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Some concepts of recursiveness on admissible ordinals

By Masaru FUKUYAMA

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There have been proposed several generalized concepts of recursiveness on domains other than the natural numbers. In this paper we show four of them virtually coincide over admissible ordinals. Some of the facts presented below have already been obtained (cf. [4], [5] etc.) but as far as the author knows their explicit proofs are published for the first time here.

§ 1. Takeuti-Kino-Tugué's concept of recursiveness ([9] and [12]).

1.1. Let be given an arbitrary ordinal α. Define $TF_n(\alpha)$ (resp. $PF_n(\alpha)$) to be the set of n-ary total (resp. partial) functions with variables ranging over α and with values in α. $Fc(\alpha)$ (resp. $Pf(\alpha)$) is the set of total (resp. partial) functions such that a) they have finitely many (possibly zero) function variables each of which ranges over $TF_n(\alpha)$ (resp. $PF_n(\alpha)$) for a fixed $n \geqq 1$; b) they have finitely many (at least one) number variables ranging over α; and c) their values are in α. Hereafter letters $a, b, c, d, e, a_1, b_1, c_1, d_1, e_1, \cdots$ denote ordinals less than an ordinal α fixed in each context.

If α is an ordinal greater than ω and closed under j[1], then we can single out the primitive recursive functions on α from $Fc(\alpha)$ by Schemata I$\sim$XII and XIII$'$ of [12], 2.1.

1.2. Let α be as in 1.1. We call a function in $Pf(\alpha)$ T-partial recursive if it is defined by the schemata obtained from Schemata I$\sim$XII, XIII$'$ by replacing each occurrence of '$=$' by '$\cong$', and the additional schema XIV. (cf. [12], 2.1.)

1.3. Again α is greater than ω and closed under j. A function in $Pf(\alpha)$ is T-partial recursive in the classical sense if it is obtained by the schemata used to introduce the T-partial recursive functions in 1.2 and the additional schemata (0_β) for each β.

(0_β). $f(a)=\beta$, where β is a fixed ordinal less than α.

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1) See, [12].

§ 2. Kripke's concept of recursiveness ([4] and [5]).

2.1. Kripke defines, for an arbitrary but fixed ordinal α, a formal calculation system $\mathcal{K}(\alpha)$. For the definition of $\mathcal{K}(\alpha)$, see [4] or [5]. We mention only the rules of inference of $\mathcal{K}(\alpha)$. We denote by $\bar{a}$ the numeral corresponding to an ordinal a $(<\alpha)$.

$R1$: to pass from an equation d containing a free variable u to the equation which results from d by substituting a numeral for u.

$R2$: to pass from an equation $r=s$ containing no free variables and an equation $t=\bar{a}$ where t is a term containing no free variables to the equation which results from $r=s$ by replacing an occurrence of t in s by $\bar{a}$.

$R3$a: to pass from an equation of the form $t_u[\bar{a}]=\bar{0}$ containing no free variables to the equation $(\exists u<\bar{b})t=\bar{0}$, where $a<b$.

$R3$b: to pass from a sequence of equations $\langle t_u[\bar{a}]=\bar{1} \mid a<b\rangle$ to the equation $(\exists u<\bar{b})t=\bar{1}$, where each equation $t_u[\bar{a}]=\bar{1}$ contains no free variables.

We call a term (an equation) containing no numerals other than $\bar{0}$ and $\bar{1}$ a term (an equation) in the strict sense.

2.2. A function $f(\in Pf(\alpha))$ is K-partial recursive (in the strict sense) if the following holds in $\mathcal{K}(\alpha)$. There exists a system E of equations (in the strict sense) such that

$$E\binom{h_1, \cdots, h_m}{g_1, \cdots, g_m}, \quad E\vdash f(\bar{a}_1, \cdots, \bar{a}_n)=\bar{a}$$
$$\text{if and only if} \quad f(h_1, \cdots, h_m, a_1, \cdots, a_n)\cong a\,. \tag{1}$$

§ 3. Kripke's recursiveness comprises Takeuti-Kino-Tugué's recursiveness.

THEOREM 1. *Let α be an ordinal greater than ω and closed under j. If $f(\in Pf(\alpha))$ is T-partial recursive (T-partial recursive in the classical sense), then f is K-partial recursive in the strict sense (K-partial recursive).*

PROOF. We can provide in order each of the following functions and operations on functions with a system of equations for calculating it using the systems for its predecessors in the list.

a) $\lambda a\cdot 0$ (Schema IIa). b) $\lambda a\cdot a$ (III). c) sg. d) $\overline{sg}$. e) Iq (IV). f) $\lambda ha_1, \cdots, a_n\cdot h(a_1, \cdots, a_n)$ (VIII). g) The composition of functions (IX). h) The additions of a variable (Xa, b; XIa, b). i) Primitive recursion (XIII). j) The μ-operation (XIV). k) $\lambda a\cdot a'$ (I). l) sup*. m) $+$. n) $\times$. o) k $(=\lambda aj(a, 0))$. p) j (VI). q) The infinitary addition. r) The bounded existential quantification. s) The bounded μ-operation (XII). t) $\lambda a\cdot\omega$ (IIb). u) g^1 (VIIa). v) g^2 (VIIb). w) $\lambda a\cdot\beta$ (0_β).

We use the operation sup* in the following sense.

$\sup^* \{f(h_1, \cdots, h_m, a_1, \cdots, a_n, b) \mid b < a\} \cong c$ if and only if

a) $f(h_1, \cdots, h_m, a_1, \cdots, a_n, b)$ is defined for every $b < a$; and

b) c is the least ordinal (less than α) that is greater than or equal to $f(h_1, \cdots, h_m, a_1, \cdots, a_n, b)$ for all $b < a$.

The infinitary addition Σ is an operation on functions defined as follows.

$$\Sigma\{f(h_1, \cdots, h_m, a_1, \cdots, a_n, b) \mid b < 0\} \cong 0;$$

$$\begin{aligned}&\Sigma\{f(h_1, \cdots, h_m, a_1, \cdots, a_n, b) \mid b < a'\}\\ &\quad\cong \{f(h_1, \cdots, h_m, a_1, \cdots, a_n, b) \mid b < a\} + f(h_1, \cdots, h_m, a_1, \cdots, a_n, a);\end{aligned}$$

$$\begin{aligned}&\Sigma\{f(h_1, \cdots, h_m, a_1, \cdots, a_n, b) \mid b < a\}\\ &\quad\cong \sup^* \{\Sigma\{f(h_1, \cdots, h_m, a_1, \cdots, a_n, b) \mid b < c\} \mid c < a\},\end{aligned}$$

if a is a limit ordinal.

We exhibit for example a system E of equations for sup*. The principal function letter and the given function letter of E are f and g respectively.

$$E\begin{cases} E_1: \text{A system of equations for } \overline{sg} \text{ with the principal function letter } \widehat{\overline{sg}}.\\ E_2: \text{A system of equations for } Iq \text{ with the principal function letter } \widehat{Iq}.\\ h(u_1, \cdots, u_n, 0, w) = 0\\ h(u_1, \cdots, u_n, v, w) = \widehat{\overline{sg}}(k_1(u_1, \cdots, u_n, v, w))\\ k_1(u_1, \cdots, u_n, v, w) = (\exists x < v) k_2(x, u_1, \cdots, u_n, w)\\ k_2(u, u_1, \cdots, u_n, w) = \widehat{Iq}(w, g(u_1, \cdots, u_n, u))\\ E_0: \text{A system of equations for the } \mu\text{-operator with the principal function}\\ \qquad \text{letter } f \text{ and the given function letter } h.\end{cases}$$

As for a system of equations for the recursion scheme the system of [**12**] can be adopted without any change. We can construct systems for a)$\sim$v) so that they are all in the strict sense but for w) this is not necessarily the case.

The proof of the 'if' part of §2(1) is a routine and the 'only if' part is accomplished by Hermes' semantical method in [2] modified to fit the case involving partial functions.

§4. Admissible ordinals.

4.1. Given a system E of equations of $K(\alpha)$, define, for each ordinal γ, a set S_γ^E of equations of $K(\alpha)$ by

$$S_0^E = E;$$

and for $\gamma > 0$, $S_\gamma^E = E \cup$ the set of all immediate consequences (by $R1 \sim R3$b) of members of $\cup\{S_\delta^E \mid \delta<\gamma\}$; when we apply R3b to a subset of $\cup\{S_\delta^E \mid \delta<\gamma\}$ we require that the ordinal bounding the existential quantifier be less than γ.

We call α admissible if $S_\alpha^E = S_{\alpha+1}^E$ for every system E of equations of $\mathcal{K}(\alpha)$.

4.2. Consider the system $\mathcal{K}(\alpha)$. We call α recursively regular if $\sup \Re f < \alpha$[2] for every K-partial recursive function f on α whose domain is a proper initial segment of α.

4.3. Denote by $\mathcal{L}$ the language of set theory introduced in [**1**], P. 231. The Δ_0-, Σ_1- and Σ-formulas of $\mathcal{L}$ are defined there, as well as the notions of a relation on a set A being $\Delta_0(\Sigma_1, \Sigma, \boldsymbol{\Delta}_0, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma})$ over A. For any Σ-formula φ of $\mathcal{L}$ and a variable x not occurring in φ, $\mathrm{Rel}(x, \varphi)$ is a formula obtained from φ by relativizing to x all the quantifiers of φ.

We introduce the next schemata in $\mathcal{L}$.

Σ_1-replacement-reflection schema:

$$\forall u \in x \exists v \varphi[u, v] \rightarrow \exists y((\forall u \in x)(\exists v \in y)\varphi[u, v] \wedge (\forall v \in y)(\exists u \in x)\varphi[u, v])$$

for all Σ_1-formulas φ with y not free in φ.

Σ-reflection schema:

$$\varphi \rightarrow \exists y \,\mathrm{Rel}(y, \varphi)$$

for all Σ-formulas φ with y not occurring in φ.

Δ_0-separation schema:

$$\forall x \exists y \forall u(u \in y \leftrightarrow u \in x \wedge \varphi[u])$$

for all Δ_0-formulas with y not free in φ.

Σ_1-replacement schema:

$$\forall u \in x \exists !\, v \varphi[u, v] \rightarrow \exists y \forall v(v \in y \leftrightarrow \exists u \in x \varphi[u, v])$$

for all Σ_1-formulas φ with y not free in φ.

A set A is admissible if it satisfies a)$\sim$d). ([**6**], 2.5.)

a) A is non-empty and transitive.

b) If p and q are in A, then $\{p, q\}$ is in A.

c) If p is in A, then the transitive closure of p is in A.

d) The Σ_1-replacement-reflection axioms hold in A.

4.4. When we treat an ε-structure $\langle U, E \upharpoonright U^2\rangle$ (where E is Gödel's E) for $\mathcal{L}$, we write simply $U \models \varphi$ for $\langle U, E \upharpoonright U^2\rangle \models \varphi$. Hereafter letters $p, q, r, h,$

2) For a set X of ordinals, $\sup X$ is the least ordinal greater than all the ordinals in X.

$e, p_1, q_1, \cdots$ denote elements of the universe of a given structure.

LEMMA 1. *Let A be transitive and the Σ_1-replacement-reflection axioms hold in A. If in addition $\forall p \forall q(p, q \in A \rightarrow p \frown q \in A)$, then the Δ_0-separation axioms and Σ_1-replacement axioms hold in A.*

LEMMA 2. (cf. [1], 2.1 and 2.2.) *Let A be admissible.*

i) *Every instance of Δ_0-separation, Σ_1-replacement and Σ-reflection schemata holds in A.*

ii) *If $p, q \in A$, then $\cup p$, $\langle p, q\rangle$, $p \times q$, $p - q$, $\mathfrak{D}p$ and $\mathfrak{R}p$ are in A.*

iii) *Every subset of A which is Σ over A is Σ_1 over A.*

iv) *If a subset X of A is Δ_1 over A and $X \subseteq a$ for some $a \in A$ then $X \in A$.*

v) *Let G be a function with domain and range subsets of A such that the (graph of) G is Σ_1 over A. If $p \in A$ is a subset of the domain of G, then $G \upharpoonright p \in A$.*

4.5. Let F be Gödel's function for generating the constructible sets.

THEOREM 2. (cf. [5].) *The next five conditions are equivalent for ordinals greater than ω.*

i) *α is admissible.*

ii) *α is recursively regular.*

iii) *α is closed under j, and for every primitive recursive relation P on α, $\forall a_1 \cdots \forall a_n \forall c \exists d(\forall a(a < c \rightarrow \exists ! b P(a, b, a_1, \cdots, a_n)) \rightarrow \forall a \forall b(a < c \wedge P(a, b, a_1, \cdots, a_n) \rightarrow b < d))$.*

iv) *$F``\alpha$ is admissible and $F``\alpha \frown On = \alpha$.*

v) *There exists an admissible set A such that $A \frown On = \alpha$.*

PROOF. 4.5.1. i) implies ii). By reductio ad absurdum. Suppose that there were a K-partial recursive function f such that $\mathfrak{D}f = a_1(<\alpha)$ and $\sup \mathfrak{R}f = \alpha$. Let g be defined as $g(a, b) \cong \sup^* \{Iq(f(c), b) \mid c < a\}$. By the method used to prove Theorem 1 construct a system E_1 of equations for g with the principal function letter l. E_1 contains a system of equations for f with the principal function letter h. Construct a system E from E_1 in the following way.

$$E \begin{cases} E_1 \\ g_1(u, v) = (\exists w < h(v)) l(u, w) \\ g_2(u) = (\exists v < u) g_1(u, v) . \end{cases}$$

Then, $g_2(a_1) = \bar{1}$ is deducible from E but $g_2(\bar{a}_1) = \bar{1} \notin S_\alpha^E$: Fix an arbitrary $c < a_1$. For any $b < f(c)$, $E \vdash l(\bar{a}_1, \bar{b}) = \bar{1}$, since $g(a_1, b) = \bar{1}$ for all b. So we have the next deduction from E.

$$\dfrac{\dfrac{\dfrac{\vdots}{g_1(\bar{a}_1, \bar{c}) = (\exists w < h(\bar{c}))\, l(\bar{a}_1, w)} \qquad \dfrac{\vdots}{h(\bar{c}) = \overline{f(c)}}}{g_1(\bar{a}_1, \bar{c}) = (\exists w < \overline{f(c)})\, l(\bar{a}_1, w)} \qquad \dfrac{\vdots}{(\exists w < \overline{f(c)})\, l(\bar{a}_1, w) = \bar{1}}}{g_1(\bar{a}_1, \bar{c}) = \bar{1}}$$

In view of the definition of S^E_γ, we know that the least ordinal γ_c such that $(\exists w < \overline{f(c)})\, l(\bar{a}_1, w) = 1 \in S^E_{\gamma_c}$ is not less than $f(c)$, hence so is the least ordinal δ_c such that $g_1(\bar{a}_1, \bar{c}) = \bar{1} \in S^E_{\delta_c}$. Therefore $\sup\{\delta_c \mid c < a_1\} \geqq \alpha$. It follows that $(\exists v < \bar{a}_1) g_1(\bar{a}_1, v) = \bar{1}$ does not belong to S^E_γ for any $\gamma < \alpha$.

4.5.2. ii) implies iii). Suppose that α had its predecessor b. As before we can construct a system of equations of $\mathcal{K}(\alpha)$ computing the function f such that $\mathfrak{D}f = 1$ and $f(0) = b$ (i. e. $f(a) \cong \mu c(a = 0 \wedge c = b)$). This is absurd, hence α is a limit ordinal. For an arbitrary ordinal $a_1 < \alpha$, by the same method the function g defined as $g(a) \cong \mu c(a < a_1' \wedge c = k(a))$ is shown to be K-partial recursive. Therefore α is closed under k. Given $a_0, a_1 < \alpha$, then $b_1 = \max(a_0, a_1) + 1 < \alpha$ and $j(a_0, a_1) < j(b_1, 0) = k(b_1) < \alpha$. Thus α is closed under j. Let P be primitive recursive and $\forall a(a < c \rightarrow \exists!\, bP(a, b, a_1, \cdots, a_n))$. Define h as $h(a) \cong \mu b(P(a, b, a_1, \cdots, a_n) \wedge a < c)$, then by Theorem 1 h is K-partial recursive and $\mathfrak{D}h = c$, and so $\sup \mathfrak{R}g < \alpha$.

4.6. We need some preparations to prove that iii) implies iv). Throughout 4.6 α is an ordinal greater than ω and closed under j.

4.6.1. LEMMA 3. ([**10**].) *There exist primitive recursive relations* $\underline{\in}$[3] *and* $\equiv$[3] *on* α *such that*

i) $a \underline{\in} b$ *if and only if* $F(a) \in F(b)$;

ii) $a \equiv b$ *if and only if* $F(a) = F(b)$.

LEMMA 4. *Put* $u(a) = \mu b < a' \forall c(c < a \wedge c \underline{\in} a \rightarrow u(c) < b)$ *and* $Or(a) \leftrightarrow \forall b \forall c(b, c \underline{\in} a \rightarrow b \underline{\in} c \vee b \equiv c \vee c \underline{\in} b) \wedge \forall b \forall c(b \underline{\in} c \wedge c \underline{\in} a \rightarrow b \underline{\in} a)$.

i) u *and* Or *are primitive recursive.*

ii) $Or(a)$ *if and only if* $F(a)$ *is an ordinal (less than* α*).*

iii) *If* $Or(a)$, *then* $\forall a_1(a_1 \equiv a \rightarrow u(a_1) = u(a))$ *and* $u(a) = F(a)$.

iv) *If* $a \in F``\alpha$, *then* $u(Od(a)) = a$. *(*Od *is Gödel's* Od.*)*

LEMMA 5. *Put* $Odr(a, b) \leftrightarrow Or(b) \wedge u(b) = a \wedge \forall c(c < b \rightarrow \neg Or(c) \vee u(c) \neq a)$.

i) Odr *is primitive recursive.*

ii) *If* $a \in F``\alpha$, *then* $Od(a) = b \leftrightarrow Odr(a, b)$.

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3) See, [**10**]. We write $\underline{\in}$ for Takeuti's primitive recursive function $\in$ in [**10**]. We draw no symbolical distinction between the usual set-theoretical notation $\{\ ,\ \}$ for unordered pairs and the corresponding Takeuti's primitive recursive function; likewise for $\langle\ ,\ \rangle$, $\cup$, $\cap$, $\times$ and $\lceil$.

4.6.2. Let $\mathfrak{X}$ be the structure $\langle \alpha, \Subset \rangle$ for $\mathcal{L}$. The notions of a relation on α being Δ_0-(Σ_1-, etc.) definable in $\mathfrak{X}$ are defined analogously to [1], P. 231.

LEMMA 6. *If a relation P is definable in $\mathfrak{X}$ by a Δ_0-formula φ, then P is primitive recursive.*

PROOF. By induction on the construction of φ. Case 3a. φ is of the form $\forall v \in u_i \varphi_1$. Let P_1 be the relation defined in $\mathfrak{X}$ by φ_1. By Lemma 1 and the definition of F,

$$a \Subset a_i \leftrightarrow \exists b(b < a_i \wedge a \equiv b). \tag{1}$$

Hence

$$\begin{aligned} P(a_1, \cdots, a_n) &\leftrightarrow \forall a(a \Subset a_i \rightarrow P_1(a_1, \cdots, a_n, a)) \\ &\leftrightarrow \forall a(a < a_i \rightarrow (a \Subset a_i \rightarrow P_1(a_1, \cdots, a_n, a))). \end{aligned}$$

(cf. [10], 3.2.4. and 3.2.30.) From this and Proposition 1 of [10], P is primitive recursive.

LEMMA 7. i) *For an arbitrary formula φ of $\mathcal{L}$,*

$$\mathfrak{X} \models \varphi[a_1, \cdots, a_n] \quad \text{if and only if} \quad F``\alpha \models \varphi[F(a_1), \cdots, F(a_n)].$$

ii) *For an arbitrary sentence φ of $\mathcal{L}$, $\mathfrak{X} \models \varphi$ if and only if $F``\alpha \models \varphi$.*

PROOF. By induction on the construction of φ using Lemma 3.

4.6.3. LEMMA 8. *Let α satisfy Theorem 2, iii). If an $n+m$-ary ($n \geqq 1$, $m \geqq 0$) relation P is definable in $\mathfrak{X}$ by a Δ_0-formula φ, then*

$$\begin{aligned} &\forall a_{n+1} \cdots \forall a_{n+m} \forall b \exists c \forall a_1 \cdots \forall a_n (\langle a_1, \cdots, a_n \rangle^{3)} \Subset c \\ &\qquad \leftrightarrow \langle a_1, \cdots, a_n \rangle \Subset b \wedge P(a_1, \cdots, a_{n+m})). \end{aligned}$$

PROOF. By induction on the construction of φ. Case 1a. φ is of the form $u_i \in u_j$. In this case $P(a_1, \cdots, a_{n+m}) \leftrightarrow a_i \Subset a_j$. Example 1. $n=3$, $i=1$; $m \geqq 1$, $j=4$. Take arbitrary a_4 and b, and put $c = b \cap (a_4 \times D(b))$[3]. Example 2. $n=3$, $j=1$; $m \geqq 1$, $i=4$. Given a_4 and b, put $c_1 = j(2, \{a_4\} \times R(b), 0)$[3], $c_2 = R(c_1^{-1} \ulcorner \{a_4\})$[3] and $c = b \cap (c_2 \times D(b))$. The remaining cases are treated similarly. Case 1b. φ is of the form $\neg \varphi_1$ where φ_1 is atomic. Let P_1 be defined by φ_1. By Case 1a we obtain, for given $a_{n+1}, \cdots, a_{n+m}$ and b, an ordinal c_1 such that

$$\langle a_1, \cdots, a_n \rangle \Subset c_1 \leftrightarrow \langle a_1, \cdots, a_n \rangle \Subset b \wedge P_1(a_1, \cdots, a_{n+m}).$$

Put $c = b \dot{-} c_1$[3]. Case 3a. φ is of the form $\exists v \in u_i \varphi_1$. Let P_1 be defined by φ_1, then

$$P(a_1, \cdots, a_{n+m}) \leftrightarrow \exists a(a \Subset a_i \wedge P_1(a_1, \cdots, a_{n+m}, a)). \tag{2}$$

Put

$$\left.\begin{aligned} &Q_1(a_1, \cdots, a_{n+m}, d) \leftrightarrow \forall a(a \Subset d \leftrightarrow a \Subset a_i \wedge P_1(a_1, \cdots, a_{n+m}, a)); \\ &Q_2(a_1, \cdots, a_{n+m}, d) \leftrightarrow Q_1(a_1, \cdots, a_{n+m}, d) \wedge \forall d_1(d_1 < d \rightarrow \neg Q_1(a_1, \cdots, a_{n+m}, d_1)). \end{aligned}\right\} \tag{3}$$

By Lemma 6 and Proposition 1 of [10], Q_1 and Q_2 are primitive recursive (cf. (1)). By Ind. Hyp. $\forall a_1 \cdots \forall a_{n+m} \exists ! d Q_2(a_1, \cdots, a_{n+m}, d)$. Given $a_{n+1}, \cdots, a_{n+m}$ and b, we obtain by Theorem 2, iii) an ordinal d_1 satisfying

$$\forall a_1' \cdots \forall a_{n+m}' \forall d(a_1', \cdots, a_{n+m}' < \max(a_{n+1}, \cdots, a_{n+m}, b)+1 \wedge Q_2(a_1', \cdots, a_{n+m}', d) \rightarrow d < d_1) . \tag{4}$$

In addition we can assume $g_0(d_1)=0$[3]. Subcase 1. $1 \leqq i \leqq n$. By Ind. Hyp. there exist c_1 and c_2 such that

$$\left.\begin{array}{l} \forall a_1 \cdots \forall a_n \forall a(\langle a, a_1, \cdots, a_n\rangle \sqsubseteq c_1 \leftrightarrow \langle a, a_1, \cdots, a_n\rangle \sqsubseteq d_1 \times b \wedge P_1(a_1, \cdots, a_{n+m}, a)) ; \\ \forall a_1 \cdots \forall a_n \forall a(\langle a, a_1, \cdots, a_n\rangle \sqsubseteq c_2 \leftrightarrow \langle a, a_1, \cdots, a_n\rangle \sqsubseteq c_1 \wedge a \sqsubseteq a_i) . \end{array}\right\} \tag{5}$$

Assume

$$\langle a_1, \cdots, a_n\rangle \sqsubseteq b \wedge a \sqsubseteq a_i \wedge P(a_1, \cdots, a_{n+m}, a) \tag{6}$$

There exists d for which $Q_2(a_1, \cdots, a_{n+m}, d)$. From (6) and (3), $a \sqsubseteq d$ (7). On the other hand we can assume $a_1, \cdots, a_n < b$ since $\langle a_1, \cdots, a_n\rangle \sqsubseteq b$ (cf. (1)). From this and (4), $d < d_1$ (8). (7) and (8) imply $a \sqsubseteq d_1$ (9), because $g_0(d_1)=0$. Hence $\langle a, a_1, \cdots, a_n\rangle \sqsubseteq c_2$, by (6), (9) and (5). Noting (2), we have shown that

$$\langle a_1, \cdots, a_n\rangle \sqsubseteq b \wedge P(a_1, \cdots, a_{n+m}) \rightarrow \langle a_1, \cdots, a_n\rangle \sqsubseteq D(c_2) .$$

The converse implication is obvious. Subcase 2 where $n+1 \leqq i \leqq n+m$ is treated similarly. Case 3b is reduced to Case 3a by taking the complement (cf. Case 1b).

LEMMA 9. *If α satisfies Theorem* 2, iii), *then it satisfies Theorem* 2, iv).

PROOF. It is clear that a) and b) of 4.3 hold with respect to $F``\alpha$. For d), noting Lemma 7, it suffices to prove that if P is a relation definable in $\mathfrak{T}$ by a Σ_1-formula[4] φ, then

$$\forall a(a \sqsubseteq c \rightarrow \exists b P(a, b)) \rightarrow \exists d(\forall a(a \sqsubseteq c \rightarrow \exists b(b \sqsubseteq d \wedge P(a, b))) \wedge \forall b(b \sqsubseteq d \rightarrow \exists a(a \sqsubseteq c \wedge P(a, b))) .$$

Let φ be of the form $\exists v \varphi_1$, where φ_1 is Δ_0. Let P_1 be defined by φ_1. Put

$$Q_1(a, e_1) \leftrightarrow \exists b \exists e(P_1(a, b, e) \wedge e_1 \equiv \langle b, e\rangle) ,$$

$$Q_2(a, e_1) \leftrightarrow Q_1(a, e_1) \wedge \forall e_2(Q_1(a, e_2) \rightarrow e_1 \leqq e_2) ,$$

$$Q_3(a, e_1, c) \leftrightarrow (a \sqsubseteq c \wedge Q_2(a, e_1)) \vee (\neg a \sqsubseteq c \wedge e_1 = 0) ,$$

$$Q_4(b, c, d_1) \leftrightarrow \exists a \exists e(a \sqsubseteq c \wedge e \sqsubseteq D(d_1) \wedge P_1(a, b, e)) .$$

Since the existential quantifiers in Q_1 can be bounded by e_1 (cf. (1)), Q_1 is obtained from a primitive recursive relation by substituting (constant) ordinals

4) See, [1].

for some of its variables. Hence so are Q_2 and Q_3. It is clear that Q_4 is $\varDelta_0$-definable in $\mathfrak{T}$. Assume $\forall a(a \sqsubseteq c \rightarrow \exists b\, P(a, b))$, then $\forall a(a < c \rightarrow \exists !\, e_1 Q_3(a, e_1, c))$. By Theorem 2, iii) we obtain d_1 satisfying $\forall a \forall e_1(a < c \wedge Q_3(a, e_1, c) \rightarrow e_1 < d_1) \wedge g_0(d_1) = 0$. By Lemma 8 there exists d such that $\forall b(b \sqsubseteq d \leftrightarrow b \sqsubseteq R(d_1) \wedge Q_4(b, c, d_1))$. Now there is no difficulty in showing that this d is the desired one.

Next we prove that $\alpha \subseteq F``\alpha$ by transfinite induction over $\alpha$. Suppose $a \subseteq F``\alpha$. By Lemma 5, ii) $\forall a_1(a_1 < a \rightarrow \exists !\, b\, Odr(a_1, b))$, hence by Theorem 2, iii) there exists d such that $\forall a_1 \forall b(a_1 < a \wedge Odr(a_1, b) \rightarrow b < d) \wedge g_0(d) = 0$. It follows that $a \subseteq F(d)$. On the other hand from the fact that the $\varSigma_1$-replacement-reflection axioms hold in $F``\alpha$, via Lemma 1, we obtain $p (\in F``\alpha)$ such that $F``\alpha \models \forall x(x \in p \leftrightarrow x \in F(d) \wedge Ord(x))$. Clearly $p (= F(d) \cap On)$ is an ordinal and $a \subseteq p$ i. e. $a \leqq p \in F``\alpha$, so $a \in F``\alpha$.

Finally we prove c). Note the following facts.

(1) If $p \in F``\alpha$, then there exists a transitive $q \in F``\alpha$ such that $p \subseteq q$. (Obvious.)

(2) If $p \in F``\alpha$, then $\cup p \in F``\alpha$. (From (1) and the $\varDelta_0$-separation axioms in $F``\alpha$; cf. Lemma 1.)

(3) $\omega \in F``\alpha$. (For, $\omega \in \alpha \subseteq F``\alpha$.)

Using (2) and the $\varSigma_1$-replacement axioms in $F``\alpha$ (cf. Lemma 1.) we can show that for an arbitrary $p \in F``\alpha$ there exists a function $G (\subseteq A)$ such that G is $\varSigma_1$ over A; $\mathfrak{D}G = \omega$; $G(0) = p$ and $G(n+1) = \cup G(n)$ for any $n < \omega$. By (2), (3) and $\varSigma_1$-replacement, $\cup G``\omega$ (= the transitive closure of $p) \in F``\alpha$. (For the existence of G, cf. the proof of Lemma 12, iii).)

4.7. v) implies iii).

4.7.1. LEMMA 10. *If A is admissible and $A \cap On = \alpha$, then α is closed under j.*

PROOF. It follows from the admissibility of A that there exists a function G such that $\mathfrak{D}G = \alpha^2$; $\mathfrak{R}G \subseteq \alpha$ and $G(a, b) = \sup \{G(a_1, b_1) \mid \langle a_1, b_1 \rangle \prec \langle a, b \rangle\}$, where $\prec$ is a well-ordering of the pairs of ordinals defined as $\langle a, b \rangle \prec \langle c, d \rangle$ if and only if $\max(a, b) < \max(c, d) \vee (\max(a, b) = \max(c, d) \wedge (b < d \vee (b = d \wedge a < c)))$. By induction over the well-ordered set $\langle \alpha^2, \prec \rangle$, $j \restriction \alpha^2 = G$. (For the existence of G, cf. the proof of Lemma 12, iii).)

4.7.2. For a function f on α with $\mathfrak{D}f = TF_{l_1}(\alpha) \times \cdots \times TF_{l_m}(\alpha) \times \alpha^n$, let $f^\#$ be a function defined by

$$f^\# = \{\langle d, h_1, \cdots, h_m, a_1, \cdots, a_n \rangle \mid h_i \in PF_{l_i}(\alpha) \cap A \;\; (1 \leqq i \leqq m) \text{ and } \langle d, h_1^c, \cdots, h_m^c, a_1, \cdots, a_n \rangle \in f\}\,.$$

(For $h \in PF_l(\alpha)$, $h^c = h \cup \{\langle 0, a_1, \cdots, a_l \rangle \mid \langle a_1, \cdots, a_l \rangle \notin \mathfrak{D}h\}$.)

The content of the next lemma is virtually included in a result of [8].

LEMMA 11. *Let A be admissible, $A \cap On = \alpha$ and f be a primitive recursive function on α with $\mathfrak{D}f = TF_{l_1}(\alpha) \times \cdots \times TF_{l_m}(\alpha) \times \alpha^n$.*

i) $\mathfrak{D}f^{\#} = (PF_{l_1}(\alpha) \cap A) \times \cdots \times (PF_{l_m}(\alpha) \cap A) \times \alpha^n$ *and* $Rf^{\#} \subseteq \alpha$.

ii) *If f has no function variables, then $f^{\#} = f$.*

iii) *$f^{\#}$ is Σ_1 over A.*

PROOF. i) and ii) are obvious. iii). By induction on the construction of f by Schemata I$\sim$XIII$'$. All the cases are easy except Cases VI and XIII. Noting that j is defined by the recursion $j(a, b) = \sup\{j(c, d) \mid \langle c, d\rangle \prec \langle a, b\rangle\}$, we can treat Case VI in the similar way to that for Case XIII. Case XIII. For simplicity, we deal with the case where f is defined from g by $f(h, a) = g(\lambda b f^a(h, b), h, a)$ with h unary. Put $B = \{\langle h, l\rangle \mid h, l \in PF_1(\alpha) \cap A$; $\mathfrak{D}l$ is an ordinal less than α, and $\forall b(b \in \mathfrak{D}l \rightarrow l(b) = g^{\#}(l \upharpoonright b, h, b)).\}$.

$$F = \{\langle b, h, a\rangle \mid \exists l(\langle h, l\rangle \in B \wedge \langle b, a\rangle \in l)\} .$$

Then we have successively the following facts.

(1) $\langle h, l\rangle \in B$, $\langle h, l_1\rangle \in B$, $\langle b, a\rangle \in l$ and $\langle b_1, a\rangle \in l_1$ imply $b = b_1$.

(2) F is a function, $\mathfrak{D}F \subseteq (PF_1(\alpha) \cap A) \times \alpha$ and $\mathfrak{R}F \subseteq \alpha$.

(3) B and F are Σ_1 over A.

(4) $\mathfrak{D}F = (PF_1(\alpha) \cap A) \times \alpha$.

(Suppose that there existed a satisfying $\forall l(\langle h, l\rangle \in B \rightarrow \neg \exists b(\langle b, a\rangle \in l))$. Denote the least such a by a_1. Then, $\forall a(a \in a_1 \rightarrow \exists!\, l(\langle h, l\rangle \in B \wedge \mathfrak{D}l = a'))$. On the other hand there exists a Σ_1-formula φ such that $\langle h, l\rangle \in B \wedge \mathfrak{D}l = a'$ if and only if $A \models \varphi[h, l, a]$. Therefore, $A \models \forall u(u \in a_1 \rightarrow \exists!\, v\varphi[h, v, u])$. By Σ_1-replacement, there exists $p (\in A)$ for which we have $A \models \forall v(v \in p \leftrightarrow \exists u(u \in a_1 \wedge \varphi[h, v, u]))$. Put $l_1 = \cup p$, then $l_1 \in PF_1(\alpha) \cap A$. Hence, there exists b_1 such that $\langle b_1, l_1, h, a_1\rangle \in g^{\#}$. Put $l_2 = l_1 \cup \{\langle b_1, a_1\rangle\}$, then $\langle h, l_2\rangle \in B$, a contradiction.)

(5) $F = f^{\#}$ (By induction over α.)

LEMMA 12. *Let A be admissible, $A \cap On = \alpha$ and P be a primitive recursive relation on α, then*

$$\forall a_1 \cdots \forall a_n \forall c \exists d(\forall a(a < c \rightarrow \exists!\, bP(a, b, a_1, \cdots, a_n)) \\ \rightarrow \forall a \forall b(a < c \wedge P(a, b, a_1, \cdots, a_n) \rightarrow b < d)) .$$

PROOF. Let P be primitive recursive. Assume the antecedent. Denote by φ a Σ_1-formula defining P in A. (cf. Lemma 11.) Then, $A \models \forall u(u \in c \rightarrow \exists!\, v\varphi[u, v])$. From this via Σ_1-replacement, there exists $p \in A$ such that $A \models \forall u(u \in p \leftrightarrow \exists v(v \in c \wedge \varphi[u, v]))$. $\cup p'$ is the desired ordinal in A.

4.7.3. From Lemmas 10 and 12, we know that Theorem 2, v) implies iii). We prove that Theorem 2, iv) implies i) in 5.4.

§ 5. Takeuti's recursiveness comprises Kripke's recursiveness.

In the remainder of this paper we treat only functions in $Pf(\alpha)$ without function variables unless otherwise stated.

5.1. We make some preparations for the arithmetization of Kripke's system.

5.1.1. LEMMA 13. *Let α satisfy Theorem 2,* iv) *and g be a function such that* a) *$\mathfrak{D}g$ is an ordinal less than α and $\mathfrak{R}g \subseteqq \alpha$; and* b) *$g \in F``\alpha$.*

Put $g^* = \{\langle c, F(j(0, b, 0))\rangle \mid \langle c, b\rangle \in g\}$. Then

i) *g^* is a function belonging to $F``\alpha$.*

ii) *If $Od(g^*) = d$, then $\forall a(a < \mathfrak{D}g \rightarrow g(a) = u(d \dagger j(0, a, 0)))$.*

$(a \dagger b = \mu c < a(\langle c, b\rangle \mathrel{\underline{\in}} a).)$

PROOF. i) Put $\mathfrak{D}g = a_1$. Since $g \in F``\alpha$, so is $\mathfrak{R}g \in F``\alpha$. On the other hand, $g^* \subseteqq \mathfrak{R}g \times F(j(0, a_1, 0))$. It remains to show that g^* is Δ_1 over $F``\alpha$, but this follows immediately from the fact that $F \restriction \alpha$ and $\lambda abcj(a, b, c)$ are $\Delta_1$ over $F``\alpha$ (for $F \restriction \alpha$, by the definition of F; for j by Lemma 11).

ii) It is obvious that if $p, q \in F``\alpha$ then $Od(\langle p, q\rangle) \equiv \langle Od(p), Od(q)\rangle$.

Let $a < a_1$ and put $b = g(a)$. $\langle Od(b), j(0, a, 0)\rangle \equiv \langle Od(b), Od(F(j(0, a, 0))\rangle \equiv Od(\langle b, F(j(0, a, 0))\rangle) \mathrel{\underline{\in}} d$ (1). Hence, $\exists b_1(b_1 < d \wedge \langle b_1, j(0, a, 0)\rangle \mathrel{\underline{\in}} d)$, so $\langle d \dagger j(0, a, 0), j(0, a, 0)\rangle \mathrel{\underline{\in}} d$. From this and (1), $d \dagger j(0, a, 0) \equiv Od(b)$. By Lemma 4, $u(d \dagger j(0, a, 0)) = u(Od(b)) = b$.

5.1.2. Let g be a function satisfying the condition a) of Lemma 13. If there is an ordinal d such that $\forall a(a < \mathfrak{D}g \rightarrow g(a) = u(d \dagger j(0, a, 0)))$, we call the least such d the index of g.

5.2. Now we arithmetize Kripke's system using primitive recursive functions. The arguments that follow are for the most part adaptations from [**12**]. We can dispense with the axiom of constructibility by the idea of Tanaka [**11**].

5.2.1. Assignment of ordinals to the primitive symbols.

	Symbols	Ordinals assigned
i)	$\bar{a}$ (numeral for a)	$j(4, a)$
ii)	v_i (i-th variable)	$j(5, i)$
iii)	f_i (i-th function letter)	$j(6, i)$
iv)	$(\exists <)$	7
v)	$=$	8

5.2.2. Assignment of ordinals to the terms etc.

We denote by $\ulcorner t \urcorner$ the ordinal assigned to the entity t. Let t be a term.

i) If t is a numeral or a variable, then $\ulcorner t \urcorner$ is already defined.

ii) If t is $f_i(t_1, \cdots, t_n)$, then $\ulcorner t \urcorner = j(j(6, i), j(\ulcorner t_1 \urcorner, \cdots, j(\ulcorner t_{n-1} \urcorner, \ulcorner t_n \urcorner) \cdots))$.

iii) If t is $(\exists v_i < t_1)t_2$, then $\ulcorner t \urcorner = j(7, j(j(5, i), j(\ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner)))$.

Let e be an equation $t_1 = t_2$. Then, $\ulcorner e \urcorner = j(8, j(\ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner))$.

Let e_0 be a system of equations $(e_1, \cdots, e_n)$. Then, $e_0 = j(\ulcorner e_1 \urcorner, \cdots, j(\ulcorner e_{n-1} \urcorner, \ulcorner e_n \urcorner) \cdots)$.

5.2.3. LEMMA 14. *Let α satisfy Theorem 2, iv). There exists a primitive recursive relation D on α such that $\exists d(D(e_1, d) \wedge [d]_* = e)$ if and only if e_1 is (an ordinal assigned to) a system of equations and e is an equation deducible from e_1.*

PROOF. We can define the primitive recursive relations V, Tm, Eq, etc. having the same meanings as the corresponding relations in [**12**]. Using these we define D by induction.

$$
\begin{aligned}
& D(e_1, d) \leftrightarrow SE(e_1) \\
& \quad \wedge [\overset{1)}{g}{}^{1}(d) = 10 \wedge Eq(g^2(d)) \wedge \exists i(0 < i < \omega \wedge (\overset{1)}{g}{}^{2}(d) = [\overset{1)}{e_1}]_i \vee g^2(d) = \nu(i, \overset{1)}{e_1}))) \\
& \quad \vee g^1(d) = 11 \wedge (Cn_1([d]_1, [\nu(2, d)]_*) \vee Cn_{3a}([d]_1, [\nu(2, d)]_*)) \wedge D(e_1, \nu(2, d)) \\
& \quad \vee g^1(d) = 12 \wedge Cn_2([d]_1, [d]_{2*}, [\nu(3, d)]_*) \wedge D(e_1, [d]_2) \wedge D(e_1, \nu(3, d)) \\
& \quad \vee g^1(d) = 13 \wedge \exists c_1 \exists c_2(c_1, c_2 < d \wedge Tm(c_1) \wedge V(c_2) \wedge \forall c(c < d \rightarrow c = c_2 \vee \neg Ct(c_1, c)) \\
& \qquad \wedge \forall c\{c < \nu(3, d) \rightarrow D(e_1, u([d]_2 \dagger j(0, c, 0)))\} \wedge \exists c_3(c_3 < d \wedge Sb(c_3, c_1, j(4, c), c_2) \\
& \qquad \wedge [u([d]_2 \dagger j(0, c, 0))]_* = j(8, j(c_3, j(4, 1))))\} \\
& \quad \wedge [d]_1 = j(8, j(j(7, j(c_2, j(j(4, \nu(3, d)), c_1))), j(4, 1)))] .
\end{aligned}
$$

Here $[\ ]_*$ is defined as $[d]_* = \begin{cases} g^2(d) & \text{if } g^1(d) = 10 \\ [d]_1 & \text{otherwise} . \end{cases}$

Note that if $d > 0$, then $[d]_2$, $\nu(2, d)$, $\nu(3, d) < d$, and if $g^2(d) > 0$, then

$$\forall c(c < \nu(3, d) \rightarrow u([d]_2 \dagger j(0, c, 0)) < d) .$$

Obviously $\exists d(D(e_1, d) \wedge [d]_* = e)$ implies that e_1 is a system of equations and e is an equation deducible from e_1. We prove the converse by induction on the definition of the deducibility. The initial case and the cases involving $R1$, $R2$ and $R3$a are easy. For each $a < b$, let be given an e_a deducible from e_1, and let e be an immediate consequence of $\langle e_a | a < b \rangle$ by $R3$b. By Ind. Hyp. and the definition of $R3$b, there exist c_1 and c_2 such that

$$\forall a(a < b \rightarrow \exists d(D(e_1, d) \wedge Sb([d]_*, c_1, j(4, a), c_2))) .$$

Put

$$R(a, d) \leftrightarrow D(e_1, d) \wedge Sb([d]_*, c_1, j(4, a), c_2) ,$$

$$R_1(a, d) \leftrightarrow R(a, d) \wedge \forall d_1(d_1 < d \rightarrow \neg R(a, d_1)) .$$

Then, $\forall a(a < b \rightarrow \exists ! dR_1(a, d))$. Hence, $g = \{\langle d, a \rangle \mid R_1(a, d) \wedge a < b\}$ is a function with domain b. Moreover g is $\boldsymbol{\Sigma}_1$ over $F``\alpha$ by Lemma 11 with $A = F``\alpha$. It follows from Lemma 2, v) that $g \in F``\alpha$, so g has the index c. Put

$d=j(13, j(e, j(c, b)))$.

5.3. LEMMA 15. *Let α be admissible. There exist a primitive recursive function U and a primitive recursive relation S_n for each n for which the following hold.*

i) *If $f(\in Pf(\alpha))$ is K-partial recursive, then there exists e such that $f(a_1, \cdots, a_n) \cong b$ if and only if $\exists d(S_n(e, a_1, \cdots, a_n, d) \wedge U(d)=b)$.*

ii) *If f is in addition K-partial recursive in the strict sense, then the above e can be taken from the natural numbers.*

PROOF. U and S_n are defined analogously to [**12**]. The e is the ordinal assigned to the system of equations calculating f. ii) is clear from 5.2.1. and 5.2.2.

THEOREM 3. *Let α be admissible. If $f(\in Pf(\alpha))$ is K-partial recursive in the strict sense (K-partial recursive), then f is T-partial recursive (T-partial recursive in the classical sense).*

PROOF. Obvious from Lemma 15.

5.4. Now we prove that Theorem 2, iv) implies i). Let α satisfy Theorem 2, iv). Define a function $f(\in Pf(\alpha))$ by induction

$$f(d) \cong \begin{cases} f(\nu(2, d))+1 & \text{if } g^1(d)=11; \\ \max(f([d]_2), f(\nu(3, d)))+1 & \text{if } g^1(d)=12; \\ \max(\sup\{f(u([d]_2 \dagger j(0, c, 0))) \mid c<\nu(3, d)\}, \nu(3, d)+1) & \\ & \text{if } g^1(d)=13 \wedge g^2(d)>0; \\ 0 & \text{otherwise}. \end{cases}$$

Using Lemmata 2 and 11 with $A=F``\alpha$, we can show, by the same argument as that for (4) of the proof of Lemma 11, that the above f is total. Next we know by induction on a that $D(e_1, d)$ and $f(d)=a$ imply $[d]_1 \in S_a^{e_1}$. Hence the desired result.

§ 6. Platek's concept of recursiveness.

We assume that α is admissible throughout the remainder of this paper. According to [1] and [6] Platek proposes in his unpublished paper the following concept of recursiveness on admissible ordinals.

$f(\in Pf(\alpha))$ is P-partial recursive (in the strict sense) if and only if f is $\boldsymbol{\Sigma}_1$ (Σ_1) over $F``\alpha$.

THEOREM 4. *If $f(\in Pf(\alpha))$ is P-partial recursive (P-partial recursive in the strict sense), then f is T-partial recursive in the classical sense (T-partial recursive).*

PROOF. This is virtually one of the results in [8]. It suffices to show that if a relation P on α is Σ_1 ($\boldsymbol{\Sigma}_1$) over $F``\alpha$, then there exists a relation Q, T-recursive (in the classical sense) such that $P(a) \leftrightarrow \exists bQ(a, b)$. Let P be defined by a Σ_1- ($\boldsymbol{\Sigma}_1$-) formula $\exists v\varphi$, where φ is Δ_0 ($\boldsymbol{\Delta}_0$). Then, $P(a)$ if and only if $F``\alpha \models \exists v\varphi[a, v]$ and $F``\alpha \models \exists v\varphi[a, v]$ if and only if $\mathfrak{T} \models \exists v\varphi[Od(a), v]$. (cf. Lemma 7.) Denote by Q_1 the relation defined by φ in $\mathfrak{T}$. By Lemma 6 Q_1 is T-recursive (in the classical sense); and

$$P(a) \leftrightarrow \exists bQ_1(Od(a), b) \leftrightarrow \exists b\exists c(Odr(a, c) \wedge Q_1(c, b)).$$

§ 7. Montague's concept of recursiveness.

7.1. Denote by $\mathcal{M}$ the higher-order language introduced in [7], P. 65. We assume that $\mathcal{M}$ has only two predicate constants: I of type $\langle 0, 0\rangle$ and P of type $\langle 1, 0\rangle$. The atomic formulas, the formulas, the elementary formulas (the counterparts in $\mathcal{M}$ of the Δ_0-formulas of $\mathcal{L}$) and the Σ_1^M-formulas (the counterparts in $\mathcal{M}$ of the Σ_1-formulas of $\mathcal{L}$) are defined in [7], P. 68. We add the notion of Σ^M-formulas, which correspond to the Σ-formulas of $\mathcal{L}$.

i) Atomic formulas are Σ^M-formulas.

ii) If φ_1 and φ_2 are Σ^M-formulas, then $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \vee \varphi_2$ are Σ^M-formulas.

iii) If φ is a Σ^M-formula, u and v are variables and $\text{type}(u)+1 = \text{type}(v)$, then $\bigwedge u(u \in v \rightarrow \varphi)$ and $\bigvee u(u \in v \wedge \varphi)$ are Σ^M-formulas.

For the definition of the typed relations R, their types (type(R)) and their extensions (R^*), see [7], P. 67.

7.2. Define a denumerable sequence $\{U_n\}_{n<\omega}$ by

$$U_0 = \alpha,$$

$$U_{n+1} = \mathfrak{P}(U_n) \cap F``\alpha.$$

The variables of type n are taken as ranging over U_n. The typed relations Id and Sup interpret the predicate constants I and P respectively.

$$\mathrm{Id} = \langle\langle 0, 0\rangle, \mathrm{Id}^*\rangle, \quad \text{where} \quad \mathrm{Id}^* = \{\langle a, a\rangle \mid a \in U_0\}.$$

$$\mathrm{Sup} = \langle\langle 1, 0\rangle, \mathrm{Sup}^*\rangle, \quad \text{where} \quad \mathrm{Sup}^* = \{\langle X, a\rangle \mid X \in U_1 \text{ and } a = \sup X\}.$$

Note that $X \subseteq \alpha$ and $X \in F``\alpha$ imply $\sup X < \alpha$.

Let φ be a formula having no free variables other than $u_1, \cdots, u_n$. If the assignment of individuals $p_1, \cdots, p_n$ to $u_1, \cdots, u_n$ satisfies φ, we write $\mathfrak{M} \models \varphi[p_1, \cdots, p_n]$. The notions of a typed relation being Σ_1^M ($\boldsymbol{\Sigma}_1^M$) definable are defined analogously to [1], P. 231.

7.3. We divert the variables of $\mathcal{M}$ to the variables of $\mathcal{L}$ by disregarding their types.

LEMMA 16. i) *For each* n, $U_n \subseteq F``\alpha$.

ii) *For each* n, *there exists a* Δ_0-*formula* $\varphi^{(n)}$ *such that* $p \in U_n$ *if and only* *if* $F``\alpha \models \varphi^{(n)}[p]$.

LEMMA 17. *Let* φ *be an elementary* (Σ_1^M) *formula having no free variables other than* $u_1, \cdots, u_n$, *and type* $(u_i) = k_i$ $(1 \leqq i \leqq n)$. *There exists a* Δ_0 (Σ_1)-*formula* φ^* *for which the followings hold.*

i) *The free variables of* φ^* *coincide with those of* φ.

ii) *For any* $p_1, \cdots, p_n$, $F``\alpha \models \varphi^*[p_1, \cdots, p_n]$ *if and only if* $p_i \in U_{k_i}$ $(1 \leqq i \leqq n)$ *and* $\mathfrak{M} \models \varphi[p_1, \cdots, p_n]$.

PROOF. The case where φ is elementary is by induction on the construction of the elementary formulas using Lemma 16, whence follows the case where φ is Σ_1^M.

LEMMA 18. *Let* φ *be a* Σ^M-*formula having no free variables other than* $u_1, \cdots, u_n$, *and type* $(u_i) = k_i$ $(1 \leqq i \leqq n)$. *There exists a* Σ_1^M-*formula* φ' *for which the following hold.*

i) *The free variables of* φ' *coincide with those of* φ.

ii) *For any* $p_i \in U_{k_i}$ $(1 \leqq i \leqq n)$, $\mathfrak{M} \models \varphi'[p_1, \cdots, p_n]$ *if and only if* $\mathfrak{M} \models \varphi[p_1, \cdots, p_n]$.

PROOF. By induction on the construction of φ. Case 3a. φ is of the form $\wedge u^j(u^j \in u_i^{j+1} \rightarrow \varphi_1)$. ($j+1 = k_i$; we denote the type of a variable by its superscript.) By Ind. Hyp. there exists a Σ_1^M-formula φ_1' satisfying i) and ii) for $\varphi = \varphi_1$. Let φ_1' be of the form $\vee w^k \psi$, where ψ is elementary, and w^{k+1} be not in ψ. Denote by φ' the Σ_1^M-formula

$$\vee w^{k+1} \wedge u^j(u^j \in u_i^{j+1} \rightarrow \vee w^k(w^k \in w^{k+1} \wedge \psi)) .$$

Then φ' is the desired formula. i) and the 'if' part of ii) are obvious. Take the Δ_0-formula ψ^* for which Lemma 17 i) and ii) hold with $\varphi = \psi$. Let $\mathfrak{M} \models \varphi[p_1, \cdots, p_n]$. Then,

$$\mathfrak{M} \models \wedge u^j(u^j \in p_i \rightarrow \vee w^k \psi[p_1, \cdots, p_n, u^j, w^k]) ,$$

$$F``\alpha \models \forall u \in p_i \exists w(\psi^*[p_1, \cdots, p_n, u, w] \wedge \varphi^{(k)}[w]) .$$

By the Σ_1-replacement-reflection and Δ_0-separation axioms in $F``\alpha$, there exist $q$ and $q_1$ in $F``\alpha$ such that

$$F``\alpha \models \forall u \in p_i \exists w \in q(\psi^*[p_1, \cdots, p_n, u, w] \wedge \varphi^{(k)}[w]) ,$$

$$F``\alpha \models \forall w(w \in q_1 \leftrightarrow w \in q \wedge \varphi^{(k)}[w]) .$$

Now it is obvious that this q_1 is an object in U_{k+1} to be assigned to w^{k+1}.

7.4. We call an n-ary function $f (\in Pf(\alpha))$, M-partial recursive (in the strict sense) if the typed relation $\langle\langle \overbrace{0, \cdots, 0}^{n+1} \rangle, f\rangle$ is Σ_1^M (Σ_1^M) definable.

THEOREM 5. *If* $f (\in Pf(\alpha))$ *is M-partial recursive (in the strict sense) then* f *is P-partial recursive (in the strict sense).*

PROOF. Immediate from Lemma 17.

7.5. We specialize A to $F``\alpha$ in 4.7.2.

LEMMA 19. *If f is a primitive recursive function on α with $\mathfrak{D}f = TF_{l_1}(\alpha) \times \cdots \times TF_{l_m}(\alpha) \times \alpha^n$, then $f^\# \subseteq U_0 \times (U_{2l_1+1} \times \cdots \times U_{2l_m+1} \times U_0^n)$. Hence, $\langle\langle 0, 2l_1+1, \cdots, 2l_m+1, \overbrace{0, \cdots, 0}^{n}\rangle, f^\#\rangle$ is a typed relation.*

LEMMA 20. *If f is a primitive recursive function on α with $\mathfrak{D}f = TF_{l_1}(\alpha) \times \cdots \times TF_{l_m}(\alpha) \times \alpha^n$, then $\langle\langle 0, 2l_1+1, \cdots, 2l_m+1, \overbrace{0, \cdots, 0}^{n}\rangle, f^\#\rangle$ is Σ_1^M definable.*

PROOF. By induction on the definition of the primitive recursive functions. Again we treat only Case XIII. It suffices to show that there exists a Σ^M-formula $\varphi[u^3, v^3]$ such that $\langle h, l\rangle \in B$ if and only if $\mathfrak{M} \models \varphi[h, l]$. Let a Σ_1^M-formula defining $g^\#$ be $\theta[u^0, u^3, v^3, v^0]$. Put

$$\psi_1[u^1] \leftrightarrow \bigwedge u^0(u^0 \in u^1 \to \bigvee w^1(\bigwedge v^0(v^0 \in w^1 \to v^0 \varepsilon u^1) \wedge Pw^1u^0))\,;$$

$$\psi_2[u^1, u^0] \leftrightarrow \psi_1[u^1] \wedge Pu^1u^0\,;$$

$$\begin{aligned}\psi_3[u^3] \leftrightarrow & \bigwedge u^2(u^2 \in u^3 \to \bigvee u^0 \bigvee v^0(u^2 = \langle u^0, v^0\rangle)) \\ & \wedge \bigwedge u^2 \bigwedge v^2(u^2, v^2 \in u^3 \to \bigvee u^0 \bigvee v^0(u^0 = 2(u^2) \\ & \qquad \wedge v^0 = 2(v^2) \wedge u^0 \neq v^0) \vee \bigvee u^0(u^0 = 1(u^2) \wedge u^0 = 1(v^2)))\,;\end{aligned}$$

$$\begin{aligned}\varphi[u^3, v^3] \leftrightarrow & \psi_3[u^3] \wedge \psi_3[v^3] \\ & \wedge \bigwedge u^2(u^2 \in v^3 \to \bigvee u^0 \bigvee u^1(u^0 = 2(u^2) \wedge \psi_2[u^1, u^0] \\ & \qquad \wedge \bigwedge v^0(v^0 \in u^1 \to \bigvee v^2(v^2 \in v^3 \wedge 2(v^2) = v^0)) \\ & \wedge \bigwedge u^2(u^2 \in v^3 \to \bigvee w^3 \bigvee u^0 \bigvee v^0(u^0 = 1(u^2) \wedge v^0 = 2(u^2) \wedge \theta[u^0, w^3, u^3, v^0] \\ & \wedge \bigwedge v^2(v^2 \in w^3 \to v^2 \varepsilon v^3 \wedge \bigvee w^0(w^0 = 2(v^2) \wedge w^0 < v^0)) \\ & \wedge \bigwedge v^2(v^2 \in v^3 \to \bigvee w^0(w^0 = 2(v^2) \wedge v^0 \leqq w^0) \vee v^2 \varepsilon w^3))))\,.\end{aligned}$$

In the above we must still replace respectively the expressions like $u^n \varepsilon u^{n+1}$, $u^0 = 1(u^2)$, $u^0 = 2(u^2)$, $u^2 = \langle u^0, v^0\rangle$, $u^0 \neq v^0$ and $u^0 < v^0$ by Σ^M-formulas defining the typed relations whose extensions are $\{\langle p, q\rangle \mid q \in U^{n+1} \wedge p \in q\}$, $\{\langle a, p\rangle \mid \exists b(p = \langle a, b\rangle)\}$, $\{\langle b, p\rangle \mid \exists a(p = \langle a, b\rangle)\}$, $\{\langle p, a, b\rangle \mid p = \langle a, b\rangle\}$, $\{\langle a, b\rangle \mid a \neq b\}$ and $\{\langle a, b\rangle \mid a < b\}$. But this is accomplished easily.

THEOREM 6. *If $f (\in Pf(\alpha))$ is T-partial recursive (T-partial recursive in the classical sense) then it is M-partial recursive in the strict sense (M-partial recursive).*

PROOF. Let f be T-partial recursive. Then there exists $e < \omega$ such that $\langle b, a_1, \cdots, a_n\rangle \in f \leftrightarrow \exists d(S_n(e, a_1, \cdots, a_n, d) \wedge U(d) = b)$. Since $e < \omega$, $\lambda a_1 \cdots a_n d$, $S_n(e, a_1, \cdots, a_n, d)$ and U are primitive recursive. By Lemma 20 there exists a Σ_1^M-formula $\varphi[u_1^0, \cdots, u_n^0, v^0, w^0]$ such that $S_n(e, a_1, \cdots, a_n, d) \wedge U(d) = b$ if and only if $\mathfrak{M} \models \varphi[a_1, \cdots, a_n, b, d]$. Then, $\langle b, a_1, \cdots, a_n \rangle \in f$ if and only if $\mathfrak{M} \models \bigvee w^0 \varphi[a_1, \cdots, a_n, b]$.

The remaining case is similar.

ADDED IN PROOF. The condition $F``\alpha \frown On = \alpha$ in Theorem 2, iv) is ultimately redundant. After having Theorem 2 in the original form we can prove that if $F``\alpha$ is admissible then $F``\alpha \frown On = \alpha$.

Tokyo Kyoiku University

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