset \forall)$$

$$\forall x\exists y\forall z\phi(\alpha(b), x, y, z),\ \exists y\forall z\phi(a, b, y, z) \qquad (\supset \exists)$$

$$\exists w\forall x\exists y\forall z\phi(w, x, y, z),\ \exists y\forall z\phi(a, b, y, z) \qquad (\supset \forall)$$

$$\exists w\forall x\exists y\forall x\phi(w, x, y, z),\ \forall x\exists y\forall z\phi(a, x, y, z) \qquad (\supset \exists)$$

$$\exists w\forall x\exists y\forall z\phi(w, x, y, z),\ \exists w\forall x\exists y\forall z\phi(w, x, y, z) \qquad (S)$$

$$\exists w\forall x\exists y\forall z\phi(w, x, y, z)$$

The midsequent is $\{\phi(\alpha(b), d, \kappa, e),\ \phi(a, b, b, c)\}$. Let f, g, be the new function symbols of $L^{\vee}$. By 2.4 we obtain

$$\phi(\alpha f(a), f\alpha f(a), \kappa, g(\alpha f(a), f\alpha f(a), \kappa)),\ \phi(a, f(a), f(a), g(a, f(a), f(a))) \qquad (\supset \exists)$$

$$\exists y\phi(\alpha f(a), f\alpha f(a), y, g(\alpha f(a), f\alpha f(a), y)),\ \exists y\phi(a, f(a), y, g(a, f(a), y)) \qquad (\supset \exists)$$

$$\exists w\exists y\phi(w, f(w), y, g(w, f(w), y)),\ \exists y\phi(a, f(a), y, g(a, f(a), y)) \qquad (\supset \exists)$$

$$\exists w\exists y\phi(w, f(w), y, g(w, f(w), y)),\ \exists w\exists y\phi(w, f(w), y, g(w, f(w), y)) \qquad (S)$$

$$\exists w\exists y\phi(w, f(w), y, g(w, f(w), y))$$

The disjunction of the midsequent is then a derivable Herbrand form for $\exists w\forall x\exists y\forall z\phi(w, x, y, z)$.

REMARK. Note that we allow applications of $(\supset\forall)$, $(\supset\exists)$ to several formulas at a time. For finite sequents this modification is obviously inessential. However, we shall also consider derivations with infinite sequents below.

3.4. DEFINITION. Given a derivation $\mathcal{D}$ and an eigenvariable v of $\mathcal{D}$, the *height* of v is

$h(v)=1+\max\{h(w): w$ is an eigenvariable distinct from v and w occurs free in the formula where v is used as an eigenvariable$\}$.

It is clear that if $\mathcal{D}$ is a derivation of a sentence φ, $h(v)$ is equal to or less than the number of instances of the rule $(\supset\forall)$ below the sequent in which v is used as an eigenvariable.

Let $\varphi=\exists x_1\forall y_1\exists x_2\forall y_2\cdots\exists x_n\forall y_n\phi(x_1,\cdots,x_n,y_1,\cdots,y_n)$ be a *fixed prenex sentence* of $L_{\omega_1,\omega}$. We shall write $\phi=\phi(\bar{x},\bar{y})$, $\breve{\varphi}=\exists\bar{x}\phi(\bar{x};\bar{f})$, $\bar{f}=\langle f_1,\cdots,f_n\rangle$.

3.5. DEFINITION. The *$\bar{f}$-rank*, $\rho_{\bar{f}}$, of a term t of the language of $\breve{\varphi}$ is defined inductively by:

(i) $\rho_{\bar{f}}(t)=0$ if $f_1,\cdots,f_n$ do not occur in t

(ii) $\rho_{\bar{f}}(g(t_1,\cdots,t_k))=\begin{cases}\sup\{\rho_{\bar{f}}(t_i): i=1,\cdots,k\}+1 & \text{if } g \text{ is one of } f_1,\cdots,f_n\\ \sup\{\rho_{\bar{f}}(t_i): i=1,\cdots,k\} & \text{otherwise}.\end{cases}$

A set of terms T is of *(uniformly) bounded $\bar{f}$-rank* if there exists a $k<\omega$ such that for every $t\in T$, $\rho_{\bar{f}}(t)\leqq k$.

In example 3.3, $h(b)=1$, $h(d)=2$, $h(e)=3$, $h(c)=2$. The terms corresponding to b, d, e, c, in the induced derivation of $\breve{\varphi}$ (in example 3.3) are $f(a)$, $f\alpha f(a)$, $g(\alpha f(a), f\alpha f(a), \kappa)$, $g(a, f(a), f(a))$ and their $\langle f, g\rangle$-ranks are exactly the heights of the corresponding eigenvariables. This is easily seen to be true in general. Since the heights of eigenvariables in a normal derivation are bounded by the number of instances of $(\supset\forall)$ below the midsequent, the Herbrand disjunction $\breve{\varphi}^{(T)}$ obtained from a normal derivation of our fixed prenex formula φ would be over a set of terms T of uniformly bounded $\bar{f}$-rank.

The above analysis of the relationship between heights of eigenvariables in a derivation and $\bar{f}$-ranks of terms depends only on the quantifier-rules. In particular, it holds as well for derivations with countably infinite sequents; e.g.,

$$\frac{\Gamma\supset\Delta\cup\{\phi_n(t_n): n<\omega\}}{\Gamma\supset\Delta\cup\{\exists x_n\phi_n(x_n): n<\omega\}}\ (\supset\exists)$$

and

$$\frac{\Gamma \supset \Delta \cup \{\phi_n(a_n) : n < \omega\}}{\Gamma \supset \Delta \cup \{\forall x_n \phi_n(x_n) : n < \omega\}} \ (\supset \forall)$$

when the a_n are distinct and not free in any formula of the conclusion.

The following will be proved for derivations in which infinite sequents are permitted.

3.6. THEOREM. *φ has a derivation in normal form $\Leftrightarrow$ there is a derivation of $\breve{\varphi}^{(T)}$ where T has bounded $\bar{f}$-rank.*

We shall also show

3.7. THEOREM. *There is a valid $L_{\omega_1,\omega}$ formula $\varphi = \exists x \forall y \phi(x, y)$, ϕ quantifier-free, such that for any collection T of terms of bounded f-rank, $\bigvee\!\!\!\bigvee_{t \in T} \phi(t, f(t))$ is not valid.*

3.8. COROLLARY. *There is a prenex formula φ of $L_{\omega_1,\omega}$ with a finite-sequent derivation such that φ has no derivation in normal form.*

REMARK. Derivations with infinite sequents are needed in 3.6 for the "if" direction. However, 3.8 follows from 3.7 and the "only if" part of 3.6.

PROOF OF 3.6. ($\Rightarrow$) This follows from the fact that the $\bar{f}$-ranks of the terms in the induced derivation of $\breve{\varphi}$ are bounded by the number of instances of ($\supset \forall$) (to possibly infinitely many formulas at a time) below the midsequent.

($\Leftarrow$) This is only sketched. Note that for derivations without cut we need only worry about the right-hand part of sequents. Let $\breve{\varphi}^{(T)} = \bigvee\!\!\!\bigvee_{t_1, \cdots, t_n \in T} \phi(\bar{t} ; \bar{f})$. Let $\langle t_n \rangle_{n<\omega}$ be a 1—1 enumeration of $T \cup \{f_i(\bar{t}) : \bar{t} \in T^{k_i}, 1 \leq i \leq n\}$, where f_i is a k_i-ary function symbol. Take $\langle w_n \rangle_{n<\omega}$ to be a 1—1 enumeration of new variables. If $\rho_{\bar{f}}(t_n) = l$, we shall say that w_n is of *height* l.

We are given

$$\begin{array}{c} \downarrow \mathscr{D} \\ \{\phi(\bar{t}^{(i)} ; \bar{f})\}_{i<\omega} \end{array}$$

where $\bar{t}^{(i)} \in T^n$. Let $\phi(\bar{x}^{(i)}, \bar{y}^{(i)})$ be the formula corresponding to $\phi(\bar{t}^{(i)} ; \bar{f})$ under the substitution of w_n for t_n. If we substitute w_n for t_n throughout $\mathscr{D}$ (the substitution is for maximal occurrences of t_n) we get a derivation

$$\begin{array}{c} \downarrow \mathscr{D}' \\ \{\phi(\bar{x}^{(i)}, \bar{y}^{(i)})\}_{i<\omega} . \end{array}$$

We now show how to obtain a derivation $\mathscr{D}''$ of φ from $\mathscr{D}'$ in a series of quantifier steps.

We shall say that a free variable in $\phi(\bar{x}^{(i)}, \bar{y}^{(i)})$ *precedes* another free variable in ϕ if it has to be quantified out first in the passage from ϕ to φ. For example, if $\varphi = \exists x \forall y \phi(x, y)$, then w_2 precedes w_1 in $\phi(w_1, w_2)$. A free variable occurs in *maximal position* in a formula if no free variable precedes it in that formula. $w_2(w_1)$ is in maximal position in $\phi(w_1, w_2)$ $(\forall y\phi(w_1, y))$. An occurrence of a free variable in the range of $\bar{y}^{(i)}$ in a formula $\phi(\bar{x}^{(i)}, \bar{y}^{(i)})$ is called an *eigenvariable occurrence.*

The following passage from $\mathcal{D}'$ to $\mathcal{D}'' \vdash \varphi$ is illustrated in example 3.3.

(1) Apply $(\supset\exists)$ to quantify out existentially the free variables of $\{\phi(\bar{x}^{(i)}, \bar{y}^{(i)})\}_{i<\omega}$ which do not occur as eigenvariables and are in maximal positions. Repeat a finite number of times until the only variables in maximal positions are eigenvariables.

(2) Now look at the free variables of greatest height. They are in maximal positions and they occur as eigenvariables, for otherwise they would be preceded by eigenvariables of greater height. The $1-1$ enumeration $\langle t_n\rangle_{n<\omega}$ insures that these variables occur in unique formulas. Thus we may use $(\supset\forall)$ to quantify them out universally. This reduces the highest rank of variables by one.

A finite number of applications of (1) and (2) leads to the desired derivation $\mathcal{D}''$ of φ.

PROOF OF 3.7. We take a language with one constant symbol c and two sequences of unary predicate symbols r_n, $n>0$, and s_m, $m \geqq 0$. Let $\sigma(k) = \sum_{i=0}^{k} i$. For any $k=0, 1, 2, \cdots$ and m with $\sigma(k) < m \leqq \sigma(k+1)$ take

$$\phi_m(x, y) = (r_m(x) \vee \sim r_m(y)) \vee (\bigvee_{i<\sigma(k)} s_i(x)) \vee \bigvee_{\sigma(k)\leqq i<m} \sim s_i(y) \vee s_m(x).$$

For example, for $k=2$, $\sigma(k)=3$, $\sigma(k+1)=6$ and ϕ_4, ϕ_5, ϕ_6 have the form

$$\phi_4(x, y) = (r_4(x) \vee \sim r_4(y)) \vee (s_0(x) \vee s_1(x) \vee s_2(x)) \vee \sim s_3(y) \vee s_4(x)$$

$$\phi_5(x, y) = (r_5(x) \vee \sim r_5(y)) \vee (s_0(x) \vee s_1(x) \vee s_2(x)) \vee \sim s_3(y) \vee \sim s_4(y) \vee s_5(x)$$

$$\phi_6(x, y) = (r_6(x) \vee \sim r_6(y)) \vee (s_0(x) \vee s_1(x) \vee s_2(x)) \vee \sim s_3(y) \vee \sim s_4(y) \vee \sim s_5(y) \vee s_6(x).$$

We take $\phi(x, y) = \bigwedge_{m>0} \phi_m(x, y)$.

The following facts (i)—(iv) concerning the ϕ_m are easily checked:

(i) $\phi_m(x, y) \vee \phi_m(y, z)$ is valid for any m.

(ii) $\phi_{m_1}(x, y) \vee \phi_{m_2}(y, z)$ is valid for $\sigma(k) < m_1 \leqq \sigma(k+1) < m_2$.

(iii) $\phi_{m_1}(x, y) \vee \phi_{m_2}(y, z)$ is valid for $\sigma(k) < m_2 < m_1 \leqq \sigma(k+1)$.

(iv) $\bigvee_{n\leqq k} \phi_{\sigma(k)+n+1}(f^n(c), f^{n+1}(c))$ is not valid for any k.

We use (i)—(iv) to prove:

(1) $\exists x\phi(x, f(x))$ is valid (so $\exists x\forall y\phi(x, y)$ is valid).

(2) $\bigvee_{t\in T} \phi(t, f(t))$ is not valid for any subset T of $\{f^n(c)\}_{n<\omega}$ of bounded f-rank.

PROOF OF (1). It suffices to show that $\bigvee_{n<\omega} \phi(f^n(c), f^{n+1}(c))$ is valid; i.e., that $\bigvee_{n<\omega} \bigwedge_{0<m<\omega} \psi_m(f^n(c), f^{n+1}(c))$ is valid. This is equivalent to the validity of

$$(*) \qquad \bigvee_{n<\omega} \psi_{\pi(n)}(f^n(c), f^{n+1}(c))$$

for each $\pi : \omega \to \omega - \{0\}$. Consider any such π. If there is an n with $\pi(n) = \pi(n+1)$, the validity of (*) follows from (i). Suppose for all n, $\pi(n) \neq \pi(n+1)$. Let k be the least natural number for which there is some n with $\pi(n) \leqq \sigma(k+1)$. Since $\sigma(0)=0$, we have $\sigma(k) < \pi(n)$, whenever $\pi(n) \leqq \sigma(k+1)$. If there is an n with $\pi(n) \leqq \sigma(k+1) < \pi(n+1)$ we can apply (ii). Otherwise, whenever $\pi(n) \leqq \sigma(k+1)$, also $\pi(n+1) \leqq \sigma(k+1)$. Hence for some n, $\sigma(k) < \pi(n+1) < \pi(n) \leqq \sigma(k+1)$; in this case (*) is valid by (iii).

PROOF OF (2). It suffices to show that for any k, $\bigvee_{n\leqq k} \bigwedge_{0<m<\omega} \psi_m(f^n(c), f^{n+1}(c))$ is not valid; this is equivalent to $\bigwedge_{\pi\in(\omega-\{0\})^{k+1}} \bigvee_{n\leqq k} \psi_{\pi(n)}(f^n(c), f^{n+1}(c))$. However, by (iv), $\bigvee_{n\leqq k} \psi_{\pi(n)}(f^n(c), f^{n+1}(c))$ is not valid for π given by $\pi(n) = \sigma(k)+n+1$. This completes the proof of 3.7.

REMARKS. (1) A slightly more complicated argument in (1) of 3.7 shows that for any $\pi : \omega \to \omega - \{0\}$, if $\pi(0) \leqq \sigma(k+1)$, then already

$$\bigvee_{n\leqq\sigma(k+1)} \psi_{\pi(n)}(f^n(c), f^{n+1}(c))$$

is valid. From this we can see that the formula $\bigwedge_{\pi\in(\omega-\{0\})^{(\omega)}} \bigvee_{n\leqq m(\pi)} \psi_{\pi(n)}(f^n(c), f^{n+1}(c))$ (where $m(\pi)$ is the least $\sigma(k+1)$ s.t. $\pi(0) \leqq \sigma(k+1)$) is valid. We can then reduce the conjunction over $(\omega-\{0\})^{(\omega)}$ to a countable conjunction. The bound is also helpful if we wish to construct a proof-tree for $\bigvee_{n<\omega} \bigwedge_{0<m<\omega} \psi_m(f^n(c), f^{n+1}(c))$.

(2) In the above example, a set of bounded f-rank is finite; however, a similar but notationally more complicated example can be given where such sets may be infinite.

(3) The above counterexample can be easily transcribed into one for a language with two binary relation symbols, one unary function symbol and one constant symbol.

§ 4. Takeuti's interpolation theorem for $L_{\kappa,\kappa}$.

Let κ be a strongly inaccessible cardinal. The formulas of $L_{\kappa,\kappa}$ are built up using negation, conjunction and disjunction of sets of formulas of cardinality less than κ and quantifiers $\exists\bar{x}$, $\forall\bar{x}$, where $\bar{x} = \langle x_\xi \rangle_{\xi<\lambda}$, $\lambda < \kappa$. The latter are called *homogeneous quantifiers* when the language is extended to $L^+_{\kappa,\kappa}$ in 4.4 below. The new function symbols of $L\check{}_{\kappa,\kappa}$ are to have arguments of cardinality less than κ and $\check{\varphi}$, $\hat{\varphi}$, $\varphi^{(T)}$ are defined as in § 1. 4.1, 4.2, 4.3 are proved by a direct extension of the arguments for $L_{\omega_1,\omega}$.

4.1. LEMMA. *Let φ be a sentence of $L_{\kappa,\kappa}$. For any $L_{\kappa,\kappa}$-structure $\mathfrak{A}$, we have*

(i) $\mathfrak{A} \models \varphi \Leftrightarrow$ *for all $\check{L}_{\kappa,\kappa}$-extensions $\langle \mathfrak{A}, \bar{f} \rangle$, $\langle \mathfrak{A}, \bar{f} \rangle \models \check{\varphi}$.*

(ii) $\mathfrak{A} \models \varphi \Leftrightarrow$ *for some $\hat{L}_{\kappa,\kappa}$-extension $\langle \mathfrak{A}, \bar{f} \rangle$, $\langle \mathfrak{A}, \bar{f} \rangle \models \hat{\varphi}$.*

4.2. THEOREM. *Let φ be a sentence of $\check{L}_{\kappa,\kappa}$, T the set of all closed terms of $\check{L}_{\kappa,\kappa}$ (the language of $\check{\varphi}$). Then φ is valid if and only if $\check{\varphi}^{(T)}$ is valid.*

4.3. LEMMA (Interpolation for propositional $L_{\kappa,\kappa}$). *Let φ, ψ be quantifier-free formulas of $L_{\kappa,\kappa}$. If $\varphi \vee \psi$ is valid, then there is a quantifier-free interpolant σ s.t. $\varphi \vee \sigma$ and $\sim \sigma \vee \psi$ are valid.*

We now consider the extension of $L_{\kappa,\kappa}$ using (possibly) *inhomogeneous quantifiers* $Q^{\tau}\bar{x}$ as in [12].

4.4. DEFINITION. (a) $L^{+}_{\kappa,\kappa}$ is obtained by closing under the operations of $L_{\kappa,\kappa}$ and the operation of forming $Q^{\tau}\bar{x}\psi(\bar{x})$ whenever $\bar{x}$ is a sequence of distinct variables $\bar{x} = \langle x_{\alpha} \rangle_{\alpha<\lambda}$, $\lambda < \kappa$, $\tau : \lambda \to \{\forall, \exists\}$, and $\psi(\bar{x}) \in L^{+}_{\kappa,\kappa}$.

(b) The *dual* τ^* of τ is given by

$$\tau^*(\alpha) = \begin{cases} \forall & \text{if } \tau(\alpha) = \exists \\ \exists & \text{if } \tau(\alpha) = \forall\,. \end{cases}$$

The *reduced* form φ^* of a formula φ of $L_{\kappa,\kappa}$ is obtained by the usual de Morgan laws and by setting $(\sim Q^{\tau}\bar{x}\psi(\bar{x}))^* = Q^{\tau^*}\bar{x}(\sim \psi(\bar{x}))^*$.

(c) Given $Q^{\tau}\bar{x}\psi(\bar{x})$, let $\psi(\bar{y}\,;\bar{f})$ be obtained from $\psi(\bar{x})$ by replacing x_{α} where $\tau(\alpha) = \exists$ by (Skolem) function symbols f of the variables in $\bar{y}$ which precede x_{α}. A sequence s from a model $\mathfrak{A}$ *satisfies* $Q^{\tau}\bar{x}\psi(\bar{x})$ if for some extension $\langle \mathfrak{A}, \bar{f} \rangle$, s satisfies $\forall \bar{y}\psi(\bar{y}\,;\bar{f})$ in $\langle \mathfrak{A}, \bar{f} \rangle$.

4.5. THEOREM (Takeuti). *Let φ, ψ be in $L_{\kappa,\kappa}$. If $\varphi \vee \psi$ is valid, then there is an interpolant $\sigma \in L^{+}_{\kappa,\kappa}$ such that $\varphi \vee \sigma$ and $(\sim \sigma)^* \vee \psi$ are valid.*

PROOF. This follows the lines of the first proof of the finitary interpolation theorem in Kreisel-Krivine [8]. We shall assume to begin with that φ, ψ have no constant or function symbols. Since $\varphi \vee \psi$ is valid, we have by 4.1 that $\check{\varphi} \vee \check{\psi}$ is valid. We can pull the existential quantifiers to the front in $\check{\varphi}$ and $\check{\psi}$ and write

$$\check{\varphi} \equiv \exists \bar{x}\varphi_1(\bar{x}\,;\bar{f})\,, \qquad \check{\psi} \equiv \exists \bar{y}\psi_1(\bar{x}\,;\bar{g})$$

where $\bar{x}$, $\bar{y}$ are disjoint sequences of variables and $\bar{f}$, $\bar{g}$ disjoint sequences of function symbols. From $\models \check{\varphi} \vee \check{\psi}$ we get $\models \exists \bar{x} \exists \bar{y}(\varphi_1(\bar{x}\,;\bar{f}) \vee \psi_1(\bar{y}\,;\bar{g}))$ and by 4.2 we then obtain

$$\models \bigvee\!\!\!\bigvee_{\langle \bar{t}, \bar{s} \rangle} (\varphi_1(\bar{t}\,;\bar{f}) \vee \psi_1(\bar{s}\,;\bar{g}))$$

where $\bar{t}$, $\bar{s}$ range over sequences from the set T of all closed terms in the language of $\breve{\varphi} \vee \breve{\psi}$. It follows that

$$\models \bigvee\!\!\!\!\bigvee_{\bar{t}} \varphi_1(\bar{t}\,;\bar{f}) \vee \bigvee\!\!\!\!\bigvee_{\bar{s}} \psi_1(\bar{s}\,;\bar{g})\,.$$

We may now use the propositional interpolation theorem 4.3 to get a quantifier-free interpolant $\sigma = \sigma(\bar{r})$ where $\bar{r} = \langle r_\alpha \rangle_{\alpha<\lambda<\kappa}$ is a 1—1 enumeration of the terms in σ all of which belong to T. We can assume that these are arranged in such a way that $\alpha < \beta$ implies r_β is not a subterm of r_α. We then have

$$\models \bigvee\!\!\!\!\bigvee_{\bar{t}} \varphi_1(\bar{t}\,;\bar{f}) \vee \sigma(\bar{r})\,, \qquad \models \sim \sigma(\bar{r}) \vee \bigvee\!\!\!\!\bigvee_{\bar{s}} \psi_1(\bar{s}\,;\bar{g})\,.$$

Since $\models \bigvee\!\!\!\!\bigvee_{\bar{t}} \varphi_1(\bar{t}\,;\bar{f}) \rightarrow \exists \bar{x} \varphi_1(\bar{x}\,;\bar{f})$ and $\models \bigvee\!\!\!\!\bigvee_{\bar{s}} \psi_1(\bar{s}\,;\bar{g}) \rightarrow \exists \bar{y} \psi_1(\bar{y}\,;\bar{g})$, we get

$$\models \breve{\varphi} \vee \sigma(\bar{r})\,, \qquad \models \sim \sigma(\bar{r}) \vee \breve{\psi}\,.$$

Define $\tau : \lambda \rightarrow \{\forall, \exists\}$ so that

$$\tau(\alpha) = \begin{cases} \exists \text{ if the outermost function symbol in } r_\alpha \text{ is in the range of } \bar{f} \\ \forall \text{ if it is in the range of } \bar{g}\,. \end{cases}$$

Let τ^* be the dual of τ, $Q^\tau \bar{x}$, $Q^{\tau^*} \bar{x}$ the associated (possibly) inhomogeneous quantifiers. We claim that

$$\models \breve{\varphi} \vee Q^\tau \bar{x} \sigma(\bar{x}) \qquad \text{and} \qquad \models Q^{\tau^*} \bar{x} \sim \sigma(\bar{x}) \vee \breve{\psi}\,.$$

We prove $\models \breve{\varphi} \vee Q^\tau \bar{x} \sigma(\bar{x})$. Let $\bar{t}$ be the subsequence from $\bar{r}$ of terms with outermost function symbol in the range of $\bar{g}$. Let $\bar{y}$ be a 1—1 enumeration of new variables corresponding to $\bar{t}$ and let $\bar{r}' = \langle r'_\alpha \rangle_{\alpha<\lambda}$ be obtained from $\bar{r}$ by substituting y_α for t_α. (The substitution is to be in the sense that for given r_α, we look for the maximal subterms with outermost function symbol in the range of $\bar{g}$ and substitute the corresponding y for these subterms.) Then the terms in $\bar{r}'$ contain no occurrences of function symbols from $\bar{g}$. Write $\bar{r}' = \langle \bar{y}, \bar{s} \rangle$, $\bar{y}$ the sequence of new variables, $\bar{s}$ the terms with function symbols from $\bar{f}$. Suppose we had a model $\langle \mathfrak{A}, \bar{f} \rangle$ s. t.

$$\langle \mathfrak{A}, \bar{f} \rangle \models \sim \breve{\varphi} \wedge \sim Q^\tau \bar{x} \sigma(\bar{x})\,.$$

Then $\langle \mathfrak{A}, \bar{f} \rangle \models \sim \breve{\varphi} \wedge \sim \forall \bar{y} \sigma(\bar{y}\,;\bar{s})$. For we can write $\sigma(\bar{y}, \bar{s}) = \sigma(\bar{y}\,;\bar{f})$ and so $\langle \mathfrak{A}, \bar{f} \rangle \models \forall \bar{y} \sigma(\bar{y}, \bar{s})$ implies $\langle \mathfrak{A}, \bar{f} \rangle \models \forall \bar{y} \sigma(\bar{y}\,;\bar{f})$ and, by definition 4.4, $\mathfrak{A} \models Q^\tau \bar{x} \sigma(\bar{x})$. But now $\langle \mathfrak{A}, \bar{f} \rangle \models \sim \breve{\varphi} \wedge \exists \bar{y} \sim \sigma(\bar{y}, \bar{s})$ implies that for some sequence $\bar{a}$ from $|\mathfrak{A}|$,

$$\langle \mathfrak{A}, \bar{f} \rangle \models \sim \breve{\varphi} \wedge \sim \sigma(\bar{a}, \bar{s})\,.$$

Since the function symbols from $\bar{g}$ are disjoint from $\bar{f}$, we are free to interpret the terms $\bar{t}$ with outermost function symbol from $\bar{g}$ so

$$\langle \mathfrak{A}, \bar{f}, \bar{g} \rangle \models \sim \breve{\varphi} \wedge \sim \sigma(\bar{t}, \bar{s})\,.$$

But then $\langle \mathfrak{A}, \bar{f}, \bar{g} \rangle \models \sim \breve{\varphi} \wedge \sim \sigma(\bar{r})$, contradicting $\models \breve{\varphi} \vee \sigma(\bar{r})$. Similarly, $\models Q^{\tau*}\bar{x} \sim \sigma(\bar{x}) \vee \breve{\psi}$.

Finally, $\models \breve{\varphi} \vee Q^{\tau}\bar{x}\sigma(\bar{x})$ implies $\models \varphi \vee Q^{\tau}\bar{x}\sigma(\bar{x})$. For suppose

$$\mathfrak{A} \models \sim \varphi \wedge \sim Q^{\tau}\bar{x}\sigma(\bar{x}).$$

Then for some extension $\langle \mathfrak{A}, \bar{f} \rangle$,

$$\langle \mathfrak{A}, \bar{f} \rangle \models \widehat{(\sim \varphi)} \wedge \sim Q^{\tau}\bar{x}\sigma(\bar{x}).$$

But then $\langle \mathfrak{A}, \bar{f} \rangle \models \sim \breve{\varphi} \wedge \sim Q^{\tau}\bar{x}\sigma(\bar{x})$. Similarly, $\models Q^{\tau*}\bar{x} \sim \sigma(\bar{x}) \vee \breve{\psi}$ implies $\models Q^{\tau*}\bar{x} \sim \sigma(\bar{x}) \vee \psi$.

The result for the case that φ, ψ have constant and function symbols is obtained by the usual technique of reducing to relational φ_1, ψ_1. T_1, S_1 express the functionality of the new relation symbols in φ_1, ψ_1 respectively. Then $\models \varphi \vee \psi$ implies

$$\models (T_1 \rightarrow \varphi_1) \vee (S_1 \rightarrow \psi_1);$$

Given an interpolant σ_1 between $T_1 \rightarrow \varphi_1$ and $S_1 \rightarrow \psi_1$, we obtain

$$T_1 \models \varphi_1 \vee \sigma_1 \text{ and } S_1 \models \sim \sigma_1 \vee \psi_1.$$

By restoring the old function symbols we then get $\models \varphi \vee \sigma$ and $\models \sim \sigma \vee \psi$.

REMARKS. (1) $Q^{\tau*}\bar{x} \sim \sigma(\bar{x})$ implies $\sim Q^{\tau}\bar{x}\sigma(\bar{x})$, but not conversely, except for structures satisfying the axiom of determinacy.

(2) 4.5 does not give interpolation for $L_{\kappa,\kappa}$ with only homogeneous quantifiers. Indeed, Malitz [9] shows that the interpolation theorem fails for this language.

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