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A remark on the character ring of a compact Lie group

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Introduction

Let G be a compact topological group, $D(G)$ the set of equivalence classes of irreducible representations of G. (In this note the representation will mean always the continuous complex representation.) The character ring $R(G)$ of G is the free abelian group generated by $D(G)$ with the ring structure induced by the tensor product of representations. In the present note we provide a method of finding a system of generators of the character ring $R(G)$ of a compact (not necessarily connected) Lie group G, assuming that the quotient group G/G_0 of G modulo the connected component G_0 of G is a cyclic group (Theorem 5). Our problem reduces to finding generators of a certain commutative semi-group in the similar way as for a compact *connected* Lie group.

By applying the theorem we can know the structure of the character ring of the orthogonal group $O(2l)$ of degree $2l$ or of the double covering group $\mathrm{Pin}(2l)$ of $O(2l)$. (See § 3 for the definition of $\mathrm{Pin}(2l)$.) Let λ^i be the i-th exterior power of the standard representation of $O(2l)$, α the 1-dimensional representation of $O(2l)$ defined by $\alpha(x)=\det x$ for $x\in O(2l)$. Let μ^l be the irreducible representation of $\mathrm{Pin}(2l)$ such that its restriction to the connected component $\mathrm{Spin}(2l)$ of $\mathrm{Pin}(2l)$ splits into the direct sum of two half-spinor representations of $\mathrm{Spin}(2l)$ and $p: \mathrm{Pin}(2l)\to O(2l)$ denote the covering homomorphism. Then we have

$$R(O(2l))=\boldsymbol{Z}[\lambda^1, \lambda^2, \cdots, \lambda^l, \alpha] \textit{ with relations } \alpha^2=1 \textit{ and } \lambda^l\alpha=\lambda^l,$$

$$R(\mathrm{Pin}(2l))=\boldsymbol{Z}[\lambda^1\circ p, \lambda^2\circ p, \cdots, \lambda^{l-1}\circ p, \mu^l, \alpha\circ p]$$

$$\textit{with relations } (\alpha\circ p)^2=1 \textit{ and } \mu^l(\alpha\circ p)=\mu^l.$$

The character ring of $O(2l)$ was formerly presented by Minami [7] by different methods.

§ 1. Induced representations.

Let G be a compact topological group. We consider the set of equivalence classes of representations of G as a subset of the character ring $R(G)$ of G and introduce an inner product (,) on $R(G)$ in such a way that $D(G)$ is an orthonormal basis of $R(G)$. For an element $\chi \in R(G)$, an element $\rho \in D(G)$ such that the integer (χ, ρ), denoted by m_ρ, is not zero is called a *component* of χ. We call m_ρ the *multiplicity* of the component ρ in χ. For a representation ρ of G, the equivalence class of ρ will be denoted by $[\rho]$.

Let $h: H \to G$ be a continuous homomorphism from a compact group H into a compact group G. Then h induces a ring homomorphism $R(G) \to R(H)$ by the composition of h, denoted by h^*, and $R(H)$ becomes an $R(G)$-module by means of the homomorphism h^*.

Let G be a compact group, H a closed subgroup of G with the finite index $[G:H]$, $i: H \to G$ the inclusion homomorphism. For a representation $\sigma: H \to GL(V)$ of H, the space

$$\Gamma(G, V)^H = \{f\colon G \to V\,;\ f(gh) = \sigma(h)^{-1} f(g) \ \ for \ \ g \in G,\ h \in H\}$$

is a complex vector space of dimension $[G, H] \dim V$. G acts linearly on $\Gamma(G, V)^H$ by $(gf)(g') = f(g^{-1}g')$ for $g, g' \in G$ and we have a representation of G on $\Gamma(G, V)^H$, which is called the *representation induced by* σ. The space $\Gamma(G, V)^H$ is naturally identified with the space of sections of the vector bundle $G \times_H V$ over G/H associated with the representation σ of H and the action of G on $\Gamma(G, V)^H$ is nothing but the one induced from the natural action of G on $G \times_H V$. The equivalence class of this representation depends only on the equivalence class of σ so that we have a map $i_*: D(H) \to R(G)$, which is linearly extended to an $R(G)$-homomorphism $i_*: R(H) \to R(G)$ (cf. Atiyah [1]). Then we have the Frobenius reciprocity:

$$(i^*\rho, \sigma) = (\rho, i_*\sigma) \qquad for \quad \rho \in R(G),\ \sigma \in R(H)\,.$$

Now we assume that *H is a normal subgroup of G with the finite index.* Then the quotient group $A = G/H$ of G modulo H is a finite group and the natural projection $\pi: G \to A$ is a homomorphism. $\hat{A}$ denotes the character group $\mathrm{Hom}(A, \boldsymbol{C}^*)$ of A. We imbed $\hat{A}$ into $D(G)$ by the product-preserving map $\alpha \mapsto \alpha \circ \pi$. Then $\hat{A}$ acts on $D(G)$, therefore on $R(G)$, by the multiplication of elements of $\hat{A}$. For a representation $\sigma: H \to GL(V)$ of H and $g \in G$, another representation $\sigma': H \to GL(V)$ of H is defined by

$$\sigma'(g') = \sigma(g^{-1}g'g) \qquad for \quad g' \in H\,.$$

The equivalence class of σ' depends only on the equivalence class of σ and on $\pi(g)$ so that A acts on $D(H)$, therefore on $R(H)$, by conjugations. The followings are immediate consequences of definitions:

(1) $$i^*(\alpha\cdot\rho)=i^*\rho \quad \text{for} \quad \alpha\in\hat{A}, \quad \rho\in R(G)\,,$$

(2) $$a\cdot(i^*\rho)=i^*\rho \quad \text{for} \quad a\in A, \quad \rho\in R(G)\,,$$

(3) $$i_*(a\cdot\sigma)=i_*\sigma \quad \text{for} \quad a\in A, \quad \sigma\in R(H)\,,$$

(4) $$\alpha\cdot(i_*\sigma)=i_*\sigma \quad \text{for} \quad \alpha\in\hat{A}, \quad \sigma\in R(H)\,.$$

THEOREM 1. (Clifford) *Let $\rho\in D(G)$. Take $\sigma_1\in D(H)$ such that $(i^*\rho, \sigma_1)>0$ and put $m(\rho)=(i^*\rho, \sigma_1)$, $\Phi_\rho=A\cdot\sigma_1\subset D(H)$. Then both $m(\rho)$ and Φ_ρ depend only on ρ and we have the decomposition*

$$i^*\rho=m(\rho)\sum_{\sigma\in\Phi_\rho}\sigma\,.$$

For the proof, see Feit [3].

Let $A\backslash D(H)$ (resp. $\hat{A}\backslash D(G)$) denotes the set of A-orbits in $D(H)$ (resp. $\hat{A}$-orbits in $D(G)$). The map $\varphi: D(G)\to A\backslash D(H)$ defined by $\rho\mapsto\Phi_\rho$ is surjective from the Frobenius reciprocity and induces a surjective map

$$\Phi:\ \hat{A}\backslash D(G)\longrightarrow A\backslash D(H)$$

in view of (1). Note that $m(\rho)$ is constant on each $\hat{A}$-orbit in $D(G)$.

THEOREM 2. (Clifford-Iwahori) *If the quotient group $A=G/H$ is commutative, then the map Φ is bijective. The inverse map of Φ is given as follows. Let $\sigma\in D(H)$. Take $\rho_1\in D(G)$ such that $(i_*\sigma, \rho_1)>0$ and put $m(\sigma)=(i_*\sigma, \rho_1)$, $\Psi_\sigma=\hat{A}\cdot\rho_1\subset D(G)$. Then both $m(\sigma)$ and Ψ_σ depend only on σ and we have the decomposition*

$$i_*\sigma=m(\sigma)\sum_{\rho\in\Psi_\sigma}\rho\,.$$

The map $\psi: D(H)\to\hat{A}\backslash D(G)$ defined by $\sigma\mapsto\Psi_\sigma$ induces a map

$$\Psi:\ A\backslash D(H)\longrightarrow\hat{A}\backslash D(G)$$

in view of (3). The map Ψ is the inverse of Φ. In particular:

1) *If A is a cyclic group, then $m(\rho)=m(\sigma)=1$ for any $\rho\in R(G)$ and $\sigma\in R(H)$.*

2) *If the order $|A|$ of A is a prime number p, then for the orbits Φ_ρ and Ψ_σ corresponding by the bijection Φ it happens one of following two cases:*

a) $|\Phi_\rho|=p$ *and* $|\Psi_\sigma|=1$,

b) $|\Phi_\rho|=1$ *and* $|\Psi_\sigma|=p$,

where $|S|$ means the cardinality of the set S.

PROOF. This theorem can be proved in the same way as the classical Clifford theorem for $A=\boldsymbol{Z}_2$ (Iwahori-Matsumoto [5]). But we give here another proof.

Let $\sigma\in D(H)$. For $\rho=\alpha\cdot\rho_1\in\Psi_\sigma$ we have $(i_*\sigma, \rho)=(\sigma, i^*(\alpha\rho_1))=(\sigma, i^*\rho_1)=(i_*\sigma, \rho_1)$. Therefore it suffices to show that if $\rho\in D(G)$ with $(i_*\sigma, \rho)>0$

then $\rho \in \Psi_\sigma$. Note that $i_* 1 = \sum_{\alpha \in \hat{A}} \alpha$ since A is commutative. It follows that

$$i_*(i^* \rho_1) = i_*((i^* \rho_1) 1) = \rho_1 (i_* 1) = \rho_1 \sum_{\alpha \in \hat{A}} \alpha = \sum_{\alpha \in \hat{A}} \alpha \cdot \rho_1 .$$

On the other hand, the Frobenius reciprocity yields that $(i^* \rho_1, \sigma) > 0$ so that ρ is a component of $i_*(i^* \rho_1)$. Thus $\rho \in \hat{A} \cdot \rho_1 = \Psi_\sigma$. The above simple proof was communicated by Professor H. Nagao.

1) See Feit [3].

2) Recall (cf. Atiyah [1]) the general equality $\sum m_\rho^2 = |A_\sigma|$ for $\sigma \in D(H)$, $i_* \sigma = \sum_{\rho \in D(G)} m_\rho \rho$ and $A_\sigma = \{a \in A ; a \cdot \sigma = \sigma\}$. In our case we have $|\Psi_\sigma| = |A_\sigma| = |A| / |\Phi_\rho|$ by the above equality and 1), so that $|\Phi_\rho| |\Psi_\sigma| = p$, which yields the statement 2). q. e. d.

Note that $m(\sigma)$ is also constant on each A-orbit in $D(H)$ in view of (3) and that $m(\rho)$ and $m(\sigma)$ take the same value on the orbits corresponding by Φ from the Frobenius reciprocity.

REMARK. We denote by $R(G)^{\hat{A}}$ (resp. $R(H)^A$) the submodule of $R(G)$ (resp. $R(H)$) of elements fixed by $\hat{A}$ (resp. A). From (2) and (4) we have $i^* R(G) \subset R(H)^A$ and $i_* R(H) \subset R(G)^{\hat{A}}$. If A is cyclic, then by Theorem 2, 1)

$$i_* R(H) = R(G)^{\hat{A}} .$$

It is also known (Atiyah [1]) that if the order $|A|$ of A is square free (A is not necessarily commutative), then

$$i^* R(G) = R(H)^A .$$

§ 2. Character ring of a compact Lie group.

Let G be a compact Lie group, G_0 the connected component of G. Take a maximal torus T_0 of G_0. Note that $D(T_0)$ is a commutative group by the tensor product. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G_0 and T_0. Take an $\mathrm{Ad}\, G$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Let Δ be the root system of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$, i. e. the set of non-zero elements α of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ such that

$$\mathfrak{g}_\alpha^C = \{X \in \mathfrak{g}^C ; [H, X] = 2\pi \sqrt{-1}\, \alpha(H) X \text{ for any } H \in \mathfrak{t}\}$$

is not zero. Take a fundamental system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ of Δ and fix it once and for all. The duality defined by means of $\langle\ ,\ \rangle$ identifies $\mathfrak{t}$ with $\mathfrak{t}^*$ so that the root system Δ may be considered as a subset of $\mathfrak{t}$. Taking a basis $\{h_1, \cdots, h_m\}$ of the center of $\mathfrak{g}$, we introduce a lexicographic order $>$ on $\mathfrak{t}^*$ by the basis $\{\alpha_1, \cdots, \alpha_l, h_1, \cdots, h_m\}$ of $\mathfrak{t}$. Such order on $\mathfrak{t}^*$ will be called a *linear order associated with* Π. We put

$$Z_0 = \{\lambda \in \mathfrak{t}^* ;\ \lambda(H) \in \boldsymbol{Z} \text{ for any } H \in \mathfrak{t} \text{ such that } \exp H = 1\}$$

and

$$D_0 = \{\lambda \in Z_0 ;\ \lambda(\alpha_i) \geqq 0 \text{ for any } \alpha_i \in \Pi\} .$$

Then Z_0 is a lattice of $\mathfrak{t}^*$ and isomorphic with $D(T_0)$ by the correspondence $\lambda \mapsto e^{2\pi\sqrt{-1}\lambda}$, where $e^{2\pi\sqrt{-1}\lambda}$ is the character of T_0 defined by $e^{2\pi\sqrt{-1}\lambda}(\exp H) = e^{2\pi\sqrt{-1}\lambda(H)}$ for $H \in \mathfrak{t}$. Thus we can introduce an order $>$ on $D(T_0)$ by means of the order $>$ on Z_0. D_0 is a commutative semi-group. We put

$$D_d(T_0) = \{e^{2\pi\sqrt{-1}\lambda} ;\ \lambda \in D_0\} .$$

An element of $D_d(T_0)$ will be called a *dominant* (with respect to Π) *irreducible representation* of T_0.

Now we define a closed subgroup T of G with the connected component T_0 by

$$T = \{g \in G ;\ \mathrm{Ad}\, g\mathfrak{t} = \mathfrak{t},\ \mathrm{Ad}\, g\Pi = \Pi\} .$$

The quotient group T/T_0 is naturally isomorphic with the quotient group G/G_0. This follows from $G_0 \cap T = T_0$, the conjugateness in G_0 of maximal tori of G_0 and that of fundamental systems of Δ under the normalizer of T_0 in G_0. We put $A = G/G_0 = T/T_0$. The adjoint representation Ad induces a homomorphism $\tau : A \to GL(\mathfrak{t})$. We define a finite subgroup C of $GL(\mathfrak{t})$ by $C = \tau A$. It leaves Z_0 and D_0 invariant so that we can define the set $Z = C\backslash Z_0$ (resp. $D = C\backslash D_0$) of C-orbits in Z_0 (resp. in D_0). We introduce a linear order $>$ on Z by defining that $\Lambda > \Lambda'$ for $\Lambda, \Lambda' \in Z$ if $\mathrm{Max}\, \Lambda > \mathrm{Max}\, \Lambda'$. We introduce also an operation $+$ on Z by defining that for $\Lambda, \Lambda' \in Z$, $\Lambda + \Lambda'$ is the C-orbit through $\mathrm{Max}\, \Lambda + \mathrm{Max}\, \Lambda'$. Note that $\mathrm{Max}\,(\Lambda + \Lambda') = \mathrm{Max}\, \Lambda + \mathrm{Max}\, \Lambda'$. The operation $+$ induces a commutative semi-group structure on D.

Now we consider the following commutative diagram of inclusions:

$$\begin{tikzcd} G & T \arrow[l, "j"'] \\ G_0 \arrow[u, "i_{G_0}"] & T_0 \arrow[l, "j_0"] \arrow[u, "i_{T_0}"'] \end{tikzcd}$$

Then we have the following commutative diagram of ring homomorphisms:

$$\begin{tikzcd} R(G) \arrow[r, "j^*"] \arrow[d, "i^*_{G_0}"'] & R(T) \arrow[d, "i^*_{T_0}"] \\ R(G_0) \arrow[r, "j^*_0"'] & R(T_0) \end{tikzcd}$$

It is classical that j_0^* is injective. The homomorphism j^* is also injective since for any $g \in G$ there exists $g_0 \in G_0$ such that $g_0 g g_0^{-1} \in T$ (Gantmacher [4]). The inclusions $\hat{A} \subset D(G)$ and $\hat{A} \subset D(T)$ defined in §1 are compatible with the injection $j^*: R(G) \to R(T)$, i.e. $j^*\alpha = \alpha$ for $\alpha \in \hat{A}$. Let $\delta \in D(T)$. From Theorem 1 there exist a positive integer $m(\delta)$ and $\Lambda_\delta \in Z$ such that

$$i_{T_0}^*\delta = m(\delta) \sum_{\lambda \in \Lambda_\delta} e^{2\pi\sqrt{-1}\lambda} .$$

A surjective map $\varphi: D(T) \to Z$ is defined by the correspondence $\delta \mapsto \Lambda_\delta$. For $\delta, \delta' \in D(T)$, we say that δ is *strictly higher than* δ' if $\varphi(\delta) > \varphi(\delta')$ and it will be denoted by $\delta \gg \delta'$. We put

$$D_d(T) = \{\delta \in D(T)\,;\ \varphi(\delta) \in D\} .$$

An irreducible representation δ of T is called *dominant* (with respect to Π) if the equivalence class of δ belongs to $D_d(T)$.

Let $\rho: G \to GL(V)$ be an irreducible representation of G. The holomorphic extension $G^C \to GL(V)$ of ρ to the complexification G^C of G or its differential $\mathfrak{g}^C \to \mathfrak{gl}(V)$ will be denoted by the same letter ρ and put

$$\begin{aligned} V_0 &= \{v \in V\,;\ \rho(X)v = 0 \textit{ for any } X \in \mathfrak{m}\} \\ &= \{v \in V\,;\ \rho(\exp X)v = v \textit{ for any } X \in \mathfrak{m}\} , \end{aligned} \tag{5}$$

where $\mathfrak{m} = \sum_{\alpha \in \Delta, \alpha > 0} \mathfrak{g}_\alpha^C$. Then V_0 is T-invariant since $\mathrm{Ad}\, T$ leaves $\mathfrak{m}$ invariant, so that we have a representation $\delta_\rho: T \to GL(V_0)$ of T. We shall prove later that δ_ρ is a dominant irreducible representation of T. The equivalence class of δ_ρ depends only on the equivalence class of ρ. It is called the *Cartan component* of ρ or of the equivalence class $[\rho]$ of ρ. The Cartan component of $\rho \in D(G)$ will be denoted by δ_ρ and the map $D(G) \to D_d(T)$ defined by $\rho \mapsto \delta_\rho$ will be denoted by γ. The classical representation theory of a compact *connected* Lie group yields the following

THEOREM 3. 1) *The Cartan component δ_ρ of $\rho \in D(G_0)$ belongs to $D_d(T_0)$ and the map $\gamma_0: D(G_0) \to D_d(T_0)$ defined by $\rho \mapsto \delta_\rho$ is an A-equivariant bijection.* (For $\lambda \in D_0$, $\gamma_0^{-1}(e^{2\pi\sqrt{-1}\lambda})$ will be denoted by ρ_λ.)

2) *For $\rho \in D(G_0)$ the Cartan component δ_ρ of ρ is the highest component among components of $j_0^*\rho \in R(T_0)$ and has the multiplicity* 1.

Now we shall prove that the former representation $\delta_\rho: T \to GL(V_0)$ of T induced by an irreducible representation $\rho: G \to GL(V)$ of G is irreducible and dominant. Let W_0 be a T-invariant subspace of V_0. Then the subspace $W = \{\rho(g_0)s\,;\ g_0 \in G_0,\ s \in W_0\}_C$ of V spanned by $\rho(G_0)W_0$ is G-invariant because of $G = TG_0$. Decompose W_0 into the direct sum of 1-dimensional T_0-invariant subspaces: $W_0 = W_1 + \cdots + W_m$, where T_0 acts on W_i by character $e^{2\pi\sqrt{-1}\lambda_i}$ of T_0 $(1 \leqq i \leqq m)$. Then the subspace V_i of W spanned by $\rho(G_0)W_i$ is a G_0-irreducible G_0-invariant subspace with the Cartan component $e^{2\pi\sqrt{-1}\lambda_i}$ and W

is the direct sum of the V_i's. Therefore we have

$$W_0=\{v\in W;\ \rho(X)v=0 \textit{ for any } X\in\mathfrak{m}\}=W\cap V_0.$$

It follows from the G-irreducibility of V that $W_0=0$ or V_0. Thus V_0 is T-irreducible. We have

$$i^*_{T_0}[\delta_\rho]=m([\delta_\rho])\sum_{\lambda\in\Lambda_{[\delta_\rho]}}e^{2\pi\sqrt{-1}\lambda}.$$

It follows from Theorem 3 and (5) that

$$i^*_{G_0}[\rho]=m([\delta_\rho])\sum_{\lambda\in\Lambda_{[\delta_\rho]}}\rho_\lambda$$

and so $\Lambda_{[\delta_\rho]}\subset D_0$. Thus δ_ρ is dominant.

Theorem 3 is true also for a general compact Lie group G in the following sense.

THEOREM 4. 1) (Kostant) *The map* $\gamma: D(G)\to D_d(T)$ *defined by* $\rho\mapsto\delta_\rho$ *is an* $\hat{A}$*-equivariant bijection such that*

$$i^*_{G_0}\rho=m(\delta_\rho)\sum_{\lambda\in\Lambda_{\delta_\rho}}\rho_\lambda.$$

Therefore we have the following commutative diagram:

$$\begin{array}{ccc}\hat{A}\backslash D(G) & \xrightarrow{\quad\Gamma\quad} & \hat{A}\backslash D_d(T)\\ \Phi_G\big\downarrow & & \big\downarrow\Phi_T\\ A\backslash D(G_0) & \xrightarrow[\quad\Gamma_0\quad]{} & A\backslash D_d(T_0)\end{array}$$

where vertical maps Φ_G *and* Φ_T *are the surjective maps defined in* § 1, *horizontal maps* Γ *and* Γ_0 *are the bijective maps induced by* γ *and* γ_0.

2) *For* $\rho\in D(G)$ *the Cartan component* δ_ρ *of* ρ *is the strictly highest component among components of* $j^*\rho\in R(T)$ *and has the multiplicity* 1.

PROOF. 1) This was stated in Kostant [6] without proof. We prove it here for the sake of completeness.

Let $\delta: T\to GL(V_0)$ be a dominant irreducible representation of T. Decompose V_0 into the direct sum of 1-dimensional T_0-invariant subspaces:

$$V_0=W_1+\cdots+W_m,$$

where T_0 acts on W_i by character $e^{2\pi\sqrt{-1}\lambda_i}\in D_d(T_0)$ $(1\leqq i\leqq m)$. Take one of the λ_i's, say λ_1, and let $\rho_1: G_0\to GL(V_1)$ be an irreducible representation of G_0 with the Cartan component $e^{2\pi\sqrt{-1}\lambda_1}$ (Theorem 3). We imbed W_1 into V_1 as a T_0-invariant subspace. The natural map $T\times_{T_0}W_1\to G\times_{G_0}V_1$ is a T-equivariant injective bundle map over $A=T/T_0=G/G_0$ so that we have a T-equivariant imbedding $\Gamma(T, W_1)^{T_0}\subset\Gamma(G, V_1)^{G_0}$. From the Frobenius reciprocity: $((i_{T_0})_* e^{2\pi\sqrt{-1}\lambda_1}, [\delta]) = (i^*_{T_0}[\delta], e^{2\pi\sqrt{-1}\lambda_1}) > 0$, we have a T-irreducible T-invariant subspace V'_0 of $\Gamma(T, W_1)^{T_0}$ and a T-equivariant isomorphism $\theta: V_0 \to V'_0$. Thus we have a T-equivariant injective homomorphism $\theta: V_0 \to \Gamma(G, V_1)^{G_0}$. The subspace V' of $\Gamma(G, V_1)^{G_0}$ spanned by $G_0\theta(V_0)$ is G-invariant because of $G = TG_0$ so that we have a representation $\rho': G \to GL(V')$ of G. For $g_0 \in G_0$, $s \in V_0$ and $t \in T$ we have

$$(\rho'(g_0)\theta(s))(t) = \theta(s)(g_0^{-1}t) = \theta(s)(t(t^{-1}g_0^{-1}t)) = \rho_1(t^{-1}g_0^{-1}t)^{-1}\theta(s)(t).$$

It follows seeing $\theta(s)(t) \in W_1$ and $\mathrm{Ad}\, t^{-1}\mathfrak{m} = \mathfrak{m}$ that

$$\rho'(X)\theta(s) = 0 \qquad \textit{for any} \quad X \in \mathfrak{m} \textit{ and } s \in V_0.$$

Thus we have

$$V'_0 = \{f \in V';\ \rho'(X)f = 0 \textit{ for any } X \in \mathfrak{m}\}, \tag{6}$$

$$i^*_{G_0}[\rho'] = \rho_{\lambda_1} + \cdots + \rho_{\lambda_m}.$$

The representation ρ' is G-irreducible since for a non-trivial G-invariant subspace W' of V', $W' \cap V'_0$ is also a non-trivial T-invariant subspace of V'_0 in view of (6). The equivalence class of ρ' depends only on the equivalence class of δ since $\lambda_i \in A\cdot\lambda_1$ for any i (Theorem 1).

Next we prove that if a dominant irreducible representation $\delta: T \to GL(V_0)$ of T is obtained from an irreducible representation $\rho: G \to GL(V)$ of G by (5), then the above obtained representation ρ' is equivalent to ρ. Let

$$i^*_{T_0}[\delta] = e^{2\pi\sqrt{-1}\lambda_1} + \cdots + e^{2\pi\sqrt{-1}\lambda_m}.$$

Then (5) implies

$$i^*_{G_0}[\rho] = \rho_{\lambda_1} + \cdots + \rho_{\lambda_m}.$$

Together with (6) we have a G_0-equivariant isomorphism $\theta: V \to V'$ which is an extension of the T-equivariant isomorphism $\theta: V_0 \to V'_0$. For $t \in T$, $g_0 \in G_0$ and $s \in V_0$ we have

$$\begin{aligned}\theta(\rho(t)\rho(g_0)s) &= \theta(\rho(tg_0t^{-1})\rho(t)s) = \rho'(tg_0t^{-1})\theta(\rho(t)s)\\ &= \rho'(tg_0t^{-1})\rho'(t)\theta(s) = \rho'(tg_0)\theta(s)\\ &= \rho'(t)\rho'(g_0)\theta(s) = \rho'(t)\theta(\rho(g_0)s).\end{aligned}$$

It follows that θ is a G-equivariant isomorphism since V is spanned by $\rho(G_0)V_0$ and $G = TG_0$. Thus we have proved that the map γ is bijective.

The other statements are clear from the construction.

2) Let

$$j^*\rho = \sum_{\delta \in D(T)} m_\delta \delta$$

and

$$i_{T_0}^*\delta = m(\delta)\sum_{\lambda\in\Lambda_\delta} e^{2\pi\sqrt{-1}\lambda}.$$

Then

$$i_{T_0}^* j^*\rho = \sum_\delta m_\delta m(\delta)\sum_{\lambda\in\Lambda_\delta} e^{2\pi\sqrt{-1}\lambda}.$$

On the other hand, we have by 1)

$$i_{G_0}^*\rho = m(\delta_\rho)\sum_{\lambda\in\Lambda_\delta}\rho_\lambda$$

so that

$$j_0^* i_{G_0}^*\rho = m(\delta_\rho)\sum_{\lambda\in\Lambda_\delta} j_0^*\rho_\lambda.$$

It follows from Theorem 3 that the highest component of $j_0^* i_{G_0}^*\rho \in R(T_0)$ is $e^{2\pi\sqrt{-1}\lambda_0}$, where $\lambda_0 = \mathrm{Max}\,\Lambda_{\delta_\rho}$, with the multiplicity $m(\delta_\rho)$. Comparing the highest components of $i_{T_0}^* j^*\rho$ and $j_0^* i_{G_0}^*\rho$, we know that $m_{\delta_\rho}=1$ and that $m_\delta \neq 0$, $\delta\neq\delta_\rho$ imply $\Lambda_\delta < \Lambda_{\delta_\rho}$. This completes the proof of 2). q. e. d.

For each $\Lambda\in Z$ the complete inverse $\varphi^{-1}(\Lambda)$ of Λ for the map $\varphi: D(T)\to Z$ defined by $\delta\mapsto\Lambda_\delta$ is a finite subset of $D(T)$ from the Frobenius reciprocity. We introduce on each $\varphi^{-1}(\Lambda)$ an arbitrary linear order and fix it once and for all. We introduce a linear order $>$ on $D(T)$ as follows: For δ, $\delta'\in D(T)$, $\delta>\delta'$ if and only if $\delta\gg\delta'$ or $\varphi(\delta)=\varphi(\delta')$, $\delta>\delta'$.

LEMMA 1. *The highest* (with respect to the above order) *component of an element* $\chi\in j^*R(G)\subset R(T)$ *belongs to* $D_d(T)$.

PROOF. Let

$$\chi = j^*\sum_{\rho\in D(G)} m_\rho\rho, \qquad \delta_{\rho_0} = \mathop{\mathrm{Max}}_{m_\rho\neq 0}\delta_\rho$$

and

$$j^*\rho = \sum_{\delta\in D(T)} m_{\rho,\delta}\delta,$$

so that

$$\chi = \sum_\rho m_\rho\sum_\delta m_{\rho,\delta}\delta.$$

It follows from Theorem 4 that the highest component of χ is δ_{ρ_0}, which belongs to $D_d(T)$, with the multiplicity m_{ρ_0}. q. e. d.

Henceforth we assume that *the quotient group* $A=G/G_0$ *is a cyclic group*. Let α be a generator of the character group $\hat{A}$ of A. Then Theorem 2 implies that $m(\delta)=1$ for any $\delta\in D(T)$ and that Φ_T and Φ_G are bijections so that $\varphi(\delta)=\varphi(\delta')$ for δ, $\delta'\in D(T)$ if and only if there exists a non-negative integer n such that $\alpha^n\delta=\delta'$.

LEMMA 2. *Let* δ, $\delta'\in D(T)$. *Then* $\delta\delta'\in R(T)$ *has the strictly highest component* δ'' *with the multiplicity* 1 *such that* $\varphi(\delta)+\varphi(\delta')=\varphi(\delta'')$.

PROOF. Let $\lambda_0=\mathrm{Max}\,\Lambda_\delta$ and $\lambda_0'=\mathrm{Max}\,\Lambda_{\delta'}$. Then the highest component of

$$(i_{T_0}^*\delta)(i_{T_0}^*\delta')=\sum_{(\lambda,\lambda')\in\Lambda_\delta\times\Lambda_{\delta'}} e^{2\pi\sqrt{-1}(\lambda+\lambda')}\in R(T)$$

is $e^{2\pi\sqrt{-1}(\lambda_0+\lambda'_0)}$ with the multiplicity 1. On the other hand, if $\delta\delta'=\sum_{\varepsilon\in D(T)} m_\varepsilon\varepsilon$ we have

$$i_{T_0}^*(\delta\delta')=\sum_\varepsilon m_\varepsilon \sum_{\mu\in\Lambda_\varepsilon} e^{2\pi\sqrt{-1}\mu}.$$

Comparing the highest components of $(i_{T_0}^*\delta)(i_{T_0}^*\delta')$ and $i_{T_0}^*(\delta\delta')$ we know that there exists $\delta''\in D(T)$ such that $m_{\delta''}=1$ and $\Lambda_\delta+\Lambda_{\delta'}=\Lambda_{\delta''}$ and that δ'' is strictly highest in $\delta\delta'$. q. e. d.

LEMMA 3. 1) *Let $\delta_1, \delta_2, \delta'\in D(T)$ such that $\delta_1\ll\delta_2$ and δ_1'' (resp. δ_2'') the strictly highest component of $\delta_1\delta'$ (resp. of $\delta_2\delta'$). Then $\delta_1''\ll\delta_2''$.*

2) *Let $\delta_1, \delta_2, \delta_1', \delta_2'\in D(T)$ such that $\delta_1\ll\delta_2$, $\delta_1'\ll\delta_2'$ and δ_1'' (resp. δ_2'') be the strictly highest component of $\delta_1\delta_1'$ (resp. of $\delta_2\delta_2'$). Then $\delta_1''\ll\delta_2''$.*

PROOF. 1) We have by Lemma 2 $\varphi(\delta_1'')=\varphi(\delta_1)+\varphi(\delta')$ and $\varphi(\delta_2'')=\varphi(\delta_2)+\varphi(\delta')$. Together with $\varphi(\delta_1)<\varphi(\delta_2)$ we have $\varphi(\delta_1'')<\varphi(\delta_2'')$.

2) We have by Lemma 2 $\varphi(\delta_1'')=\varphi(\delta_1)+\varphi(\delta_1')$ and $\varphi(\delta_2'')=\varphi(\delta_2)+\varphi(\delta_2')$. Together with the inequalities $\varphi(\delta_1)<\varphi(\delta_2)$ and $\varphi(\delta_1')<\varphi(\delta_2')$ we have $\varphi(\delta_1'')<\varphi(\delta_2'')$. q. e. d.

LEMMA 4. *Let χ_i $(1\leqq i\leqq m)$ be an element of $R(G)$ such that $j^*\chi_i$ has the strictly highest component $\delta_i\in D_d(T)$ with the multiplicity 1 and n_i $(1\leqq i\leqq m)$ be a non-negative integer. Then $j^*(\chi_1^{n_1}\cdots\chi_m^{n_m})$ has the strictly highest component $\delta\in D_d(T)$ with the multiplicity 1 such that $\varphi(\delta)=n_1\varphi(\delta_1)+\cdots+n_m\varphi(\delta_m)$.*

PROOF. The existence of the strictly highest component δ follows from Lemma 3. The other statements follow from Lemma 2. q. e. d.

THEOREM 5. *Assume that $A=G/G_0$ is a cyclic group. Let $\{\Lambda_1,\cdots,\Lambda_m\}$ be a system of generators of the semigroup D, δ_i $(1\leqq i\leqq m)$ an element of $D_d(T)$ with $\varphi(\delta_i)=\Lambda_i$, χ_i $(1\leqq i\leqq m)$ an element of $R(G)$ such that $j^*\chi_i$ has the strictly highest component δ_i with the multiplicity* 1. (Existence of such χ_i is assured by Theorem 4.) *Let α be a generator of the character group $\hat{A}$ of A. Then the character ring $R(G)$ of G is generated by $\chi_1,\cdots,\chi_m,\alpha$.*

PROOF. Take any element $\chi\in R(G)$. Let δ_0 be the highest component of $j^*\chi$. m_{δ_0} denotes the multiplicity of δ_0 in $j^*\chi$. Since $\delta_0\in D_d(T)$ (Lemma 1), we have non-negative integers $n_1,\cdots,n_m$ such that $n_1\Lambda_1+\cdots+n_m\Lambda_m=\varphi(\delta_0)$. On the other hand, Lemma 4 implies that $j^*(\chi_1^{n_1}\cdots\chi_m^{n_m})$ has the strictly highest component, say δ, with the multiplicity 1 such that $n_1\Lambda_1+\cdots+n_m\Lambda_m=\varphi(\delta)$. Thus there exists a non-negative integer n such that $\alpha^n\delta=\delta_0$. It follows that the highest component of $j^*(m_{\delta_0}\alpha^n\chi_1^{n_1}\cdots\chi_m^{n_m})=m_{\delta_0}\alpha^n j^*(\chi_1^{n_1}\cdots\chi_m^{n_m})$ is δ_0 with the multiplicity m_{δ_0}. Therefore the highest component of $j^*(\chi-m_{\delta_0}\alpha^n\chi_1^{n_1}\cdots\chi_m^{n_m})$ is lower than δ_0. Thus we can show inductively that χ is a polynomial of $\chi_1,\cdots,\chi_m,\alpha$ with coefficients in $\boldsymbol{Z}$, recalling that j^* is injective. q. e. d.

THEOREM 6. *Assume that* $A=G/G_0$ *is a cyclic group. Let* $\{\Lambda_1, \cdots, \Lambda_m\}$ *be an independent system of* D, *i.e.* $\sum_{i=1}^{m} n_i\Lambda_i = \sum_{i=1}^{m} n'_i\Lambda_i$, *where the* n_i, n'_i *are non-negative integers, implies* $n_i=n'_i$ *for any* i. *Let* δ_i $(1\leqq i\leqq m)$ *be an element of* $D_d(T)$ *with* $\varphi(\delta_i)=\Lambda_i$, χ_i *an element of* $R(G)$ *such that* $j^*\chi_i$ *has the strictly highest component* δ_i *with the multiplicity* 1. *Then the system* $\{\chi_1, \cdots, \chi_m\}$ *has no relations in* $R(G)$.

PROOF. Let $F\in \boldsymbol{Z}[X_1, \cdots, X_m]$ be a relation for $\{\chi_1, \cdots, \chi_m\}$, i.e. $F=\sum a_{n_1\cdots n_m}X_1^{n_1}\cdots X_m^{n_m}$ $(a_{n_1\cdots n_m}\in \boldsymbol{Z})$ satisfies $F(\chi_1, \cdots, \chi_m)=0$. Suppose that $F\neq 0$. Let $\sum n_i^0\Lambda_i$ be the highest among the $\sum n_i\Lambda_i$ such that $a_{n_1\cdots n_m}\neq 0$. The assumption for $\{\Lambda_1, \cdots, \Lambda_m\}$ implies the uniqueness of such $(n_1^0, \cdots, n_m^0)$, so that $a_{n_1\cdots n_m}\neq 0$, $(n_1, \cdots, n_m)\neq(n_1^0, \cdots, n_m^0)$ imply $\sum n_i\Lambda_i<\sum n_i^0\Lambda_i$. It follows from Lemma 4 that $j^*F(\chi_1, \cdots, \chi_m)$ has the strictly highest component with the multiplicity $a_{n_1^0\cdots n_m^0}$, which contradicts $F(\chi_1, \cdots, \chi_m)=0$. q.e.d.

§ 3. Character rings of $O(n)$ and Pin (n).

We recall the notion of the group Pin (n) (Atiyah-Bott-Shapiro [2]). Let C_n be the Clifford algebra over $\boldsymbol{R}$ of degree n associated with the positive definite quadratic form, i.e. the associative algebra over $\boldsymbol{R}$ generated by $1, e_1, \cdots, e_n$ with relations $e_i^2=1$ $(1\leqq i\leqq n)$ and $e_ie_j+e_je_i=0$ $(1\leqq i<j\leqq n)$. C_n^* denotes the group of invertible elements of C_n. For $i=0, 1$, C_n^i denotes the subspace of C_n spanned by the $e_{i_1}e_{i_2}\cdots e_{i_r}$ $(1\leqq i_1<i_2<\cdots<i_r\leqq n,\ r\equiv i \bmod 2)$. Then C_n^0 is a subalgebra of C_n and $C_n=C_n^0+C_n^1$ (direct sum). Let ι be an automorphism of C_n defined by $x^0+x^1\mapsto x^0-x^1$ for $x^i\in C_n^i$ $(i=0, 1)$, $x\mapsto x^t$ be an anti-automorphism of C_n defined by $e_{i_1}e_{i_2}\cdots e_{i_r}\rightarrow e_{i_r}\cdots e_{i_2}e_{i_1}$ $(1\leqq i_1<i_2<\cdots<i_r\leqq n)$. Put $\bar{x}=\iota(x^t)$ for $x\in C_n$ and define the *norm* ν of $x\in C_n$ by $\nu(x)=\bar{x}x$. We shall identify $\boldsymbol{R}^n$ with the subspace of C_n spanned by $e_1, \cdots, e_n$ and $\boldsymbol{R}$ with that spanned by 1. Then we have a natural homomorphism p from the *twisted Clifford group* Γ_n defined by

$$\Gamma_n=\{s\in C_n^*\ ;\ \iota(s)\boldsymbol{R}^n s^{-1}\subset \boldsymbol{R}^n\}$$

into $GL(n, \boldsymbol{R})$. It is known that the norm ν induces a homomorphism $\nu: \Gamma_n\rightarrow \boldsymbol{R}^*$. We put

$$\mathrm{Pin}\,(n)=\{s\in \Gamma_n\ ;\ |\nu(s)|=1\}$$

and

$$\mathrm{Spin}\,(n)=\mathrm{Pin}\,(n)\cap C_n^0.$$

Both Pin (n) and Spin (n) are compact Lie groups with respect to the topology induced by that of C_n. For $n\geqq 2$, Spin (n) is the connected component of Pin (n) and Pin (n)/Spin $(n)\cong \boldsymbol{Z}_2$. For $n\geqq 3$, Spin (n) is simply connected. We have the following exact sequences:

$$1 \longrightarrow \boldsymbol{R}^* \longrightarrow \Gamma_n \xrightarrow{p} O(n) \longrightarrow 1$$

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Pin}\,(n) \xrightarrow{p} O(n) \longrightarrow 1$$

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Spin}\,(n) \xrightarrow{p} SO(n) \longrightarrow 1$$

where $\boldsymbol{Z}_2 = \{\pm 1\} \subset \boldsymbol{R}^*$. Our $\mathrm{Pin}\,(n)$ is slightly different from $\mathrm{Pin}\,(n)$ in [**2**], which was defined by the Clifford algebra associated with the *negative* definite quadratic form. For example, our $\mathrm{Pin}\,(1)$ is isomorphic with $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, while $\mathrm{Pin}\,(1)$ in [**2**] is isomorphic with $\boldsymbol{Z}_4$.

If $n = 2l+1$ is odd, we have isomorphisms

$$O(2l+1) \cong SO(2l+1) \times \boldsymbol{Z}_2$$

and

$$\mathrm{Pin}\,(2l+1) \cong \mathrm{Spin}\,(2l+1) \times \boldsymbol{Z}_2$$

so that we shall confine ourselves to consider $O(2l)$ or $\mathrm{Pin}\,(2l)$.

Let first $G = O(2l)$ $(l \geqq 1)$. Then $G_0 = SO(2l)$ and $A \cong \boldsymbol{Z}_2$.

$$T_0 = \left\{ \begin{pmatrix} r(t_1) & & \\ & \ddots & \\ & & r(t_l) \end{pmatrix} ; \quad t_i \in \boldsymbol{R} \right\}$$

where

$$r(t_i) = \begin{pmatrix} \cos 2\pi t_i & -\sin 2\pi t_i \\ \sin 2\pi t_i & \cos 2\pi t_i \end{pmatrix}$$

is a maximal torus of G_0. The Lie algebra $\mathfrak{t}$ of T_0 is identified with

$$\mathfrak{t} = \left\{ H(x_1, \cdots, x_l) = \begin{pmatrix} R(x_1) & & \\ & \ddots & \\ & & R(x_l) \end{pmatrix} ; \quad x_i \in \boldsymbol{R} \right\}$$

where

$$R(x_i) = \begin{pmatrix} 0 & -2\pi x_i \\ 2\pi x_i & 0 \end{pmatrix}.$$

The linear form on $\mathfrak{t}$ taking value x_i at $H(x_1, \cdots, x_l)$ will be denoted by x_i $(1 \leqq i \leqq l)$. Then the root system is

$$\Delta = \{\pm(x_i \pm x_j) ; \ 1 \leqq i < j \leqq l\}$$

and

$$\Pi = \{\alpha_i = x_i - x_{i+1} \ (1 \leqq i \leqq l-1), \ \alpha_l = x_{l-1} + x_l\}$$

is a fundamental system of Δ. The order $x_1 > \cdots > x_l > 0$ is a linear order associated with Π. We have

$$D_0=\{\sum m_i x_i\,;\ m_i\in \boldsymbol{Z},\ m_1\geqq m_2\geqq \cdots \geqq m_{l-1}\geqq |m_l|\}\,.$$

$\tau\colon A\to C$ is an isomorphism and $C\cong \boldsymbol{Z}_2$ is generated by the transformation τ_0 of $\mathfrak{t}^*$ defined by $\tau_0 x_i=x_i$ $(1\leqq i\leqq l-1)$ and $\tau_0 x_l=-x_l$. Thus the set

$$\{\sum m_i x_i\,;\ m_i\in \boldsymbol{Z},\ m_1\geqq m_2\geqq \cdots \geqq m_{l-1}\geqq m_l\geqq 0\}$$

is a complete set of representatives of $D=C\backslash D_0$. If we put $\Lambda_i=C(x_1+\cdots+x_i)$ $(1\leqq i\leqq l)$, i.e.

$$\Lambda_i=\begin{cases}\{x_1+\cdots+x_i\} & 1\leqq i\leqq l-1\\ \{x_1+\cdots+x_{l-1}+x_l,\ x_1+\cdots+x_{l-1}-x_l\} & i=l\,,\end{cases}$$

then $\{\Lambda_1,\cdots,\Lambda_m\}$ is a system of generators of the semigroup D. Let $\rho_0\in D(O(2l))$ be the equivalence class of the standard representation of $O(2l)$, $\lambda^i(\rho_0)\in D(O(2l))$ the i-th exterior power of ρ_0. Then

$$i^*_{SO(2l)}\lambda^i(\rho_0)=\begin{cases}\rho_{x_1+\cdots+x_i} & 1\leqq i\leqq l-1\\ \rho_{x_1+\cdots+x_{l-1}+x_l}+\rho_{x_1+\cdots+x_{l-1}-x_l} & i=l\end{cases}$$

so that $\varphi(\delta_{\lambda^i(\rho_0)})=\Lambda_i$ $(1\leqq i\leqq l)$. Moreover the determinant representation $\alpha\in D(O(2l))$ generates $\hat{A}$. It follows from Theorem 5 that $R(O(2l))$ is generated by $\lambda^1(\rho_0),\cdots,\lambda^l(\rho_0),\alpha$. More precisely we have

THEOREM 7.

$$R(O(2l))=\boldsymbol{Z}[\lambda^1(\rho_0),\cdots,\lambda^l(\rho_0),\alpha]$$

with relations $\alpha^2=1$ *and* $\lambda^l(\rho_0)\alpha=\lambda^l(\rho_0)$.

PROOF. Let $R=\boldsymbol{Z}[\lambda^1(\rho_0),\cdots,\lambda^l(\rho_0)]$ be the subring of $R(O(2l))$ generated by $\{\lambda^i(\rho_0)\,;\ 1\leqq i\leqq l\}$. Then $R(O(2l))$ is generated by α over R.

The highest components of $j_0^* i^*_{SO(2l)}\lambda^i(\rho_0)$ are $e^{2\pi\sqrt{-1}\lambda}$ for $\lambda=x_1+\cdots+x_i$ $(1\leqq i\leqq l)$, which are linearly independent. It follows from Theorem 6 that $\{i^*_{SO(2l)}\lambda^i(\rho_0)\,;\ 1\leqq i\leqq l\}$ has no relations in $R(SO(2l))$. Therefore $\{\lambda^i(\rho_0)\,;\ 1\leqq i\leqq l\}$ has no relations and the homomorphism $i^*_{SO(2l)}$ is injective on R.

Thus it remains to prove that the ideal

$$I=\{F\in R[X]\,;\ F(\alpha)=0\}$$

of $R[X]$ is generated by X^2-1 and $\lambda^l(\rho_0)X-\lambda^l(\rho_0)$. Since the first polynomial clearly belongs to I and the second belongs to I in view of Theorem 2, 2), it suffices to show that if $F=fX+g$ $(f,g\in R)$ is a polynomial in I with degree 1, then $g=-f$ and f is divisible by $\lambda^l(\rho_0)$. From $0=i^*_{SO(2l)}F(\alpha)=i^*_{SO(2l)}f+i^*_{SO(2l)}g=i^*_{SO(2l)}(f+g)$ and that $i^*_{SO(2l)}$ is injective on R, we have $g=-f$ and thus $f\alpha=f$. Let $f=h+k\lambda^l(\rho_0)$, where $h\in \boldsymbol{Z}[\lambda^1(\rho_0),\cdots,\lambda^{l-1}(\rho_0)]$ and $k\in R$. $f\alpha=f$ implies $h\alpha=h$. We shall show that $h=0$. Suppose that

$$h=\sum a_{n_1\cdots n_{l-1}}\lambda^1(\rho_0)^{n_1}\cdots\lambda^{l-1}(\rho_0)^{n_{l-1}} \qquad (a_{n_1\cdots n_{l-1}}\in \boldsymbol{Z})$$

is not zero. Let $\sum_{i=0}^{l-1} n_i^0(x_1+\cdots+x_i)$ be the highest among the $\sum_{i=1}^{l-1} n_i(x_1+\cdots+x_i)$ such that $a_{n_1\cdots n_{l-1}}\neq 0$. It follows from Lemma 4 that j^*h has the strictly highest component, say δ, with the multiplicity 1 such that $\Lambda_\delta=\left\{\sum_{i=1}^{l-1} n_i^0(x_1+\cdots+x_i)\right\}$. On the other hand, $h\alpha=h$ implies that $\alpha\delta=\delta$, which contradicts $|\Lambda_\delta|=1$ in view of Theorem 2, 2). q. e. d.

Now let $G=\mathrm{Pin}(2l)$ $(l\geqq 1)$. Then $G_0=\mathrm{Spin}(2l)$ and $A=G/G_0\cong \boldsymbol{Z}_2$. We have $\nu=\alpha\circ p$ and $\hat{A}$ is generated by ν. Let $\hat{\rho}_0\in D(\mathrm{Pin}(2l))$ be the equivalence class of the covering homomorphism $p:\mathrm{Pin}(2l)\to O(2l)$ and $\lambda^i(\hat{\rho}_0)\in D(\mathrm{Pin}(2l))$ the i-th exterior power of $\hat{\rho}_0$. Take an irreducible C_{2l}^{0C}-module M_0 and let $M=C_{2l}^C\otimes_{C_{2l}^{0C}} M_0$, where C_{2l}^{0C} (resp. C_{2l}^C) denotes the complexification of C_{2l}^0 (resp. of C_{2l}). Then M is an irreducible $\mathrm{Pin}(2l)$-module, whose equivalence class will be denoted by μ^l. The restriction $i^*_{\mathrm{Spin}(2l)}\mu^l$ is the sum of two half-spinor representations of $\mathrm{Spin}(2l)$. Then we have the following theorem in the same way as for $O(2l)$, but replacing $x_1+\cdots+x_{l-1}\pm x_l$ by $\frac{1}{2}(x_1+\cdots+x_{l-1}\pm x_l)$.

THEOREM 8. *$R(\mathrm{Pin}(2l))=\boldsymbol{Z}[\lambda^1(\hat{\rho}_0),\cdots,\lambda^{l-1}(\hat{\rho}_0),\mu^l,\nu]$ with relations $\nu^2=1$ and $\mu^l\nu=\nu$.*

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