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Mappings into compact complex manifolds with negative first Chern class

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§ 1. Introduction

The purpose of this paper is to prove the following[1)]

THEOREM. *Let V be an n-dimensional compact complex manifold with negative first Chern class. Let* $\mathcal{D}=\{(z^1, \cdots, z^n)\in \boldsymbol{C}^n;\ |z^1|<1, \cdots, |z^n|<1\}$ *and* $\mathcal{D}^*=\{(z^1, \cdots, z^n)\in \mathcal{D};\ z^1\neq 0\}$. *If a holomorphic mapping* $f: \mathcal{D}^*\to V$ *is non-degenerate at some point, then f is a meromorphic mapping from $\mathcal{D}$ into V.*

COROLLARY 1. *Let V be as above. Let M be an n-dimensional complex manifold and A an analytic subvariety of M. If a holomorphic mapping f of $M-A$ into V is non-degenerate at some point, then f is a meromorphic mapping from M into V.*

COROLLARY 2. *Let V be as above. Let A be an analytic subvariety of V. Then every holomorphic transformation of $V-A$ extends to a holomorphic transformation of V.*

By a theorem of Kodaira, the assumption that the first Chern class of V be negative is equivalent to the condition that the canonical line bundle K_V is ample, i.e., the line bundle K_V^m, for some positive integer m, has sufficiently many holomorphic sections to induce an imbedding of V into a complex projective space. If this holds already for $m=1$, i.e., K_V itself has sufficiently many sections to induce an imbedding of V into a projective space, then K_V is said to be very ample. Under the assumption that K_V is very ample, the theorem above has been proved by Griffiths [1].

§ 2. The punctured disk D^*

The upper half-plane

$$H=\{w=u+iv\in \boldsymbol{C};\ v>0\}$$

is a universal covering space of the punctured disk

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1) For a generalization, see the Addendum to this paper.

$$D^* = \{z \in \boldsymbol{C};\ 0 < |z| < 1\}$$

and the covering projection $p: H \to D^*$ is given by

$$z = p(w) = e^{2\pi i w}.$$

A fundamental domain is given by

$$F = \{w \in H;\ 0 < u < 1\}.$$

Up to a constant factor, the Bergman metric ds_H^2 of H and the corresponding area element μ_H are given by

$$ds_H^2 = \frac{dw d\bar{w}}{v^2} \qquad \mu_H = \frac{du \wedge dv}{v^2}.$$

We denote by $ds_{D^*}^2$ and μ_{D^*} the metric and the area element on D^* defined by

$$p^*(ds_{D^*}^2) = ds_H^2, \qquad p^*(\mu_{D^*}) = \mu_H.$$

LEMMA 2.1. *Let* $0 < a < 1$ *and* $D_a^* = \{z \in \boldsymbol{C}; 0 < |z| < a\}$. *Then the area of* D_a^* *with respect to* μ_{D^*} *is finite, i.e.,*

$$\int_{D_a^*} \mu_{D^*} < \infty.$$

PROOF. The subset of the fundamental domain F which corresponds to D_a^* is given by

$$F_b = \{w \in F;\ v > b\},$$

where b is a certain positive number (which can be determined from the value a). Then

$$\int_{D_a^*} \mu_{D^*} = \int_{F_b} \mu_H = \int_b^\infty dv \int_0^1 \frac{du}{v^2} = \frac{1}{b}$$

QED.

Let

$$\mathcal{D}^* = D^* \times D \times \cdots \times D \subset \boldsymbol{C}^n.$$

If we denote by ds_D^2 the Bergman metric of $D = \{z \in \boldsymbol{C}; |z| < 1\}$ and by μ_D the corresponding area element of D, then the Bargman metric $ds_{\mathcal{D}^*}^2$ of $\mathcal{D}^*$ and the corresponding volume element $\mu_{\mathcal{D}^*}$ are given by

$$ds_{\mathcal{D}^*}^2 = ds_{D^*}^2 + ds_D^2 + \cdots + ds_D^2,$$

$$\mu_{\mathcal{D}^*} = \mu_{D^*} \wedge \mu_D \wedge \cdots \wedge \mu_D.$$

The following lemma is an immediate consequence of Lemma 2.1.

LEMMA 2.2. *Let* $0 < a < 1$ *and* $\mathcal{D}_a^* = D_a^* \times D_a \times \cdots \times D_a \subset \mathcal{D}^*$, *where* $D_a^* = \{z \in \boldsymbol{C}; 0 < |z| < a\}$ *and* $D_a = \{z \in \boldsymbol{C};\ |z| < a\}$. *Then*

$$\int_{\mathcal{D}_a^*} \mu_{\mathcal{D}^*} < \infty .$$

§ 3. Schwarz lemma for a compact complex manifold with negative first Chern class

Let V be a compact complex manifold of dimension n. Let μ_V be a volume element; it is an everywhere positive $2n$-form. In terms of a local coordinate system $z^1, \cdots, z^n$ of V, μ_V may be written in the form

$$\mu_V = i^n W dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n ,$$

where W is a positive function. To this volume element, we associate the *Ricci tensor*

$$\sum R_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta , \qquad \text{where} \quad R_{\alpha\bar{\beta}} = -\partial^2 \log W / \partial z^\alpha \partial \bar{z}^\beta .$$

If there exists a volume element μ_V such that the associated Ricci tensor $(R_{\alpha\bar{\beta}})$ is negative definite, then we say that the first Chern class $c_1(V)$ of V is negative (denoted by $c_1(V) < 0$). (This is equivalent to say that the canonical line bundle of V is ample, (see [4])).

LEMMA 3.1. *Let V be an n-dimensional compact complex manifold with $c_1(V) < 0$. Let μ_V be a volume element of V such that the associated Ricci tensor is negative definite. Let $\mu_{\mathcal{D}^*}$ be the volume element of $\mathcal{D}^* = D^* \times D \times \cdots \times D$ $(= D^* \times D^{n-1})$ defined in § 2. Then there exists a positive constant c such that, for every holomorphic mapping $f: \mathcal{D}^* \to V$, the inequality*

$$c \cdot f^*(\mu_V) \leqq \mu_{\mathcal{D}^*}$$

holds.

REMARK. In the following, we shall replace the volume element μ_V by $c\mu_V$, so that $f^*(\mu_V) \leqq \mu_{\mathcal{D}^*}$ for every holomorphic mapping $f: \mathcal{D}^* \to V$ (i.e., f is volume-decreasing). The Ricci tensor associated to $c \cdot \mu_V$ is the same as the Ricci tensor associated to μ_V.

PROOF. Let

$$\mathcal{H} = H \times D \times \cdots \times D \ (= H \times D^{n-1}) ,$$

where H denotes the upper-half plane as in § 2. Take the volume element

$$\mu_{\mathcal{H}} = \mu_H \wedge \mu_D \wedge \cdots \wedge \mu_D$$

on $\mathcal{H}$. Let $p: \mathcal{H} \to \mathcal{D}^*$ the covering projection corresponding to the covering projection $p: H \to D^*$. Then

$$p^*(\mu_{\mathcal{D}^*}) = \mu_{\mathcal{H}} .$$

Hence, it suffices to prove $(f \circ p)^*(\mu_V) \leqq \mu_{\mathcal{H}}$. (For this will imply $p^* f^*(\mu_V) \leqq p^*(\mu_{\mathcal{D}^*})$ and hence $f^*(\mu_V) \leqq \mu_{\mathcal{D}^*}$ since p is a local diffeomorphism). In other words, it suffices to prove that every holomorphic mapping $h: \mathcal{H} \to V$ is volume-decreasing, i.e.,

$$h^*(\mu_V) \leqq \mu_{\mathcal{H}}.$$

But this has been already proved in [**3**; Theorem 4.4 of Chapter II] and also in [**2**; Theorem 3]. (Since $\mathcal{H}$ is biholomorphic with an n-dimensional poly-disk and V is a *compact* complex manifold with volume element μ_V whose Ricci tensor is negative definite, all the assumptions in the quoted theorems are satisfied). QED.

§ 4. Proof of Theorem

Let V be a compact complex manifold of dimension n with volume element μ_V. Let K_V denote the canonical line bundle of V. Fix a positive integer m. For each holomorphic section s of K_V^m, we define a non-negative $2n$-form $s\bar{s}/\mu_V^{m-1}$ as follows. In terms of a local coordinate system $z^1, \cdots, z^n$ of V, we write locally

$$s = S \cdot (dz^1 \wedge \cdots \wedge dz^n)^m,$$

$$\mu_V = i^n W dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

where S is a holomorphic function and W is a positive function. Then

$$s\bar{s}/\mu_V^{m-1} = i^n \frac{|S|^2}{W^{m-1}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n.$$

It is easy to see that $s\bar{s}/\mu_V^{m-1}$ is defined independent of the choice of a local coordinate system.

LEMMA 4.1. *Let V be an n-dimensional compact complex manifold with $c_1(V) < 0$ and let μ_V be a volume element whose associated Ricci tensor is negative definite. Let m be a positive integer and s a holomorphic cross section of the line bundle K_V^m. Let $\mathcal{D}^* = D^* \times D^{n-1}$ and $\mathcal{D}_a^* = D_a^* \times D_a^{n-1}$ be as in § 2. Then, for every holomorphic mapping $f: \mathcal{D}^* \to V$, we have*

$$\int_{\mathcal{D}_a^*} f^*(s\bar{s}/\mu_V^{m-1}) < \infty.$$

PROOF. Put

$$s\bar{s}/\mu_V^{m-1} = h \cdot \mu_V.$$

Then h is a non-negative function on V. If we denote by M the maximum value of h on V, then

$$\int_{\mathcal{D}_a^*} f^*(h \cdot \mu_V) \leqq M \int_{\mathcal{D}_a^*} f^*(\mu_V) \leqq M \int_{\mathcal{D}_a^*} \mu_{\mathcal{D}^*} < \infty$$

where the second inequality is a consequence of Lemma 3.1 and the third inequality follows from Lemma 2.2. QED.

In the same way as we defined $s\bar{s}/\mu_V^{m-1}$, we can define a non-negative $2n$-form $f^*s\cdot\overline{f^*s}/\mu_{\mathcal{D}^*}^{m-1}$ on $\mathcal{D}^*$. From Lemma 3.1 and Lemma 4.1, we obtain

LEMMA 4.2. *With the notations in Lemma* 4.1, *we have*

$$\int_{\mathcal{D}_a^*} f^*s\cdot\overline{f^*s}/\mu_{\mathcal{D}^*}^{m-1}<\infty .$$

In the remainder of this section, we shall be concerned only with the domain $\mathcal{D}^*$ and not with the manifold V. We use $z^1, \cdots, z^n$ as a natural coordinate system in C^n so that $\mathcal{D}^*$ is defined by $0<|z^1|<1$, $|z^2|<1, \cdots, |z^n|<1$. Then we can write

$$f^*s=\varphi\cdot(dz^1\wedge\cdots\wedge dz^n)^m ,$$

where φ is a holomorphic function on $\mathcal{D}^*$. Our aim is to prove that φ is meromorphic in the polydisk $\mathcal{D}=D\times\cdots\times D$ $(=D^n)$.

If we write

$$\mu_{\mathcal{D}^*}=i^nKdz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n ,$$

then

$$f^*s\cdot\overline{f^*s}/\mu_{\mathcal{D}^*}^{m-1}=i^n\frac{|\varphi|^2}{K^{m-1}}dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n .$$

Since $\mu_{\mathcal{D}^*}=\mu_{D^*}\wedge\mu_D\wedge\cdots\wedge\mu_D$, we may write

$$K=K_1\cdot K_2 ,$$

where K_1 is a function of z^1 and K_2 is a function of $z^2, \cdots, z^n$. Moreover, in terms of the polar coordinate system $z^1=re^{i\theta}$, we can write

$$iK_1dz^1\wedge d\bar{z}^1=\frac{dr\wedge d\theta}{r(\log r)^2}, \qquad K_1=\frac{1}{2r^2(\log r)^2} .$$

This follows from the formula for μ_H in § 2 and from $p^*(\mu_{D^*})=\mu_H$. Since φ is holomorphic in $\mathcal{D}^*=D^*\times D^{n-1}$, we may write φ in a Laurent series in z^1 as follows:

$$\varphi(z^1, \cdots, z^n)=\sum_{j=-\infty}^{\infty}A_j(z^2, \cdots, z^n)(z^1)^j ,$$

where each A_j is holomorphic in $z^2, \cdots, z^n$. We want to prove that $A_j=0$ for $j\leqq -m$.

Since

$$\int_0^{2\pi}(z^1)^j(\bar{z}^1)^k d\theta=0 \qquad \text{for } j\neq k ,$$

we obtain

$$\int_{\mathscr{D}_a^*} i^n \frac{|\varphi|^2}{K^{m-1}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$$

$$= \sum_j \left\{ \int_{D_a^*} i \frac{|z^1|^{2j}}{K_1^{m-1}} dz^1 \wedge d\bar{z}^1 \right\} \left\{ \int_{D_a^{n-1}} i^{n-1} \frac{|A_j|^2}{K_2^{m-1}} dz^2 \wedge d\bar{z}^2 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \right\}.$$

According to Lemma 4.2, this integral exists and is finite. It suffices therefore to prove

$$I_j = \int_{D_a^*} i \frac{|z^1|^{2j}}{K_1^{m-1}} dz^1 \wedge d\bar{z}^1 = \infty \qquad \text{for } j \leqq -m.$$

In terms of the polar coordinate system, this integral I_j is given by

$$I_j = \int_0^{2\pi} \int_0^a 2^m r^{2m+2j-1} (\log r)^{2m-2} dr\, d\theta$$

$$\geqq 2\pi \cdot 2^m (\log a)^{2m-2} \int_0^a r^{2m+2j-1} dr$$

and hence $I_j = \infty$ for $j \leqq -m$. This implies $A_j = 0$ for $j \leqq -m$.

We summarize what we have proved in the following

LEMMA 4.3. *Let V be an n-dimensional compact complex manifold with $c_1(V) < 0$. Let K_V be the canonical line bundle of V. Let m be a positive integer. Let s be a holomorphic section of K_V^m. If f is a holomorphic mapping of $\mathscr{D}^* = D^* \times D^{n-1}$ into V, then*

$$f^* s = \varphi \cdot (dz^1 \wedge \cdots \wedge dz^n)^m,$$

where $z^1, \cdots, z^n$ is the natural coordinate system in $\boldsymbol{C}^n$ and φ is a holomorphic function in $\mathscr{D}^*$ of the form

$$\varphi(z^1, \cdots, z^n) = \sum_{j=-(m-1)}^{\infty} A_j(z^2, \cdots, z^n)(z^1)^j.$$

Here all A_j are holomorphic functions of $z^2, \cdots, z^n$ in D^{n-1}.

If $c_1(V) < 0$, then a theorem of Kodaira [4] implies that there is a positive integer m such that the line bundle K_V^m admits a sufficiently many holomorphic sections to induce an imbedding of V into a complex projective space. More precisely, let $s_0, s_1, \cdots, s_N$ be a basis for the space $H^0(V, K_V^m)$ of holomorphic sections of K_V^m. Then the mapping $z \in V \rightarrow (s_0(z), s_1(z), \cdots, s_N(z)) \in P_N(\boldsymbol{C})$ is an imbedding. Applying Lemma 4.3 to $s_0, \cdots, s_N$, we obtain our theorem (here we use the non-degeneracy of f).

§ 5. Proofs of Corollaries

To prove Corollary 1, let B be the set of singular points of A. If s is a holomorphic section of K_V^m, then $f^* s$ is meromorphic in $M - B$ by Lemma 4.3.

Since $\dim B \leqq \dim V-2$, f^*s is meromorphic also in M. The remainder of the proof is the same as that of the theorem.

To prove Corollary 2, let f be a biholomorphic mapping of $V-A$ onto itself. By Corollary 1, f is a bimeromorphic mapping of V onto itself. By a theorem of Peters [5] (see [3; Ch. VIII, § 2, Example 1]), f is a biholomorphic mapping of V onto itself.

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Addendum

In his letter of June 5, Professor Kodaira kindly remarked that our theorem can be generalized as follows:

Let V be an n-dimensional compact algebraic manifold of general type. Let M be an n-dimensional complex manifold and A an analytic subvariety of M. If a holomorphic mapping f of $M-A$ into V is non-degenerate at some point, then f is a meromorphic mapping from M into V.

A projective algebraic manifold V of dimension n is said to be of *general type* if

$$\sup \lim_{m\to+\infty} \frac{1}{m^n} \dim H^0(V, K^m) > 0,$$

where K denotes the canonical line bundle of V.

To understand the significance of the condition above, we prove the following lemma. (This lemma will be needed also in the proof of the generalized theorem).

LEMMA 1. *Let F be any line bundle over a projective algebraic manifold V of dimension n. Then*

$$\sup \lim_{m\to+\infty} \frac{1}{m^n} \dim H^0(V, F^m) < \infty .$$

PROOF. Choose an ample line bundle L such that FL is also ample. If m is large enough so that L^m is very ample, then we have an exact sequence

$$0 \longrightarrow H^0(V, F^m) \longrightarrow H^0(V, F^m L^m) \longrightarrow H^0(S, (F^m L^m)_S) \longrightarrow ,$$

where S is a non-singular positive divisor of V obtained as the zero set of a general holomorphic section of L^m, and $(F^m L^m)_S$ denotes the restriction of $F^m L^m$ to S (for the proof of this exact sequence, see Kodaira-Spencer, On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski, Proc. Nat. Acad. Sci. USA, 39 (1953), 1273–1278 or Hirzebruch, Topological methods in algebraic geometry, p. 130). From this exact sequence, we obtain

$$\dim H^0(V, F^m) \leqq \dim H^0(V, F^m L^m) \qquad \text{for } m \geqq m_0 .$$

Since $F^m L^m$ is ample, Kodaira's vanishing theorem implies

$$\dim H^0(V, F^m L^m) = \chi(V, F^m L^m) \qquad \text{for } m \geqq m_0 .$$

On the other hand, we have (see p. 150 of Hirzebruch's book)

$$\chi(V, F^m L^m) = a_0 + a_1 m + \cdots + a_n m^n ,$$

where $a_0, a_1, \cdots, a_n$ are rational numbers determined by V and FL (in particular, $n!\, a_n = (c_1(FL))^n[V]$, where $c_1(FL)$ denotes the Chern class of FL). Hence,

$$\sup \lim_{m \to +\infty} \frac{1}{m^n} \dim H^0(V, F^m) \leqq a_n . \qquad \text{QED.}$$

The main step in the generalization lies in the following construction of a projective imbedding of V by Kodaira.

LEMMA 2. *Let V be a projective algebraic manifold of general type and L a very ample line bundle over V. Then there exists a positive integer m such that $H^0(V, K^m L^{-1}) \neq 0$.*

PROOF. Let X be a non-singular positive divisor of V obtained as the zero set of a general holomorphic section of L. As in the proof of Lemma 1, we have an exact sequence:

$$0 \longrightarrow H^0(V, K^m L^{-1}) \longrightarrow H^0(V, K^m) \longrightarrow H^0(X, K_X^m) \longrightarrow ,$$

where K_X denotes the restriction of the canonical bundle K of V to X. Since $\dim H^0(V, K^m)$ is of order m^n by assumption and since $\dim H^0(X, K_X^m)$ is of order at most m^{n-1}, it follows that $H^0(V, K^m L^{-1}) \neq 0$ if m is sufficiently large. QED.

Let α be a nontrivial holomorphic section of $K^m L^{-1}$. Then we obtain an injection

$$\varphi \in H^0(V, L) \longrightarrow \alpha \cdot \varphi \in H^0(V, K^m) .$$

This means that although K^m is not very ample, we can still obtain a projective imbedding of V by considering the subspace of $H^0(V, K^m)$ consisting of those sections which are divisible by α.

Let $\varphi_0, \varphi_1, \cdots, \varphi_N$ be a basis for $H^0(V, L)$. Then $\alpha\bar{\alpha}\cdot\sum_j \varphi_j\bar{\varphi}_j$ may be considered as a C^∞ section of $(K^m L^{-1})L\otimes(\bar{K}^m\bar{L}^{-1})\bar{L}=K^m\otimes\bar{K}^m$ and can be locally expressed as follows:

$$|a(z)|^2\sum_j|h_j(z)|^2(i^n dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n)^m,$$

where $a(z)$, $h_0(z)$, $h_1(z)$, $\cdots$, $h_N(z)$ are locally defined holomorphic functions corresponding to $\alpha, \varphi_0, \varphi_1, \cdots, \varphi_N$, and $z^1, \cdots, z^n$ is a local coordinate system in V. We consider now a volume element μ_V defined by

$$\mu_V=|a(z)|^{2/m}\cdot W\cdot i^n dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n,$$

where

$$W=(\sum_j|h_j(z)|^2)^{1/m}.$$

We set

$$R_{\alpha\bar{\beta}}=-\partial^2\log W/\partial z^\alpha\partial\bar{z}^\beta.$$

Since $\varphi_0, \varphi_1, \cdots, \varphi_N$ define an imbedding of V into $P_N(\boldsymbol{C})$, it follows that $(R_{\alpha\bar{\beta}})$ is negative definite. Since $a(z)$ is holomorphic, we have

$$R_{\alpha\bar{\beta}}=-\partial^2\log(|a(z)|^{2/m}W)/\partial z^\alpha\partial\bar{z}^\beta$$

wherever $a(z)$ is different from zero. This fact, as Professor Kodaira points out, makes the Schwarz lemma (Lemma 3.1) valid even in the present situation without any change in the proof. Then the remainder of the proof is the same as before except for the following technical point.

Let s be a holomorphic section of K^m divisible by α, i.e., an element in the image of $H^0(V, L)\to H^0(V, K^m)$. In the proof of Lemma 4.1, we defined a non-negative function h on V by

$$s\bar{s}/\mu_V^m=h.$$

Since $s\bar{s}$ and μ_V^m have the common factor $|a(z)|^2$ and since $|a(z)|^2$ is the only contributing factor to the zeros of μ_V^m, h is well defined everywhere on V. This completes the proof of the generalized theorem.

We take this opportunity to give a slightly different and, perhaps, conceptually more natural proof of our theorem. Let s be a holomorphic section of K^m divisible by α. Then

LEMMA 3.

$$\int_{\mathcal{D}_a^*}f^*(s\bar{s})^{1/m}<\infty.$$

This lemma replaces Lemmas 4.1 and 4.2.

PROOF. Since both μ_V and $(s\bar{s})^{1/m}$ have the common factor of $|a(z)|^{2/m}$ and since $|a(z)|^{2/m}$ is the only contributing factor to the zeros of μ_V, we can define a non-negative function h on V by

$$h=(s\bar{s})^{1/m}/\mu_V .$$

Let M be the maximum value of h on V. Then

$$\int_{\mathcal{D}_a^*} f^*(s\bar{s})^{1/m} \leqq \int_{\mathcal{D}_a^*} f^*(h\mu_V) \leqq M\int_{\mathcal{D}_a^*} f^*(\mu_V) \leqq M\int_{\mathcal{D}_a^*} \mu_{\mathcal{D}^*} < \infty ,$$

where the third inequality is a consequence of the Schwarz lemma. QED.

In order to complete the proof, we shall conclude from Lemma 3 that the section f^*s is meromorphic on $\mathcal{D}_a$. It is easy to see that the problem reduces to the following lemma. We learned the proof from Mr. Shintani.

LEMMA 4. *If $f(z)=\sum_{n=-\infty}^{+\infty} a_n z^n$ is a holomorphic function on the punctured unit disk $D^*=\{z\in \boldsymbol{C}\,;\, 0<|z|<1\}$ such that*

$$\int_{D^*} |f(z)|^{2/m} dxdy < \infty$$

for a positive integer m, then f is meromorphic of order $m-1$ at the origin, i.e., $f(z)=\sum_{n=-(m-1)}^{+\infty} a_n z^n$.

PROOF. The problem can be easily reduced to the case where f is of the form

$$f(z)=\sum_{n\leqq -m} a_n z^n .$$

Then we have to show that $f(z)\equiv 0$. Assuming the contrary, let p be the least integer, $p\geqq m$, such that $a_{-p}\neq 0$. If we write

$$f(z)=z^{-p}\sum_{n\leqq -p} a_n z^{n+p}$$

and put

$$g(z)=\sum_{n=0}^{\infty} a_{-n-p} z^n ,$$

then

$$f(z)=z^{-p} g\left(\frac{1}{z}\right)$$

so that $g(z)$ is an entire function with $g(0)\neq 0$. If we put

$$w=1/z \qquad \text{and} \qquad w=u+iv ,$$

then

$$(*) \qquad \int_{D^*} |f(z)|^{2/m} dxdy = \int_{|w|>1} |g(w)|^{2/m} |w|^{(2p-4m)/m} dudv .$$

If we introduce the polar coordinate system $w=re^{i\theta}$, then this integral is equal to

$$(**)\qquad \int_1^\infty\int_0^{2\pi}|g(re^{i\theta})|^{2/m}r^{(2p-3m)/m}d\theta dr\,.$$

Since $|g(w)|^{2/m}=\exp\left(\frac{2}{m}\log|g(w)|\right)$ is subharmonic, we have

$$\int_0^{2\pi}g(re^{i\theta})^{2/m}d\theta\geqq|g(0)|^{2/m}\neq 0\,.$$

Hence, the integral (**) is greater than or equal to

$$|g(0)|^{2/m}\int_1^\infty r^{(2p-3m)/m}dr=\infty$$

because $(2p-3m)/m\geqq-1$. This is a contradiction. QED.

CONCLUDING REMARKS. Given a complex manifold X of dimension n, we define an intrinsic measure μ_X on X as follows ([**3**; Chapter IX]). Let μ be the measure on the unit ball D_n in $\boldsymbol{C}^n$ defined by the Bergman metric of D_n. Given a Borel set B in X, choose holomorphic mappings $f_i\colon D_n\to X$ and Borel sets E_i in D_n for $i=1, 2, \cdots$ such that $B\subset\bigcup_i f_i(E_i)$. Then the measure $\mu_X(B)$ of B is defined by

$$\mu_X(B)=\inf\sum_i\mu(E_i)\,,$$

where the infimum is taken over all possible choices for f_i and E_i. This intrinsic measure possesses the following two properties: (1) if $X=D_n$, then $\mu_X=\mu$; (2) if X and Y are complex manifolds of the same dimension and $f\colon X\to Y$ is a holomorphic mapping, then

$$\mu_Y(f(B))\leqq\mu_X(B)\qquad\text{for every Borel set } B \text{ in } X.$$

Moreover, μ_X is the largest measure on X such that

$$\mu_X(f(E))\leqq\mu(E)\qquad\text{for every Borel set } E \text{ in } D_n$$

and for every holomorphic mapping $f\colon D_n\to X$.

We say that a complex manifold X is *measure-hyperbolic* if $\mu_X(B)$ is nonzero for every nonempty open subset B of X. Some examples of measure-hyperbolic manifolds are given in [**3**]. The fact that the Schwarz lemma (for volume) holds for an algebraic manifold of general type means that an algebraic manifold of general type is measure-hyperbolic.

On the other hand, the intrinsic measure μ_{C^n} for $\boldsymbol{C}^n$ is identically zero. If f is a holomorphic mapping from $\boldsymbol{C}^n$ into a measure-hyperbolic manifold X of dimension n, then for any Borel set B of $\boldsymbol{C}^n$ we have $\mu_X(f(B))\leqq\mu_{C^n}(B)=0$. This implies that $f(B)$ has no interior points, i.e., f is everywhere degenerate. This may be considered as a generalized little Picard theorem. (For another version of the little Picard theorem, see [**3**; Chapter V]). For a certain class of algebraic manifolds containing those of general type, Kodaira has obtained inequalities of Nevanlinna type, which also imply the little Picard theorem for the algebraic manifolds of general type.

Now the following questions arise.

(1) Let X be an n-dimensional complex manifold with the property that every holomorphic mapping $f: \boldsymbol{C}^n \rightarrow X$ is everywhere degenerate. Is X measure-hyperbolic? (The answer is probably negative unless we assume that X is compact. Compare [**3**; Chapter IX, § 3]).

(2) Are there measure-hyperbolic algebraic manifolds which are not of general type?

The main theorem of the present paper may be considered as a result of the type of the great Picard theorem (Compare [**3**; Chapter VI]).

(3) Does our theorem generalize to a compact measure-hyperbolic manifold?