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On the index of a semi-free S^1-action

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§1. Introduction.

Let G be a compact Lie group, M^n a closed smooth n-manifold and $\varphi: G\times M^n \to M^n$ a smooth action. Then the fixed point set is a disjoint union of smooth k-manifolds F^k, $0\leqq k\leqq n$.

P.E. Conner and E.E. Floyd [2] obtained several properties of fixed point sets of smooth involutions and one of their results is the following.

Suppose that $T: M^{2k}\to M^{2k}$ is a smooth involution on a closed manifold of odd Euler characteristic. Then some component of the fixed point set is of dimension $\geqq k$.

Now we consider semi-free smooth S^1-actions on oriented manifolds and the purpose of this paper is to show the following results.

THEOREM 1.1. *Let M^n be an oriented closed smooth n-manifold and $\varphi: S^1\times M^n\to M^n$ a semi-free smooth action. Then each k-dimensional fixed point set F^k can be canonically oriented and the index of M^n is the sum of indices of F^k, that is,*

$$I(M^n)=\sum_{k=0}^{n} I(F^k)\,.$$

THEOREM 1.2. *Suppose that $\varphi: S^1\times M^{4k}\to M^{4k}$ is a semi-free smooth S^1-action on an oriented closed manifold of non-zero index. Then some component of the fixed point set is of dimension $\geqq 2k$.*

§2. Semi-free S^1-action.

Let S^1 and D^2 denote the unit circle and the unit disk in the field of complex numbers. Regard S^1 as a compact Lie group. Let M^n be an oriented closed smooth n-manifold and $\varphi: S^1\times M^n\to M^n$ a smooth action. The action φ is called semi-free if it is free outside the fixed point set. Then we have the following ([4], Lemma 2.2).

LEMMA 2.1. *The normal bundle of each component of the fixed point set in M^n has naturally a complex structure, such that the induced S^1-action on this bundle is a scalar multiplication.*

From this lemma, a codimension of each component of the fixed point set in M^n is even. Let ν^k denote the complex normal bundle to F^{n-2k}. Then ν^k is canonically oriented and F^{n-2k} can be so oriented that the bundle map $\tau(F^{n-2k})\oplus\nu^k\to\tau(M^n)$ is orientation preserving, where $\tau(M)$ denotes the tangent bundle of M.

For each complex vector bundle ξ over an oriented closed smooth manifold X, let $\boldsymbol{S}(\xi)$ and $\boldsymbol{CP}(\xi)$ denote the sphere bundle and the complex projective bundle associated to ξ, respectively. Then the orientations of $\boldsymbol{S}(\xi)$ and $\boldsymbol{CP}(\xi)$ are induced by those of X and ξ. And we have the following result.

LEMMA 2.2. *Let M^n be an oriented closed smooth n-manifold, $\varphi: S^1\times M^n\to M^n$ a semi-free smooth action and F^{n-2k} an oriented $(n-2k)$-dimensional fixed point set. Let ν^k denote the complex normal bundle to F^{n-2k}. Then*

$$\text{(a)}\qquad \sum_{k\geqq 1}[\boldsymbol{CP}(\nu^k)]=0$$

and

$$\text{(b)}\qquad [M^n]=\sum_{k\geqq 0}[\boldsymbol{CP}(\nu^k\oplus\theta^1)]$$

in the oriented cobordism ring Ω_*, where θ^1 is a trivial complex line bundle.

PROOF. For (a), we may suppose $F^n=\phi$. Let N_k be a S^1-invariant tubular neighborhood of F^{n-2k} (see [2], § 22) mutually disjoint for $k\geqq 1$. Then $B^n=M^n-\bigcup_k \operatorname{Int} N_k$ is a regularly embedded invariant submanifold with boundary, on which S^1 acts freely and the boundary of the orbit manifold B^n/S^1 is a disjoint union of $\boldsymbol{CP}(\nu^k)$ for $k\geqq 1$. This shows (a).

Next, we define two actions τ_1, τ_2 of S^1 on $D^2\times M^n$ by

$$\tau_1(\lambda, (z, x))=(\lambda z, x)\,,$$

$$\tau_2(\lambda, (z, x))=(\lambda z, \varphi(\lambda, x))$$

where λ and z represent complex numbers in S^1 and D^2 respectively and $x\in M^n$. Restricting to $S^1\times M^n$ we obtain induced actions $(\tau_1, S^1\times M^n)$ and $(\tau_2, S^1\times M^n)$ which we shall show to be equivariantly diffeomorphic. Define $f: S^1\times M^n\to S^1\times M^n$ by

$$f(\lambda, x)=(\lambda, \varphi(\lambda, x))\,.$$

It is easy to check that f is an equivariant diffeomorphism.

Now from the disjoint union $(\tau_1, D^2\times M^n)\cup(\tau_2, -D^2\times M^n)$, we form an oriented closed smooth $(n+2)$-manifold M^{n+2} and a smooth S^1-action τ on M^{n+2} by identifying the boundaries $(\tau_1, S^1\times M^n)$ and $(\tau_2, S^1\times M^n)$ via f. This construction is due to Conner and Floyd ([2], P. 119). Note that in $(\tau_1, D^2\times M^n)$, S^1 acts freely on $D^2\times M^n-(0\times M^n)$, and leaves every point of $0\times M^n$ stationary. Also in $(\tau_2, D^2\times M^n)$, S^1 acts freely on $D^2\times M^n-(0\times M^n)$, while the isotropy subgroup at $(0, x)$ is precisely the isotropy subgroup for (φ, M^n)

at x. Thus the action τ on M^{n+2} is semi-free and the equation (a) on this action implies (b). q. e. d.

§ 3. Index of a complex projective bundle.

Now we consider the index of $CP(\xi^k)$, the total space of the complex projective bundle associated to a complex k-plane bundle ξ^k over an oriented closed manifold V^n. For this purpose we prepare the following known result.

LEMMA 3.1. *Let M be a real, symmetric, nonsingular matrix of the form*

$$M=\begin{pmatrix} 0 & 0 & L \\ 0 & A & * \\ {}^tL & * & * \end{pmatrix}$$

where A, L are square matrices (empty matrix is admitted for A). Then there exists a nonsingular matrix T such that

$${}^tTMT=\begin{pmatrix} E & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -E \end{pmatrix}$$

where E is an identity matrix. Here, as always, we denote by ${}^tP$ the transpose of P.

PROOF. Suppose

$$M=\begin{pmatrix} 0 & 0 & L \\ 0 & A & B \\ {}^tL & {}^tB & C \end{pmatrix}.$$

Then the matrix

$$T=\begin{pmatrix} \frac{Y+E}{\sqrt{2}} & 0 & \frac{Y-E}{\sqrt{2}} \\ \frac{X}{\sqrt{2}} & E & \frac{X}{\sqrt{2}} \\ \frac{L^{-1}}{\sqrt{2}} & 0 & \frac{L^{-1}}{\sqrt{2}} \end{pmatrix}$$

is a desired matrix, where

$$X=-A^{-1}BL^{-1},$$

$$Y=\frac{1}{2}{}^tL^{-1}({}^tBA^{-1}B-C)L^{-1}.$$ q. e. d.

THEOREM 3.2.

$$I(CP(\xi^k))=\frac{1+(-1)^{k-1}}{2}\cdot I(V^n)\,.$$

PROOF. In fact this is an immediate consequence of [1], but we give a proof for the completeness. It suffices to prove this theorem in the case of $\dim CP(\xi^k)=n+2(k-1)=4m$ for some m.

Let $u\in H^2(CP(\xi^k)\,;Z)$ be the first Chern class of the canonical line bundle over $CP(\xi^k)$, then the cohomology ring $H^*(CP(\xi^k)\,;R)$ with real coefficients is a free $H^*(V^n\,;R)$ module with basis $\{1, u, u^2, \cdots, u^{k-1}\}$ by the theorem of Leray-Hirsch (cf. [3], P. 258). Let $\{v_i^s\}$ be a basis for $H^s(V^n\,;R)$, then as a basis for $H^{2m}(CP(\xi^k)\,;R)$ we can take $\{v_j^{2(m-t)}u^t\}$, $\left(\max\left(0, m-\frac{n}{2}\right)\leqq t\leqq \min(m, k-1)\right)$.

For an oriented manifold M^n, set

$$\langle x, y\rangle=(x\cup y)[M^n] \qquad \text{for} \quad x, y\in H^*(M^n\,;R)$$

where $[M^n]$ is the fundamental class of $H_n(M^n\,;Z)$. Then

$$\langle v_i^{2(m-s)}u^s, v_j^{2(m-t)}u^t\rangle=\begin{cases}\langle v_i^{2(m-s)}, v_j^{2(m-t)}\rangle & \text{if} \quad s+t=k-1\,,\\ 0 & \text{if} \quad s+t<k-1\,.\end{cases}$$

Arrange the basis $\{v_j^{2(m-t)}u^t\}$ in increasing order of t. Then the matrix of coefficients $\langle v_i^{2(m-s)}u^s, v_j^{2(m-t)}u^t\rangle$ has the form in Lemma 3.1. Therefore

$$I(CP(\xi^k))=\begin{cases}I(V^n) & \text{if} \quad k \text{ is odd}\\ 0 & \text{if} \quad k \text{ is even}\end{cases}$$

by Lemma 3.1. This completes the proof of Theorem 3.2.

§ 4. Indices of fixed point sets.

In this section we prove Theorem 1.1. Under the notations in Lemma 2.2, we have

$$\sum_{k:\,\text{odd}} I(F^{n-2k})=0\,,$$

$$\sum_{k:\,\text{even}} I(F^{n-2k})=I(M^n)$$

from Lemma 2.2 and Theorem 3.2. Thus

$$I(M^n)=\sum_{k=0}^{[\frac{n}{2}]} I(F^{n-2k})\,.$$

This completes the proof, since the codimension of each component of the fixed point set is even.

§ 5. Dimension of fixed point sets.

In this section we prove Theorem 1.2. Let

$$\Delta : \Omega_n(\boldsymbol{CP}^\infty) \longrightarrow \Omega_{n-2}(\boldsymbol{CP}^\infty)$$

be the Smith homomorphism (cf. [2], § 26) and

$$i_* : \Omega_n(\boldsymbol{BU}(k)) \longrightarrow \Omega_n(\boldsymbol{BU}(k+1))$$

a homomorphism induced by the canonical inclusion map $i : \boldsymbol{BU}(k) \to \boldsymbol{BU}(k+1)$. Let

$$\partial : \Omega_n(\boldsymbol{BU}(k)) \longrightarrow \Omega_{n+2k-2}(\boldsymbol{CP}^\infty)$$

be a homomorphism as follows (cf. [4], § 3). To each complex vector bundle ξ^k we have a line bundle $\hat{\xi}$ associated to the principal S^1-bundle $\boldsymbol{S}(\xi^k) \to \boldsymbol{CP}(\xi^k)$, then $\partial([\xi^k]) = [\hat{\xi}]$. Then we have the following commutative diagram (cf. [2], 26.4)

$$\begin{array}{ccc} \Omega_n(\boldsymbol{BU}(k)) & \xrightarrow{\partial} & \Omega_{n+2k-2}(\boldsymbol{CP}^\infty) \\ \downarrow i_* & & \uparrow \Delta \\ \Omega_n(\boldsymbol{BU}(k+1)) & \xrightarrow{\partial} & \Omega_{n+2k}(\boldsymbol{CP}^\infty) . \end{array}$$

And we obtain the following result by the same way as in the case of [2; Theorem 27.3].

LEMMA 5.1. *Let $\varphi : S^1 \times M^n \to M^n$ be a semi-free smooth S^1-action on an oriented closed manifold of non-zero index, and let ν^k denote the complex normal bundle to $(n-2k)$-dimensional fixed point set F^{n-2k}. There exists a k such that $[\nu^k]$ is not in the image of*

$$i_* : \Omega_{n-2k}(\boldsymbol{BU}(k-1)) \longrightarrow \Omega_{n-2k}(\boldsymbol{BU}(k)) .$$

Since

$$i_* : \Omega_m(\boldsymbol{BU}(k-1)) \cong \Omega_m(\boldsymbol{BU}(k)) \qquad \text{for} \quad m \leqq 2(k-1) ,$$

Theorem 1.2 is an immediate corollary of the above result.

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