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On Toeplitz operators

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Let L^2 and L^∞ denote the Lebesgue spaces of square integrable and essentially bounded functions with respect to normalized Lebesgue measure on the unit circle in the complex plane. Let H^2 and H^∞ denote the corresponding Hardy spaces. For ϕ in L^∞, *the Toeplitz operator induced by* ϕ is the operator T_ϕ on H^2 defined by $T_\phi f = P(\phi f)$; here P stands for the orthogonal projection in L^2 with range H^2.

The purpose of this paper is to prove an inversion theorem (Theorem 2) of T_f for f in a class of subalgebras A_ϕ of $H^\infty + C$, and then we can determine (Theorem 3) the spectrum of T_f, for any unitary function f in A_ϕ. We recall that the linear span $H^\infty + C$ of H^∞ and C is a closed subalgebra of L^∞ [**4**, Theorem 2], where C stands for the space of continuous complex valued functions on the unit circle. This algebra can also be characterized as the subalgebra of L^∞ generated by H^∞ and the function $\bar{z}$. Let $\mathscr{B}$ denote the algebra of bounded operators on H^2, $\mathscr{K}$ the uniformly closed two-sided ideal of compact operators in $\mathscr{B}$, and π the homomorphism of $\mathscr{B}$ onto $\mathscr{B}/\mathscr{K}$. An operator B in $\mathscr{B}$ is said to be a Fredholm operator if B has a closed range and both a finite dimensional kernel and cokernel. It is known [**1**] that this is equivalent to $\pi(B)$ being an invertible element of $\mathscr{B}/\mathscr{K}$. If B is a Fredholm operator, then the index $\mathrm{ind}(B)$ is defined $\mathrm{ind}(B) = \dim[\ker B] - \dim[\mathrm{coker}\, B]$. In general for a Fredholm operator B the statement $\mathrm{ind}(B) = 0$ does not imply that B is invertible. For Toeplitz operators, however, the situation is simpler as was shown by Coburn [**2**].

LEMMA 1. *If ϕ is in L^∞ such that T_ϕ is a Fredholm operator and* $\mathrm{ind}(T_\phi) = 0$, *then T_ϕ is invertible.*

Stampfli observed in [**5**] that $T_\phi T_z - T_z T_\phi$ is at most one dimensional for any ϕ in L^∞ and hence compact. Therefore, $T_f T_g - T_g T_f$ is a compact operator for any f and g in $H^\infty + C$ and $T_f T_\phi - T_\phi T_f$ is a compact operator for any ϕ in L^∞ if f is in C.

LEMMA 2. *Let f be in $H^\infty + C$, then $T_h T_f - T_{hf}$ is a compact operator for every h in L^∞.*

PROOF. Since f is in $H^\infty + C$, we can write $f = f_1 + f_2$ where f_1 in H^∞ and f_2 in C. Consider

$$T_h T_f = T_h T_{f_1+f_2} = T_h T_{f_1} + T_h T_{f_2} = T_{hf_1} + T_{hf_2} + K,$$

where K is a compact operator, since f_1 is in H^∞ and f_2 is in C.

Hence

$$T_h T_f = T_{h(f_1+f_2)} + K = T_{hf} + K,$$

for any h in L^∞.

The proof is complete.

If f is a conformal map of the open unit disk onto a simply connected domain such that f is not continuous on the closed unit disk and real part of f is continuous everywhere in the closed unit disk, therefore $\bar{f} = 2\,\mathrm{Re}\, f - f$ is in $H^\infty + C$ but $\bar{f}$ is not continuous, so there are many discontinuous conjugate analytic functions in $H^\infty + C$, hence by Lemma 2, it is easily seen that $T_h T_g - T_g T_h$ is compact for every h in L^∞ does not imply that g is in C. Let D be the collection of all discontinuous conjugate analytic functions in $H^\infty + C$.

For each ϕ in D, let A_ϕ be the uniformly closed subalgebra of L^∞ generated by C and ϕ, hence $C \subseteqq A_\phi \subseteqq H^\infty + C$. Let Ψ_ϕ denote the C^*-subalgebra of $\mathcal{B}$ generated by the operators T_f with f in A_ϕ. Then we have the following theorem.

THEOREM 1. *Ψ_ϕ contains $\mathcal{K}$ as a two-sided ideal and $\Psi_\phi/\mathcal{K}$ is isometrically isomorphic to A_ϕ.*

PROOF. Since Ψ_ϕ contains the C^*-algebra generated by the unilateral shift of multiplicity one, it follows from [3] that Ψ_ϕ contains $\mathcal{K}$ and $\mathcal{K}$ is an ideal in any algebra of $\mathcal{B}$ containing it. Since the commutator of T_f and T_g for f and g in Ψ_ϕ is a compact operator by Lemma 2. Thus the linear span of the operators of the form $T_f + K$, where f is in A_ϕ and K is in $\mathcal{K}$, is an algebra. In fact, it is a C^*-algebra which follows from Coburn's observation that $\|T_f + K\| \geqq \|T_f\|$ for any Toeplitz operator T_f. Therefore, $\Psi_\phi/\mathcal{K}$ is commutative and the mapping $T_f + K \leftrightarrow f$ is an isometrical isomorphism of $\Psi_\phi/\mathcal{K}$ onto A_ϕ. The proof is complete.

COROLLARY. *If f is in A_ϕ, then T_f is a Fredholm operator if and only if f is invertible in A_ϕ.*

PROOF. If f is invertible in A_ϕ, then $\pi(T_f)$ is invertible in $\Psi_\phi/\mathcal{K}$ and hence T_ϕ is a Fredholm operator.

If T_f is a Fredholm operator, then $\pi(T_f)$ is invertible in $\mathcal{B}/\mathcal{K}$, so is $\pi(T_f)^*$, and so $\pi(T_f)^*\pi(T_f)$ is invertible in $\mathcal{B}/\mathcal{K}$. Since $\pi(T_f)^{-1} = (\pi(T_f)^*\pi(T_f))^{-1}\pi(T_f)^*$, it suffices to show that $(\pi(T_f)^*\pi(T_f))^{-1}$ belongs to $\Psi_\phi/\mathcal{K}$.

Since the Gelfand transform of $\pi(T_f)^*\pi(T_f)$ is $\widehat{\pi(T_f)^*\pi(T_f)} = |\widehat{\pi(T_f)}|^2 \geqq 0$ on the maximal ideal space of $\Psi_\phi/\mathcal{K}$, $(\lambda - \pi(T_f)^*\pi(T_f))^{-1}$ belongs to $\Psi_\phi/\mathcal{K}$ for $\lambda < 0$. Since $(\lambda - \pi(T_f)^*\pi(T_f))^{-1}$ converges to $-(\pi(T_f)^*\pi(T_f))^{-1}$ in $\mathcal{B}/\mathcal{K}$ as $\lambda \to 0_-$, we obtain $(\pi(T_f)^*\pi(T_f))^{-1}$ belongs to $\Psi_\phi/\mathcal{K}$. This completes the proof.

From Lemma 1 and Corollary to Theorem 1, we obtain our main theorem.

THEOREM 2. *If f is in A_ϕ, then T_f is invertible if and only if f is invertible in A_ϕ and* $\operatorname{ind}(T_f)=0$.

From Corollary to Theorem 1 and Theorem 2, we can determine the spectrum of T_f, $\sigma(T_f)$, if f is a unitary function in A_ϕ.

THEOREM 3. *If f is in A_ϕ and $|f|=1$* a.e., *then*

(i) *if T_f is not invertible, $\sigma(T_f)$ is the closed unit disk, and*

(ii) *if T_f is invertible, $\sigma(T_f)$ is the essential range of f.*

PROOF OF (i). Case 1. Suppose that f^{-1} is in A_ϕ, that is, 0 does not belong to $\sigma_{A_\phi}(f)$, where $\sigma_{A_\phi}(f)$ denotes the spectrum of f as an element of the subalgebra A_ϕ. It is well known that the boundary of $\sigma_{A_\phi}(f)$ equals the boundary of $\sigma_{L^\infty}(f)$. Since $\sigma_{L^\infty}(f)$ is contained in the unit circle, no point in the open unit disk belongs to $\sigma_{A_\phi}(f)$. This implies that $f-\lambda$ is invertible in A_ϕ for every $|\lambda|<1$. Therefore $T_{f-\lambda}$ is a Fredholm operator by Corollary to Theorem 1. Since T_f is not invertible, $\operatorname{ind}(T_f)\neq 0$ by Lemma 1, hence $\operatorname{ind}(T_{f-\lambda})\neq 0$. Therefore $T_{f-\lambda}$ fails to be invertible for all λ such that $|\lambda|<1$, and it follows that $\sigma(T_f)$ is the closed unit disk.

Case 2. Suppose that f^{-1} is not in A_ϕ, that is, 0 is in $\sigma_{A_\phi}(f)$. Hence by the same argument as in Case 1, we have $\sigma_{A_\phi}(f)$ is the closed unit disk. Therefore $\sigma(T_f)$ is the closed unit disk by Theorem 2.

PROOF OF (ii). f^{-1} is in A_ϕ, since T_f is invertible by assumption. Hence by the same argument as in Case 1 of (i), we have $T_{f-\lambda}$ is a Fredholm operator for every λ such that $|\lambda|<1$. Since $\operatorname{ind}(T_{f-\lambda})=\operatorname{ind}(T_f)=0$ by assumption, $T_{f-\lambda}$ is invertible by Theorem 2. Therefore $\sigma(T_f)$ is contained in the unit circle and by the same argument as in Case 1 of (i) again, we have $\sigma(T_f)$ is the essential range of f. The proof is thus complete.

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