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An isometric immersion of a homogeneous Riemannian manifold of dimension 3 in the hyperbolic space

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Introduction.

In the previous paper [1], the present author investigated a homogeneous Riemannian manifold admitting an isometric immersion in the euclidean space or the hyperbolic space. But recently an error is pointed out by Prof. S. Tanno. Precisely the calculation at the first line in page 408 is incorrect. It must be corrected as follows;

$$0=d\omega_{12}=\Omega_{12}-\omega_{13}\wedge\omega_{32}=(K_1+K-b_3c_3)\omega_1\wedge\omega_2=(K_1+2K)\omega_1\wedge\omega_2 .$$

Hence if $K\neq 0$, there are no contradictions. So Lemma 3.5 and also Theorem B are valid only if the dimension of M is greater than 3.

The purpose of this paper is to determine a structure of a connected homogeneous Riemannian manifold of dimension 3 which admits an isometric immersion of the type number 2 in the 4 dimensional hyperbolic space.

Let G be a group of all matrices of the following type

$$\begin{pmatrix} e^t & 0 & -\xi \\ 0 & e^{-t} & \eta \\ 0 & 0 & 1 \end{pmatrix} \qquad \xi,\ \eta,\ t\in R .$$

Then (ξ, η, t) can be considered as a coordinate system of G. In this coordinate system we give a Riemannian metric on G by

$$ds^2=e^{-2t}d\xi^2+e^{2t}d\eta^2+dt^2 .$$

This metric is invariant by left translations of G and with this metric G can be considered as a homogeneous Riemannian manifold of dimension 3. In this paper we call this Riemannian manifold a B-manifold. (Although the definition of a B-manifold given in § 1 is different from the above definition, it will be shown in Theorem 1 that both definitions are equivalent.) The main results of this paper are the following:

(1) A connected homogeneous Riemannian manifold M of dimension 3 admits an isometric immersion in H^4 of the type number 2, if and only if

M is a B-manifold.

(2) An isometric immersion of a B-manifold M in H^4 of the type number 2 exists and is unique in the following sense; if f_1 and f_2 are isometric immersions of M in H^4 of the type number 2, there exist an isometry k of H^4 and an isometry g of M such that $k \circ f_1 = f_2 \circ g$.

§ 1. B-manifold.

Throughout this paper, we shall assume that M is a connected homogeneous Riemannian manifold of dimension 3 and G is the identity component of the group of all isometries of M, unless otherwise stated.

Since G acts on the bundle $F(M)$ of the orthogonal frames of M, we can consider the orbits in $F(M)$ under G or the G-orbits in $F(M)$. If α is a matrix in $O(3)$ and S is a G-orbit in $F(M)$, then $R_\alpha(S)$ is also a G-orbit in $F(M)$. Conversely, if S and S' are two G-orbits in $F(M)$, there exists a matrix α in $O(3)$ such that $R_\alpha(S) = S'$.

If the restrictions of the canonical forms ω_1, ω_2, ω_3 to a G-orbit S in $F(M)$ satisfy the following relations;

$$d\omega_1 = \omega_1 \wedge \omega_3, \qquad d\omega_2 = -\omega_2 \wedge \omega_3, \qquad d\omega_3 = 0\,, \tag{1.1}$$

we call S an orbit of type B. A frame u in $F(M)$ is said to be of type B, if the orbit $G(u)$ of u is an orbit of type B. Moreover if there exists a G-orbit of type B in $F(M)$, M is called a B-manifold. We shall prove, in the next section, that M admits an isometric immersion in H^4 of the type number 2 if and only if M is a B-manifold. The purpose of this section is to give the complete determination of the structure of a B-manifold.

LEMMA 1.1. *A G-orbit S in $F(M)$ is of type B if and only if the restrictions of the connection forms ω_{ij} and the canonical forms ω_i to S satisfy the following relations;*

$$\omega_{12} = 0, \qquad \omega_{31} = \omega_1, \qquad \omega_{32} = -\omega_2\,. \tag{1.2}$$

PROOF. If S is of type B, (1.2) is obtained by the uniqueness of the connection forms. Conversely, if (1.2) is satisfied, then (1.1) is an easy consequence of the structure equations.

For a frame $u \in F(M)$, a mapping φ_u of G into $F(M)$ is defined by $\varphi_u(g) = g(u)$ for $g \in G$. Then φ_u is a diffeomorphism of G onto the orbit $G(u)$ of u and we have the following relations:

$$\varphi_u \circ L_g = g \circ \varphi_u, \qquad \varphi_u \circ R_g = \varphi_{g(u)} \qquad \text{for } g \in G\,, \tag{1.3}$$

$$R_\alpha \circ \varphi_u = \varphi_{u\alpha} \qquad \text{for } \alpha \in O(3)\,. \tag{1.4}$$

LEMMA 1.2. *If M is a B-manifold, an isotropy subgroup of G at a point* *of M is discrete and the dimension of G is equal to* 3.

PROOF. Let H be an isotropy subgroup of G at a point p of M and take a frame u of type B at p. $\varphi_u(H)$ is contained in the intersection of $G(u)$ and the fiber $\pi^{-1}(p)$ where π is a projection of $F(M)$ onto M. Let X be a vector of the Lie algebra of H. Then $d\varphi_u(X)$ is tangent to the fibre $\pi^{-1}(p)$ at u, so $\omega_i(d\varphi_u(X))=0$ $(i=1, 2, 3)$. On the other hand $d\varphi_u(X)$ is tangent to the orbit $G(u)$, using (1.2), we obtain $\omega_{ij}(d\varphi_u(X))=0$ $(i, j=1, 2, 3)$. Therefore we have $d\varphi_u(X)=0$ and thus $X=0$. This means that the Lie algebra of H consists of only null element and consequently H is discrete and $\dim G=3$.

Let us denote the Lie algebra of G by $\mathfrak{g}$. For a frame u in $F(M)$, we can attach a basis E_1^u, E_2^u, E_3^u of $\mathfrak{g}$ by

$$\omega_i(d\varphi_u(E_j^u))=\delta_{ij} \qquad (i, j=1, 2, 3)\,. \tag{1.5}$$

LEMMA 1.3. *For a frame $u\in F(M)$, $g\in G$ and $\alpha=(\alpha_{ij})\in O(3)$, we have*

$$E_j^{g(u)\alpha}=\sum_{i=1}^{3}\alpha_{ij}\operatorname{Ad}(g)E_i^u \qquad (j=1, 2, 3)\,. \tag{1.6}$$

PROOF. On account of (1.3) and (1.4) we have

$$\varphi_{g(u)\alpha}=R_\alpha\circ\varphi_u\circ R_g\,.$$

Therefore, using a formula

$$R_\alpha^*\omega_k=\sum_{h=1}^{3}\alpha_{hk}\omega_h\,,$$

we have

$$\begin{aligned}\omega_k(d\varphi_{g(u)\alpha}(\sum_{i=1}^{3}\alpha_{ij}\operatorname{Ad}(g)E_i^u)) &= \omega_k(dR_\alpha d\varphi_u dR_g(\sum_{i=1}^{3}\alpha_{ij}dR_g^{-1}dL_g(E_i^u)))\\ &=\sum_{i=1}^{3}\alpha_{ij}\omega_k(dR_\alpha d\varphi_u dL_g(E_i^u))\\ &=\sum_{h,i=1}^{3}\alpha_{ij}\alpha_{hk}\omega_h(dg d\varphi_u(E_i^u))\\ &=\sum_{h,i=1}^{3}\alpha_{ij}\alpha_{hk}\omega_h(d\varphi_u(E_i^u))\\ &=\sum_{i=1}^{3}\alpha_{ij}\alpha_{ik}\\ &=\delta_{jk}\,.\end{aligned}$$

Thus by the definition of $E_j^{g(u)\alpha}$ we have the lemma.

LEMMA 1.4. *If $u\in F(M)$ is a frame of type B, we have*

$$[E_1^u, E_2^u]=0, \qquad [E_1^u, E_3^u]=-E_1^u, \qquad [E_2^u, E_3^u]=E_2^u\,. \tag{1.7}$$

PROOF. Put $\bar{\omega}_i=\varphi_u^*\omega_i$ $(i=1, 2, 3)$, $\bar{\omega}_i$ are the left invariant 1-forms on G.

If $[E_j^u, E_k^u]=\sum_{i=1}^{3} C_{jk}^i E_i^u$, we have

$$d\bar{\omega}_i=-\frac{1}{2}\sum_{j,k=1}^{3} C_{jk}^i \bar{\omega}_j \wedge \bar{\omega}_k .$$

Since u is a frame of type B, (1.1) shows us that $C_{jk}^i=0$ except $C_{13}^1=-1$ and $C_{23}^1=1$. Therefore the lemma is proved.

LEMMA 1.5. *If M is a B-manifold, the commutator subgroup G' of G is a 2 dimensional simply connected closed normal abelian subgroup of G and the commutator ideal $\mathfrak{g}'=[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is spanned by E_1^u and E_2^u where u is a frame of type B.*

PROOF. Let u be a frame of type B. From Lemma 1.4 we know that $\mathfrak{g}'$ is an abelian ideal of $\mathfrak{g}$ spanned by E_1^u and E_2^u. Hence the commutator subgroup G' of G is a 2 dimensional connected abelian normal subgroup of G. If the closure $\bar{G}'$ of G' is distinct from G', the dimension of G' must be 3. So $G'=G$ which is a contradiction because G is not abelian.

If $\exp X$ is an identity element of G for $X\in\mathfrak{g}'$, we have $\mathrm{Ad}(\exp X)E_3^u=E_3^u$. On the other hand if we put $X=\xi E_1^u+\eta E_2^u$, we have from Lemma 1.4, $\mathrm{Ad}(\exp X)E_3^u=-\xi E_1^u+\eta E_2^u+E_3^u$. Therefore we obtain $\xi=\eta=0$. This shows that G' is simply connected and the lemma is proved.

LEMMA 1.6. *If M is a B-manifold and u is a frame of type B, then any element $g\in G$ can be written uniquely as*

$$g=g_0\exp tE_3^u=\exp\xi E_1^u\cdot\exp\eta E_2^u\cdot\exp tE_3^u \tag{1.8}$$

where $g_0=\exp\xi E_1^u\cdot\exp\eta E_2^u\in G'$.

PROOF. Since E_1^u, E_2^u, E_3^u form a basis of $\mathfrak{g}$, it is well known that there exists a neighborhood U of the identity element of G such that an element g of U can be written as (1.8).

Since G is generated by U and G' is a normal subgroup of G it is easily shown that any element of G is also written as (1.8). If $g_0\exp tE_3^u=g_0'\exp t'E_3^u$ for some elements $g_0, g_0'\in G'$ and real numbers t and t', we have $\exp(t-t')E_3^u=g_0^{-1}g_0'\in G'$. Then $\mathrm{Ad}(\exp(t-t')E_3^u)X=X$ for any $X\in\mathfrak{g}'$. But from Lemma 1.4 we have $\mathrm{Ad}(\exp(t-t')E_3^u)E_1^u=e^{t-t'}E_1^u$, which implies that $t=t'$ and therefore $g_0=g_0'$. This completes the proof of Lemma 1.6.

LEMMA 1.7. *If M is a B-manifold, the action of G on M is simply transitive.*

PROOF. Let u be a frame of type B. Assume that $g\in G$ leaves a point $\pi(u)$ fixed. Since $g(u)$ is also a frame at $\pi(u)$, there exists an orthogonal matrix $\alpha=(\alpha_{ij})\in O(3)$ such that $g(u)=u\alpha$. Then from Lemma 1.3 we have $E_j^{g(u)}=\mathrm{Ad}(g)E_j^u$. On the other hand $E_j^{g(u)}=E_j^{u\alpha}$ implies that $E_j^{g(u)}=\sum_{i=1}^{3}\alpha_{ij}E_i^u$ from Lemma 1.3. This means that the matrix representation of $\mathrm{Ad}(g)$ with respect to the basis E_1^u, E_2^u, E_3^u is the orthogonal matrix $\alpha=(\alpha_{ij})$. If we put

$$g=\exp \xi E_1^u \cdot \exp \eta E_2^u \cdot \exp t E_3^u ,$$

we have

$$\mathrm{Ad}(g)E_1^u=e^t E_1^u, \quad \mathrm{Ad}(g)E_2^u=e^{-t}E_2^u, \quad \mathrm{Ad}(g)E_3^u=-\xi E_1^u+\eta E_2^u+E_3^u .$$

Therefore the matrix representation of $\mathrm{Ad}(g)$ is

$$\begin{pmatrix} e^t & 0 & -\xi \\ 0 & e^{-t} & \eta \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.9}$$

It is easily seen that the matrix (1.9) is orthogonal if and only if $\xi=\eta=t=0$. This shows that g is the identity element of G. The lemma is proved.

The correspondence of $g=\exp \xi E_1^u \cdot \exp \eta E_2^u \cdot \exp t E_3^u$ and the matrix (1.9) define an isomorphism of the group G into the group of matrices of degree 3. The corresponding group is just the group introduced in the introduction.

THEOREM 1. *A Riemannian manifold M is a B-manifold if and only if there exists a diffeomorphism of M and R^3 such that the Riemannian metric of M is expressed in this diffeomorphism by*

$$ds^2=e^{-2t}d\xi^2+e^{2t}d\eta^2+dt^2 . \tag{1.10}$$

PROOF. Assume that M is a B-manifold and fix a frame u of type B. Then a correspondence

$$(\xi, \eta, t) \leftrightarrow \pi\varphi_u(\exp \xi E_1^u \cdot \exp \eta E_2^u \cdot \exp t E_3^u)$$

of R^3 and M defines a diffeomorphism of R^3 and M. Identifying R^3 and M by this diffeomorphism, we have

$$\begin{cases} \partial/\partial\xi = dgd\pi d\varphi_u(\mathrm{Ad}(g^{-1})E_1^u), \quad \partial/\partial\eta = dgd\pi d\varphi_u(\mathrm{Ad}(g^{-1})E_2^u), \\ \partial/\partial t = dgd\pi d\varphi_u(E_3^u), \end{cases} \tag{1.11}$$

where $g=\exp \xi E_1^u \cdot \exp \eta E_2^u \cdot \exp t E_3^u$. From Lemma 1.3 we have

$$\mathrm{Ad}(g^{-1})E_1^u=e^{-t}E_1^u , \qquad \mathrm{Ad}(g^{-1})E_2^u=e^t E_2^u .$$

Therefore (1.11) is reduced to

$$\partial/\partial\xi=e^{-t}dgd\pi d\varphi_u(E_1^u), \quad \partial/\partial\eta=e^t dgd\pi d\varphi_u(E_2^u), \quad \partial/\partial t=dgd\pi d\varphi_u(E_3^u) .$$

It is easily seen that the frame $g(u)$ is written as

$$g(u)=(\pi\varphi_u(g)\, ;\ dgd\pi d\varphi_u(E_1^u),\ dgd\pi d\varphi_u(E_2^u),\ dgd\pi d\varphi_u(E_3^u)) .$$

So the metric of M can be written as (1.10) in this coordinate system.

Conversely assume that there exists a diffeomorphism of M and R^3 with the property stated in the theorem, then M is isometric to a B-manifold by the first part of the theorem, and thus M is a B-manifold.

§ 2. Isometric immersions and B-manifold.

In this section we shall assume that M admits an isometric immersion f of the type number 2 into the hyperbolic space H^4. H^4 is the 4 dimensional simply connected complete Riemannian manifold of constant curvature -1. Then, as in [1], we have a bundle isomorphism $\tilde{f}$ of $F(M)$ into the bundle $F_0(H^4)$ of the oriented frames of H^4.

If we put

$$\phi_i=\tilde{f}^*\theta_{4i} \qquad (i=1, 2, 3)$$

where θ_{AB} $(A, B=1, 2, 3, 4)$ are the connection forms in $F_0(H^4)$, then we have

$$\phi_i=\sum_{j=1}^{3} H_{ij}\omega_j, \quad H_{ij}=H_{ji} \qquad (i, j=1, 2, 3)\,. \tag{2.1}$$

H_{ij} $(i, j=1, 2, 3)$ are the functions on $F(M)$ and we call them the coefficients of the second fundamental forms of the immersion f.

The following formulas are useful.

$$d\phi_i+\sum_{j=1}^{3}\omega_{ij}\wedge\phi_j=0 \qquad (i=1, 2, 3)\,, \tag{2.2}$$

$$\Omega_{ij}=d\omega_{ij}+\sum_{k=1}^{3}\omega_{ik}\wedge\omega_{kj}=-\omega_i\wedge\omega_j+\phi_i\wedge\phi_j \qquad (i, j=1, 2, 3)\,, \tag{2.3}$$

where Ω_{ij} $(i, j=1, 2, 3)$ are the curvature forms.

If $H_{3j}=0$ $(j=1, 2, 3)$ at a frame $u\in F(M)$, u is called a frame of type A (with respect to the immersion f). If a G-orbit S in $F(M)$ contains a frame of type A, S is called a G-orbit of type A. By Lemma 3.1 in [1], we know that every frame contained in a G-orbit of type A is also of type A. Also, by Lemma 3.2 in [1], the restrictions of the connection forms and the canonical forms to a G-orbit of type A satisfy the following relations;

$$\begin{cases} \omega_{31}=a\omega_1+b\omega_2 \\ \omega_{32}=c\omega_1-a\omega_2 \end{cases} \tag{2.4}$$

where a, b, c are constant. Moreover, in the proof of Lemma 3.4 in [1], we have obtained

$$2aH_{12}-bH_{11}+cH_{22}=0 \tag{2.5}$$

on a G-orbit of type A.

In the following Lemma 2.1, 2.2, 2.3 and 2.4 we shall assume that S is a G-orbit in $F(M)$ of type A and all the functions and the differential forms are assumed to be restricted to S.

LEMMA 2.1. *The constants a, b, c in (2.4) satisfy*

$$a^2+bc=1 \tag{2.6}$$

and we have

$$\omega_{12}=0 \tag{2.7}$$

on S.

PROOF. By the exterior differentiation of (2.4) we have

$$-2a\omega_{12}\wedge\omega_2+\{(b+c)\omega_{12}-(a^2+bc-1)\omega_3\}\wedge\omega_1=0\,, \tag{2.8}$$

$$2a\omega_{12}\wedge\omega_1+\{(b+c)\omega_{12}+(a^2+bc-1)\omega_3\}\wedge\omega_2=0\,. \tag{2.9}$$

Making an exterior product of (2.8) with ω_1 we have

$$2a\omega_{12}\wedge\omega_1\wedge\omega_2=0\,.$$

Therefore, there exist constants λ and μ such that

$$a\omega_{12}=\lambda\omega_1+\mu\omega_2\,. \tag{2.10}$$

Substituting (2.10) in the first term of (2.8) and (2.9), we have

$$\{2\lambda\omega_2+(b+c)\omega_{12}-(a^2+bc-1)\omega_3\}\wedge\omega_1=0\,,$$

$$\{-2\mu\omega_1+(b+c)\omega_{12}+(a^2+bc-1)\omega_3\}\wedge\omega_2=0\,.$$

From these equations we can obtain

$$2\lambda\omega_2+(b+c)\omega_{12}-(a^2+bc-1)\omega_3=\alpha\omega_1\,, \tag{2.11}$$

$$-2\mu\omega_1+(b+c)\omega_{12}+(a^2+bc-1)\omega_3=\beta\omega_2\,, \tag{2.12}$$

for some constants α and β. Subtracting (2.12) from (2.11), we have

$$2\mu\omega_1+2\lambda\omega_2-2(a^2+bc-1)\omega_3=\alpha\omega_1-\beta\omega_2\,.$$

Therefore we obtain

$$a^2+bc=1,\qquad \alpha=2\mu,\qquad \beta=-2\lambda$$

and consequently we have

$$(b+c)\omega_{12}=2\mu\omega_1-2\lambda\omega_2\,. \tag{2.13}$$

Comparing (2.10) and (2.13) we have easily

$$a(\lambda^2+\mu^2)=0\qquad\text{and}\qquad(b+c)(\lambda^2+\mu^2)=0\,. \tag{2.14}$$

If $a=0$ and $b+c=0$, then $1=a^2+bc=-b^2\leqq 0$ which is a contradiction. So one of a or $b+c$ does not vanish which implies that $\lambda=\mu=0$ and from (2.10) or (2.13) we get $\omega_{12}=0$.

COROLLARY. *The dimensions of G and a G-orbit in F(M) are equal to 3.*

PROOF. From Lemma 2.1 we see that on a G-orbit S of type A, the connection forms are the linear combinations of the canonical forms. Then in the same way as the proof of Lemma 1.2, we can show that the isotropy subgroup of G at a point of M is discrete. Then the corollary is an easy consequence.

LEMMA 2.2. *On a G-orbit S of type A, we have*

$$H_{11}H_{22}-(H_{12})^2=2\,. \tag{2.15}$$

PROOF. From Lemma 2.1 the curvature form Ω_{12} is calculated as

$$\begin{aligned}\Omega_{12}&=d\omega_{12}+\omega_{13}\wedge\omega_{32}\\&=-(a\omega_1+b\omega_2)\wedge(c\omega_1-a\omega_2)\\&=(a^2+bc)\omega_1\wedge\omega_2\\&=\omega_1\wedge\omega_2\,.\end{aligned}$$

On the other hand, from (2.3), we have

$$\begin{aligned}\Omega_{12}&=-\omega_1\wedge\omega_2+\phi_1\wedge\phi_2\\&=(-1+H_{11}H_{22}-(H_{12})^2)\omega_1\wedge\omega_2\,.\end{aligned}$$

Therefore we can obtain (2.15).

LEMMA 2.3. *On a G-orbit S of type A, we have*

$$\left\{\begin{aligned}dH_{11}&=(aH_{11}+cH_{12})\omega_3\\dH_{22}&=(bH_{12}-aH_{22})\omega_3\\dH_{12}&=(bH_{11}-aH_{12})\omega_3\,.\end{aligned}\right. \tag{2.16}$$

PROOF. From the corollary of Lemma 2.1, we know that every 1-form on S is a linear combination of ω_1, ω_2, ω_3. We can put on S

$$dH_{ij}=\sum_{k=1}^{3}H_{ijk}\omega_k\qquad(i,j=1,2)\,,$$

then we have

$$H_{ijk}=H_{jik}\,. \tag{2.17}$$

From (2.2) and Lemma 2.1, we see that $d\phi_1=d\phi_2=0$ on S. Therefore we have

$$\begin{aligned}(-H_{112}+H_{121})\omega_1\wedge\omega_2+(-H_{113}+aH_{11}+cH_{12})\omega_1\wedge\omega_3&\\+(-H_{123}+bH_{11}-aH_{12})\omega_2\wedge\omega_3&=0\\(-H_{212}+H_{221})\omega_1\wedge\omega_2+(-H_{213}+aH_{12}+cH_{22})\omega_1\wedge\omega_3&\\+(-H_{223}+bH_{12}-aH_{22})\omega_2\wedge\omega_3&=0\end{aligned}$$

From these equations, we obtain

$$H_{113}=aH_{11}+cH_{12},\quad H_{123}=bH_{11}-aH_{12},\quad H_{223}=bH_{12}-aH_{22} \tag{2.18}$$

and

$$H_{112}=H_{121},\qquad H_{212}=H_{221}\,. \tag{2.19}$$

Therefore, to complete the proof of the lemma, it is sufficient to show the following;

$$H_{111}=H_{112}=H_{121}=H_{122}=H_{221}=H_{222}=0\,. \tag{2.20}$$

Differentiating (2.15) and (2.5) and using (2.17) and (2.19), we have

$$H_{22}H_{111}+H_{11}H_{221}-2H_{12}H_{112}=0 \tag{2.21}$$

$$H_{22}H_{112}+H_{11}H_{222}-2H_{12}H_{221}=0 \tag{2.22}$$

$$2aH_{112}-bH_{111}+cH_{221}=0 \tag{2.23}$$

$$2aH_{221}-bH_{112}+cH_{222}=0\,. \tag{2.24}$$

Multiplying b to (2.21) and H_{22} to (2.23) and making a sum of these equations, we have

$$(bH_{11}+cH_{22})H_{221}+2(aH_{22}-bH_{12})H_{112}=0\,.$$

Substituting $bH_{11}=2aH_{12}+cH_{22}$ in the first term, we obtain

$$(aH_{22}-bH_{12})H_{112}+(cH_{22}+aH_{12})H_{221}=0\,. \tag{2.25}$$

Similarly, we obtain from (2.22) and (2.24)

$$(aH_{12}-bH_{11})H_{112}+(cH_{12}+aH_{11})H_{221}=0\,. \tag{2.26}$$

The determinant of the coefficients of H_{112} and H_{221} in (2.25) and (2.26) is

$$\begin{vmatrix} aH_{22}-bH_{12} & cH_{22}+aH_{12} \\ aH_{12}-bH_{11} & cH_{12}+aH_{11} \end{vmatrix}=\begin{vmatrix} H_{22} & H_{12} \\ H_{12} & H_{11} \end{vmatrix}\begin{vmatrix} a & c \\ -b & a \end{vmatrix}=2\neq 0\,.$$

Therefore we obtain $H_{112}=H_{221}=0$. Because of (2.15), H_{11} and H_{22} do not vanish, so, from (2.21) and (2.22), we can obtain $H_{111}=H_{222}=0$. This completes the proof of the lemma.

LEMMA 2.4. *In the coefficients in* (2.4) *we have* $b=c$.

PROOF. Differentiating the first and the third equations in (2.16), we have

$$(b-c)(aH_{11}+cH_{12})=0\,,$$

$$(b-c)(bH_{11}-aH_{12})=0\,.$$

If $b\neq c$, we have $aH_{11}+cH_{12}=bH_{11}-aH_{12}=0$ which implies that $H_{11}=H_{12}=0$ from (2.6). But this is a contradiction to (2.15) and therefore we have $b=c$.

THEOREM 2. *If a connected homogeneous Riemannian manifold M of dimension 3 admits an isometric immersion f in H^4 of the type number 2, M is a B-manifold and any G-orbit in $F(M)$ of type B is also an orbit of type A with respect to f. Let S be a G-orbit of type B, the coefficients H_{ij} of the second fundamental form of f satisfy;* (1) $H_{12}=0$ *on S,* (2) *there exists a frame u_0 in S such that*

$$(2.27)\qquad \begin{cases} H_{11}(g_0\cdot\exp tE_3^{u_0}(u_0))=\pm\sqrt{2}\,e^t \\ H_{22}(g_0\cdot\exp tE_3^{u_0}(u_0))=\pm\sqrt{2}\,e^{-t} \end{cases}$$

where g_0 is an element of the commutator subgroup G' of G and t is a real number.

PROOF. Let S be a G-orbit in $F(M)$ of type A with respect to f. If we denote the restrictions of the connection forms and the canonical forms to S by ω_{ij} and ω_i ($i, i=1, 2, 3$) respectively, from Lemma 2.1 and 2.4, we have

$$(2.28)\qquad \omega_{12}=0,\quad \omega_{31}=a\omega_1+b\omega_2,\quad \omega_{32}=b\omega_1-a\omega_2$$

and

$$a^2+b^2=1\,.$$

From (2.28), we have also

$$(2.29)\qquad d\omega_1=(a\omega_1+b\omega_2)\wedge\omega_3,\quad d\omega_2=(b\omega_1-a\omega_2)\wedge\omega_3,\quad d\omega_3=0\,.$$

If we put $a=\cos 2\theta$ and $b=\sin 2\theta$, we can make an orthogonal matrix $\alpha=(\alpha_{ij})\in O(3)$ by

$$\alpha=\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $S'=R_\alpha(S)$ and denote the restrictions of the canonical forms to S' by ω'_i ($i=1, 2, 3$). Then we have

$$\begin{aligned} R_\alpha^*\omega'_1 &= \sum_{i=1}^{3}\alpha_{i1}\omega_i=\cos\theta\cdot\omega_1+\sin\theta\cdot\omega_2 \\ R_\alpha^*\omega'_2 &= \sum_{i=1}^{3}\alpha_{i2}\omega_i=-\sin\theta\cdot\omega_1+\cos\theta\cdot\omega_2 \\ R_\alpha^*\omega'_3 &= \omega_3\,. \end{aligned}$$

From these equations and (2.29), we have

$$\begin{aligned} R_\alpha^* d\omega'_1 &= \cos\theta\cdot d\omega_1+\sin\theta\cdot d\omega_2 \\ &= \{(\cos 2\theta\cdot\cos\theta+\sin 2\theta\cdot\sin\theta)\omega_1+(\sin 2\theta\cdot\cos\theta-\cos 2\theta\cdot\sin\theta)\omega_2\}\wedge\omega_3 \\ &= (\cos\theta\cdot\omega_1+\sin\theta\cdot\omega_2)\wedge\omega_3 \\ &= R_\alpha^*(\omega'_1\wedge\omega'_3)\,, \end{aligned}$$

which implies that $d\omega_1'=\omega_1'\wedge\omega_3'$. Similarly we obtain $d\omega_2'=-\omega_2'\wedge\omega_3'$ and $d\omega_3'=0$. Therefore S' is a G-orbit of type B and consequently M is a B-manifold.

Now changing the notation, assume that S is a G-orbit of type B. From (2.3) and Lemma 1.1, we have on S

$$\Omega_{12}=d\omega_{12}+\omega_{13}\wedge\omega_{32}=\omega_1\wedge\omega_2\,. \tag{2.30}$$

On the other hand, from (2.3), we have also

$$\begin{aligned}\Omega_{12}=(-1+H_{11}H_{22}-(H_{12})^2)\omega_1\wedge\omega_2+(H_{11}H_{23}-H_{13}H_{12})\omega_1\wedge\omega_3\\ +(H_{12}H_{23}-H_{13}H_{22})\omega_2\wedge\omega_3\,.\end{aligned} \tag{2.31}$$

Comparing (2.30) and (2.31), we obtain

$$H_{11}H_{22}-(H_{12})^2=2 \tag{2.32}$$

and

$$\begin{cases} H_{11}H_{23}-H_{12}H_{13}=0\\ H_{12}H_{23}-H_{22}H_{13}=0\,. \end{cases} \tag{2.33}$$

From (2.33), we can easily obtain $H_{13}=H_{23}=0$ on S. Then by the assumption of the type number of f, we have also $H_{33}=0$ on S. This means that S is an orbit of type A with respect to f. Moreover since $b=c=0$ in (2.5), we have $H_{12}=0$ on S.

Let u be a frame in S. Let X be an element of the commutator ideal $\mathfrak{g}'$ of $\mathfrak{g}$ and put $x_t=\exp tX$. Then from Lemma 2.3. we have

$$\frac{d}{dt}H_{11}(x_t(u))=H_{11}(x_t(u))\omega_3(d\varphi_u(\dot{x}_t))\,.$$

Since $\dot{x}_t=dL_{x_t}(X)$, we have

$$\omega_3(d\varphi_u(\dot{x}_t))=\omega_3(d\varphi_u dL_{x_t}(X))=\omega_3(d\varphi_u(X))=0\,,$$

because $\mathfrak{g}'$ is spanned by E_1^u and E_2^u. Therefore we obtain

$$H_{11}(x_t(u))=H_{11}(u)\,. \tag{2.34}$$

Similarly we can obtain

$$H_{22}(x_t(u))=H_{22}(u)\,. \tag{2.35}$$

(2.34) and (2.35) mean that H_{11} and H_{22} are constant on the G'-orbit in S.

Next we put $y_t=\exp tE_3^u$, we have from Lemma 2.3,

$$\frac{d}{dt}H_{11}(y_t(u))=H_{11}(y_t(u)),\qquad \frac{d}{dt}H_{22}(y_t(u))=-H_{22}(y_t(u))\,,$$

from which we have

$$H_{11}(y_t(u))=H_{11}(u)e^t,\quad H_{22}(y_t(u))=H_{22}(u)e^{-t}\,. \tag{2.36}$$

Put $t_0=\frac{1}{2}(\log H_{22}(u)-\log H_{11}(u))$ and $u_0=y_{t_0}(u)$, we have

$$H_{11}(u_0)=H_{22}(u_0)=\pm\sqrt{2}\,. \tag{2.37}$$

Consequently, for $g_0\in G'$ and a real number t we obtain

$$\begin{cases} H_{11}(g_0\exp tE_3^{u_0}(u_0))=H_{11}(\exp tE_3^{u_0}(u_0))=\pm\sqrt{2}\,e^t \\ H_{22}(g_0\exp tE_3^{u_0}(u_0))=H_{22}(\exp tE_3^{u_0}(u_0))=\pm\sqrt{2}\,e^{-t}\,. \end{cases}$$

This completes the proof of the theorem.

A frame u_0 in Theorem 2 is called a frame of type B_0.

§ 3. Existence and uniqueness of the immersion

In § 2, we have seen that if a homogeneous Riemannian manifold of dimension 3 admits an isometric immersion in H^4 of the type number 2, it is a B-manifold. Thus to study an isometric immersion of such a Riemannian manifold it is sufficient to consider a B-manifold.

The following theorem gives a complete determination of the isometric immersion of a B-manifold in H^4 of the type number 2.

THEOREM 3. *A B-manifold M admits a unique isometric immersion in H^4 of the type number 2, here the uniqueness is in the following sense: If f_1 and f_2 are isometric immersions of M of type number 2, there exist an isometry k of H^4 and an isometry g of M such that $k\circ f_1=f_2\circ g$.*

PROOF. First we prove the uniqueness. Let f_1 and f_2 be isometric immersions of M in H^4 of the type number 2. We denote the coefficients of the second fundamental forms of f_1 and f_2 by $H_{ij}^{(1)}$ and $H_{ij}^{(2)}$ respectively.

Fix a G-orbit S in $F(M)$ of type B. Then from Theorem 2, there are frames in S of type B_0 with respect to f_1 and f_2 which are denoted by u_1 and u_2 respectively and put $u_2=g(u_1)$ $(g\in G)$. Put $f=f_2\circ g$. Then f is also an isometric immersion of M in H^4. Denoting the coefficients of the second fundamental form of f by H_{ij}, we can easily obtain the following relations of H_{ij} and $H_{ij}^{(2)}$:

$$H_{ij}=H_{ij}^{(2)}\circ g\,.$$

For $u\in S$, we take $x\in G$ such that $u=x(u_1)$ and put $x=x_0\exp tE_3^{u_1}$ $(x_0\in G')$. Then we have

$$H_{ij}(u)=H_{ij}^{(2)}(gu)=H_{ij}^{(2)}(gx(u_1))=H_{ij}^{(2)}(gx_0g^{-1}\cdot\exp t\,\mathrm{Ad}(g)E_3^{u_1}\cdot g(u_1))\,.$$

From Lemma 1.3 and 1.5, we see that

$$gx_0g^{-1}\in G' \qquad \text{and} \qquad \mathrm{Ad}\,(g)E_3^{u_1}=E_3^{u_2}\,.$$

Hence we obtain

$$H_{11}(u)=H_{11}^{(2)}(gx_0g^{-1}\exp tE_3^{u_2}(u_2))=\pm\sqrt{2}\,e^t$$

$$H_{22}(u)=H_{22}^{(2)}(gx_0g^{-1}\exp tE_3^{u_2}(u_2))=\pm\sqrt{2}\,e^{-t}\,.$$

On the other hand we also obtain

$$H_{11}^{(1)}(u)=H_{11}^{(1)}(x_0\exp tE_3^{u_1}(u_1))=\pm\sqrt{2}\,e^t$$

$$H_{22}^{(1)}(u)=H_{22}^{(1)}(x_0\exp tE_3^{u_1}(u_1))=\pm\sqrt{2}\,e^{-t}\,.$$

Therefore we have $H_{ij}=H_{ij}^{(1)}$ on S. Then, from Theorem 1.1 in [**1**], there exists an isometry k of H^4 such that $k\circ f_1=f=f_2\circ g$.

To show the existence of the immersion, it suffices to give an example.

From Theorem 1, M is covered by one coordinate system (ξ, η, t) such that the Riemannian metric ds of M is written as

$$ds^2=e^{-2t}d\xi^2+e^{2t}d\eta^2+dt^2\,.$$

H^4 is also covered by one coordinate system (x_1, x_2, x_3, x_4) such that the Riemannian metric $d\tilde{s}$ of H^4 is written as

$$d\tilde{s}^2=\sum_{i=1}^4 dx_i^2-\frac{\left(\sum_{i=1}^4 x_i dx_i\right)^2}{1+\sum_{i=1}^4 x_i^2}\,.$$

In these coordinate systems of M and H^4, consider an immersion of M in H^4 given by

$$x_1=\frac{e^{-t}}{\sqrt{2}}\cos\sqrt{2}\,\xi,\qquad x_2=\frac{e^{-t}}{\sqrt{2}}\sin\sqrt{2}\,\xi,$$
$$x_3=\frac{e^{t}}{\sqrt{2}}\cos\sqrt{2}\,\eta,\qquad x_4=\frac{e^{t}}{\sqrt{2}}\sin\sqrt{2}\,\eta\,. \tag{3.1}$$

The straightforward calculation shows that this immersion is isometric and the type number is 2. Thus the theorem is proved.

The immersion given by (3.1) is not an imbedding but a covering map. So, from the uniqueness of the immersion, we have the following corollary.

COROLLARY. *A homogeneous Riemannian manifold of dimension* 3 *does not admit an isometric imbedding in* H^4 *of the type number* 2.

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Bibliography

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