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# Local existence and analyticity of hyperfunction solutions of partial differential equations of first order in two independent variables

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## §1. Introduction.

Let $P$ be a differential operator of first order in two independent variables $x$ and $y$,

$$P=a(x, y)\frac{\partial}{\partial x}+b(x, y)\frac{\partial}{\partial y}+c(x, y).$$

Here we assume that the coefficients $a$, $b$ and $c$ are (complex-valued) real analytic functions defined in an open set $\Omega$ in $\boldsymbol{R}^2$, and that

$$|a(x, y)|+|b(x, y)|\neq 0.$$

In this paper we shall study conditions for the local existence and analyticity of hyperfunction solutions of the equation $Pu=f$. The basic facts about the theory of hyperfunctions may be found in [2], [4]. We denote by $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{O}$ the sheaves of real analytic functions, hyperfunctions, and holomorphic functions, respectively.

Let $p$ be the principal part of $P$. We define the $k$-th commutator $c_p^k$ of $p$ by induction:

$$c_p^0=\bar{p}=\text{the operator with complex conjugate coefficients},$$

$$c_p^k=[p, c_p^{k-1}]=pc_p^{k-1}-c_p^{k-1}p.$$

Let $k_p(x, y)$ denote the first value of $k$ for which $c_p^k$ is not proportional to $p$ at the point $(x, y)$. If $c_p^k$ is proportional to $p$ for all values of $k$, we define $k_p(x, y)$ to be $\infty$. Note that $P$ is elliptic at $(x, y)$, if and only if $k_p(x, y)=0$. It is easily seen that $k_p(x, y)$ does not depend on the choice of local coordinates, and that it is invariant under multiplication of $P$ by a non-vanishing function.

Our main results are the following two theorems which state the relation between the parity of $k_p(x, y)$ and the analyticity and existence of hyperfunction solutions of $Pu=f$.

THEOREM A. (Analyticity of solutions). *The following two conditions on $P$ are equivalent:*

(a) *For every open subset $\omega$ of $\Omega$, if $u \in \mathcal{B}(\omega)$ and $Pu \in \mathcal{A}(\omega)$, then $u \in \mathcal{A}(\omega)$.*

(A) *At every point $(x, y)$ of $\Omega$, $k_p(x, y)$ is even.*

THEOREM B. (Local existence of solutions). *The following two conditions on $P$ are equivalent:*

(b) *For every point $(x, y)$ of $\Omega$, there exists a neighborhood $\omega$ of $(x, y)$ such that $P\mathcal{B}(\omega) = \mathcal{B}(\omega)$.*

(B) *At every point $(x, y)$ of $\Omega$, $k_p(x, y)$ is either even or $\infty$.*

Recently M. Sato [**1**], [**5**] has proved the analyticity of hyperfunction solutions of elliptic differential equations. As to the problem of existence, P. Schapira [**6**] has given an example of equation without solutions in the space of hyperfunctions. Later Schapira [**7**], [**8**] has shown that for differential equations of first order in any number of independent variables, the condition ($P'$) of Nirenberg and Treves [**3**] is a necessary and sufficient condition for the local existence of hyperfunction solutions. Schapira uses the technique of *a priori* inequalities, whereas our proof is based on the behavior of characteristic curves in the complex domain.

## § 2. Characteristic curves in the complex domain.

We extend the functions $a$, $b$, and $c$ to complex values of the independent variables $x$ and $y$. From now on $x$ and $y$ will denote complex variables.

We shall study the behavior of the characteristic curves in the complex domain. They are solutions of the system of equations

$$\frac{dx}{ds} = a(x, y), \qquad \frac{dy}{ds} = b(x, y), \tag{2.1}$$

where $s$ is a complex variable. Let $(x_0, y_0)$ be a point of $\Omega$ and let $x = x(s, t)$, $y = y(s, t)$ be a solution of (2.1) containing a parameter $t$ such that $x(0, 0) = x_0$, $y(0, 0) = y_0$ and the Jacobian $\partial(x, y)/\partial(s, t)$ evaluated at $(0, 0)$ is different from zero. If the domain of definition $V$ of $x(s, t)$ and $y(s, t)$ is a sufficiently small open neighborhood of $(0, 0)$ in $\boldsymbol{C}^2$, then the mapping $(s, t) \to (x(s, t), y(s, t))$ is a one-to-one transformation from $V$ onto a complex open neighborhood $U$ of $(x_0, y_0)$ and has a holomorphic inverse.

THEOREM 1. *The condition* (A) *holds in $\Omega$, if and only if,*

($\alpha$) *Every point of $\Omega$ has a neighborhood in which every characteristic curve has at most one* real *point.*

*The condition* (B) *holds in $\Omega$, if and only if,*

($\beta$) *Every point of $\Omega$ has a neighborhood in which every characteristic curve either has at most one* real *point or else is a* real *curve.*

PROOF. Let $(x_0, y_0)$ be a point of $\Omega$. It is possible to introduce new *real* local coordinates in a neighborhood of $(x_0, y_0)$, so that the operator takes the form

$$p = a'(x, y)\Big(\frac{\partial}{\partial x} + ib'(x, y)\frac{\partial}{\partial y}\Big),$$

where $a'(x, y)$ is a non-vanishing complex-valued analytic function, and $b'(x, y)$ is a *real*-valued analytic function [3]. Since the conditions (A), (B), $(\alpha)$ and $(\beta)$ are local and invariant under a *real* change of variables and multiplication of $p$ by a non-vanishing function, we may suppose that $p$ has the form

$$p = \frac{\partial}{\partial x} + ib(x, y)\frac{\partial}{\partial y}, \tag{2.2}$$

where $b(x, y)$ is a *real*-valued analytic function.

For the operator $p$ of the form (2.2), $c_p^k$ becomes

$$c_p^k = \Big(-2i\frac{\partial^k b}{\partial x^k} + \sum_{j=0}^{k-1} d_{jk}\frac{\partial^j b}{\partial x^j}\Big)\frac{\partial}{\partial y}, \qquad k \geqq 1,$$

so that

$$k_p(x, y) = \min\Big\{k\,;\ \frac{\partial^k b(x, y)}{\partial x^k} \neq 0\Big\}. \tag{2.3}$$

Hence, at a real point $(x, y)$ on an integral curve $y = \varphi(x)$ of the characteristic equation $dy/dx = ib(x, y)$, we have

$$k_p(x, y) = \min\{k\,;\ \operatorname{Im}\varphi^{(k+1)}(x) \neq 0\} \tag{2.4}$$

and

$$\operatorname{Im}\varphi^{(k+1)}(x) = \frac{\partial^k b(x, y)}{\partial x^k} \qquad \text{when } k = k_p(x, y). \tag{2.5}$$

(A) $\Rightarrow$ $(\alpha)$. It follows from (2.3) that, if the condition (A) holds, the sign of the function $b(x, y)$ does not vary with $x$ and $y$. Neither does the sign of $(\partial/\partial x)^{k_p} b(x, y)$. Suppose that, on a characteristic curve $y = \varphi(x)$, there were more than one real points. When $x$ moves along the real axis, it follows from (2.4) that every zero of $\operatorname{Im}\varphi(x)$ is of odd order $k_p+1$. Hence the sign of $\operatorname{Im}\varphi^{(k_p+1)}(x)$ changes at successive zeros of $\operatorname{Im}\varphi(x)$ on the real axis. In view of (2.5), this is a contradiction.

(B) $\Rightarrow$ $(\beta)$. Suppose that the condition (B) holds in $\Omega$. Let $(x_0, y_0)$ be a point of $\Omega$ which we may assume to be $(0, 0)$ without loss of generality. Since $b(x, y)$ is an analytic function, it follows from (2.3) that, for fixed $y$, the sign of $b(x, y)$ does not vary with $x$. Hence, if we choose a neighborhood $\omega$ of $(0, 0)$ of sufficiently small width in the direction of the $y$-axis, the sign of $b(x, y)$ does not vary with $x$ and $y$ in each of the two regions $\omega^+ = \omega \cap \{y > 0\}$ and $\omega^- = \omega \cap \{y < 0\}$.

When $b(x, y)$ does not change sign in the whole of $\omega$, by the same argument as in the proof of (A)$\Rightarrow(\alpha)$, we prove that if a characteristic curve is not a real curve, then it has at most one real point.

When $b(x, y)$ changes sign, $b(x, 0)$ vanishes identically and there is a constant $C>0$ such that $|b(x, y)| \leqq C|y|$. Hence for every characteristic curve $y=\varphi(x)$ except $y=0$ we have

$$|\arg \varphi(x_2)-\arg \varphi(x_1)| \leqq \left|\int_{x_1}^{x_2} \frac{\varphi'(x)dx}{\varphi(x)}\right| \leqq C|x_2-x_1|.$$

Let the diameter of $\omega$ be smaller than $\pi/C$. Suppose that there were a characteristic curve which is not a real curve and has more than one real points. Then the sign of $\operatorname{Im} \varphi^{(k_p+1)}(x)$ changes at successive zeros $x_1$ and $x_2$ of $\operatorname{Im} \varphi(x)$ on the real axis and so does the sign of $(\partial/\partial x)^{k_p} b(x, y)$ in view of (2.5). Hence $\varphi(x_1)$ and $\varphi(x_2)$ must be of opposite signs. It follows therefore that $|x_2-x_1| \geqq \pi/C$. Contradiction.

$(\beta) \Rightarrow$ (B). Suppose that the condition (B) does not hold in $\Omega$. Then there is a point $(x_0, y_0)$ in $\Omega$ such that $k_p(x_0, y_0)$ is odd. We may suppose $(x_0, y_0) =(0, 0)$. We denote by $\varphi(x, t)$ the solution of the equation $dy/dx=ib(x, y)$ such that $\varphi(0, t)=it$. We have then

$$\varphi(x, 0)=idx^{k_0+1}+\cdots,$$

where $d=(\partial/\partial x)^{k_0}b(0, 0)/(k_0+1)!$, $k_0=k_p(0, 0)$. Hence

$$\varphi(x, t)=(i+o(1))t+(id+o(1))x^{k_0+1}, \tag{2.6}$$

so that for sufficiently small real $x$ and real $t$ we have

$$|\operatorname{Im} \varphi(x, t)-t-dx^{k_0+1}| \leqq (|t|+|d|x^{k_0+1})/2.$$

When $dt<0$ it follows that $\operatorname{Im} \varphi(x, t)$ lies between $(t+3dx^{k_0+1})/2$ and $(3t+dx^{k_0+1})/2$. Thus if $t$ is sufficiently small, the characteristic curve $y=\varphi(x, t)$ has more than one real points and is not a real curve.

$(\alpha) \Rightarrow$ (A). Suppose that the condition (A) does not hold in $\Omega$. Then there is a point $(x_0, y_0)$ in $\Omega$ such that $k_p(x_0, y_0)$ is either odd or $\infty$. If $k_p(x_0, y_0)$ is odd, $(\beta)$ is not valid. If $k_p(x_0, y_0)=\infty$, the characteristic curve passing through $(x_0, y_0)$ is a real curve. In either case, in any neighborhood of $(x_0, y_0)$, we can find a characteristic curve having more than one real points.

## § 3. Some lemmas.

First note that the statements of Theorems A and B are local and invariant under a real change of variables and multiplication of $P$ by a non-vanishing function.

The Cauchy-Kovalevsky theorem implies that the equation $ph+c=0$ has an analytic solution in sufficiently small open sets. If we set $u=ve^{h}$, the equation $Pu=f$ is transformed into $pv=e^{-h}f$. Hence we need only prove the Theorems A and B for homogeneous operators $p$.

As to Theorem A, since we can locally solve $pv=e^{-h}f$ with $v$ analytic, it is enough to prove the equivalence of the condition (A) to the following.

$(a_0)$ For every open subset $\omega$ of $\Omega$, if $u\in\mathscr{B}(\omega)$ and $pu=0$, then $u\in\mathscr{A}(\omega)$.

Let $\omega$ be an open set in $\boldsymbol{R}^2$ and let $U$ be a complex neighborhood of $\omega$, that is, an open set in $\boldsymbol{C}^2$ which contains $\omega$ as a relatively closed subset. We use the following notation:

$$U_1=\{(x,y)\in U\,;\,\operatorname{Im}x\neq 0\}\,,\qquad U_2=\{(x,y)\in U\,;\,\operatorname{Im}y\neq 0\}\,,$$
$$U_1^{\pm}=\{(x,y)\in U\,;\,\operatorname{Im}x\gtrless 0\}\,,\qquad U_2^{\pm}=\{(x,y)\in U\,;\,\operatorname{Im}y\gtrless 0\}\,,$$
$$U^{\sigma}=\{(x,y)\in U\,;\,(\operatorname{Im}x,\operatorname{Im}y)\text{ is in the }\sigma\text{th quadrant of }\boldsymbol{R}^2\}\,.$$

If we choose a Stein neighborhood $U$ such that $U\cap\boldsymbol{R}^2=\omega$, then $\mathscr{B}(\omega)$ may be identified with $\mathcal{O}(U_1\cap U_2)/(\mathcal{O}(U_1)+\mathcal{O}(U_2))$. Hence a hyperfunction $u\in\mathscr{B}(\omega)$ is represented by a holomorphic function defined in $U_1\cap U_2$ which we call a defining function of $u$. We denote by $u^{\sigma}$ the restriction to $U^{\sigma}$ of a defining function. If each $u^{\sigma}$ can be analytically continued across $\omega$, that is, if there is a complex open set $W$ containing $\omega$ such that each $u^{\sigma}$ has a holomorphic extension to $W\cap U^{\sigma}$, then the hyperfunction $u$ is a real analytic function. Conversely, if $u$ is a real analytic function and all $u^{\sigma}$ except one vanish identically, then $u^{\sigma}$ can be analytically continued across $\omega$.

From now on $p$ will denote both the operator on $\mathscr{B}$ and on $\mathcal{O}$. If we choose $U$ such that $p\mathcal{O}(U_i)=\mathcal{O}(U_i)$, the conditions $(a_0)$ and $(b)$ can be transformed into conditions expressed in terms of defining functions of hyperfunctions.

LEMMA 1. *Let $U$ be a Stein neighborhood of $\omega\subset\Omega$ such that $p\mathcal{O}(U_i)=\mathcal{O}(U_i)$, $i=1,2$.*

*The following two conditions are equivalent:*

$(a_0)_\omega$ *If $u\in\mathscr{B}(\omega)$ and $pu=0$, then $u\in\mathscr{A}(\omega)$.*

$(\tilde{a}_0)_\omega$ *For each $\sigma$, if $u^{\sigma}\in\mathcal{O}(U^{\sigma})$ and $pu^{\sigma}=0$, then $u^{\sigma}$ can be analytically continued across $\omega$.*

*The following two conditions are equivalent:*

$(b)_\omega$ $p\mathscr{B}(\omega)=\mathscr{B}(\omega)$.

$(\tilde{b})_\omega$ *For each $\sigma$, $p\mathcal{O}(U^{\sigma})=\mathcal{O}(U^{\sigma})$.*

PROOF. $(\tilde{a}_0)_\omega\Rightarrow(a_0)_\omega$. Let $u\in\mathscr{B}(\omega)$ be represented by $u^{\sigma}\in\mathcal{O}(U^{\sigma})$. Then $pu=0$ means $pu^{\sigma}=e_1+e_2$, $e_i\in\mathcal{O}(U_i)$. By hypothesis on $U$, there exist $u_i\in\mathcal{O}(U_i)$ such that $pu_i=e_i$. Then $u$ is represented also by $u'^{\sigma}=u^{\sigma}-u_1-u_2$ for which we have $pu'^{\sigma}=0$. It follows from $(\tilde{a}_0)_\omega$ that $u'^{\sigma}$ can be analytically continued across $\omega$. Hence $u \in \mathcal{A}(\omega)$.

$(b)_\omega \Rightarrow (\tilde{b})_\omega$. Given $f^\sigma \in \mathcal{O}(U^\sigma)$, it follows from $(b)_\omega$ that we can find $u^\sigma \in \mathcal{O}(U^\sigma)$ such that $pu^\sigma - f^\sigma \in \mathcal{O}(U_1) + \mathcal{O}(U_2)$. Thus there exist $e_i \in \mathcal{O}(U_i)$ such that $pu^\sigma - f^\sigma = e_1 + e_2$. By hypothesis on $U$, there exist $u_i \in \mathcal{O}(U_i)$ such that $pu_i = e_i$. If we set $u'^\sigma = u^\sigma - u_1 - u_2$, then $u'^\sigma \in \mathcal{O}(U^\sigma)$ and $pu'^\sigma = f^\sigma$.

The implications $(a_0)_\omega \Rightarrow (\tilde{a}_0)_\omega$ and $(\tilde{b})_\omega \Rightarrow (b)_\omega$ are obvious.

In the construction of a Stein neighborhood $U$ satisfying the hypothesis of Lemma 1 and also in the proof of Theorem B we need the following lemma.

LEMMA 2. *Let $G$ be an open set in the space of two complex variables $(x, t)$. If every section $G(t)$ of $G$ by $t = \text{const}$ is simply-connected, then $(\partial/\partial x)\mathcal{O}(G) = \mathcal{O}(G)$.*

PROOF. We denote by $\pi$ the projection $(x, t) \rightarrow t$. Since every $G(t)$ is simply-connected, there exists an open covering $\{N_i\}$ of $\pi(G)$ such that $(\partial/\partial x)\mathcal{O}(G_i) = \mathcal{O}(G_i)$, where $G_i = G \cap \pi^{-1}(N_i)$. For any $g \in \mathcal{O}(G)$, we can then find a solution $v_i \in \mathcal{O}(G_i)$ of $(\partial/\partial x)v_i = g$. Since $(\partial/\partial x)(v_i - v_j) = 0$ in $G_i \cap G_j$ and all the sections $G(t)$ are connected, $v_i(x, t) - v_j(x, t)$ is a function of $t$ alone which we denote by $w_{ij}(t)$, $t \in N_i \cap N_j$. Since the first Cousin problem has a solution in any open set in the complex plane, we can find $w_i(t) \in \mathcal{O}(N_i)$ so that $w_{ij} = w_i - w_j$ in $N_i \cap N_j$. If we set $v(x, t) = v_i(x, t) - w_i(t)$ in $G_i$, then $v \in \mathcal{O}(G)$ and $(\partial/\partial x)v = g$. QED.

We keep the notation used in the proof of Theorem 1. The change of variables $(x, y) \rightarrow (x, x+y)$ transforms the differential operator into

$$p = \frac{\partial}{\partial x} + (1 + ib(x, y-x))\frac{\partial}{\partial y}$$

and the equation of characteristic curves into $y = \psi(x, t) \equiv x + \varphi(x, t)$. If the domain of definition $V$ of $\psi(x, t)$ is a sufficiently small open neighborhood of $(0, 0)$ in $\boldsymbol{C}^2$, then the mapping $(x, t) \rightarrow (x, \psi(x, t))$ is a one-to-one transformation from $V$ onto a complex open neighborhood $U$ of $(0, 0)$ and has a holomorphic inverse. Set $V = X \times T$, where $X$ and $T$ are rectangles about 0 in the complex plane. Then $U$ is a Stein manifold.

Under the *complex* change of variables $(x, y) \rightarrow (x, t)$, we have the following table of corresponding quantities:

| | |
|---|---|
| $pu(x, y) = f(x, y)$; | $(\partial/\partial x)v(x, t) = g(x, t)$, |
| $U, U_i, U_i^\pm, U^\sigma$; | $V, V_i, V_i^\pm, V^\sigma$, |
| characteristic curve $C_t: y = \psi(x, t)$; | $V(t) = X$, |
| $C_t \cap U_i^\pm, C_t \cap U^\sigma$; | $V_i^\pm(t), V^\sigma(t)$. |

We denote by $x'$ and $x''$ the real and imaginary part of $x$, respectively.

Since $(\partial/\partial x'')\operatorname{Im}\psi(x'+ix'', t)=1$ when $(x, t)=(0, 0)$, we can solve the equation $\operatorname{Im}\psi(x'+ix'', t)=0$ for $x''$ and write $x''=\xi(x', t)$, if $V$ is taken sufficiently small. Thus we have $V_1^{\pm}(t)=\{x\in X; x''\gtreqless 0\}$, $V_2^{\pm}(t)=\{x\in X; x''\gtreqless\xi(x', t)\}$ and $V^{\sigma}(t)=\{x\in X; x''\gtreqless 0$ and $x''\gtreqless\xi(x', t)\}$, so $V_1^{\pm}(t)$ and $V_2^{\pm}(t)$ are simply-connected. In view of Lemma 2, $U$ satisfies the hypothesis of Lemma 1.

## § 4. Proof of Theorem A.

First note that real points on the characteristic curve $y=\psi(x, t)$ correspond to zeros of $\xi(x', t)$ and that the characteristic curve is a real curve, if and only if $\xi(x', t)$ vanishes identically. When $k_0=k_p(0, 0)$ is finite, we obtain from (2.6)

$$\psi(x, t)=(i+o(1))t+x+(id+o(1))x^{k_0+1}.$$

Hence, if $t$ is real, $\xi(x', 0)=-dx'^{k_0+1}+\cdots$ and $\xi(0, t)=-t+\cdots$, so that we have

$$\xi(x', t)=-(1+o(1))t-(d+o(1))x'^{k_0+1}. \tag{4.1}$$

When $k_0=\infty$, $\xi(x', 0)$ vanishes identically.

(A)$\Rightarrow$($a_0$). Suppose that the condition (A) holds in a neighborhood of (0, 0). Since $k_0+1$ is odd, it follows from (4.1) with $t=0$ that $V^{\sigma}(0)\neq\emptyset$ for every $\sigma$. Furthermore, in view of ($\alpha$), for every $t\in T$, $\xi(x', t)$ has at most one zero. Hence $V^{\sigma}(t)$ is connected or empty.

Let $u^{\sigma}\in\mathcal{O}(U^{\sigma})$ be a solution of $pu^{\sigma}=0$. We shall show that $u^{\sigma}$ can be analytically continued to a neighborhood of (0, 0). Under the change of variables $(x, y)\to(x, t)$, there corresponds to $u^{\sigma}$ a solution $v^{\sigma}\in\mathcal{O}(V^{\sigma})$ of $(\partial/\partial x)v^{\sigma}=0$. Since all the sections $V^{\sigma}(t)$ are connected, $v^{\sigma}(x, t)$ is a function of $t$ alone, so it can be analytically continued to $\tilde{V}^{\sigma}=\{(x, t)\in V; V^{\sigma}(t)\neq\emptyset\}$. Since $V^{\sigma}(0)\neq\emptyset$, $\tilde{V}^{\sigma}$ is an open neighborhood of (0, 0). Thus $u^{\sigma}$ can be continued to a neighborhood $\tilde{U}^{\sigma}$ of (0, 0).

($a_0$)$\Rightarrow$(A). If $k_0$ is either odd or $\infty$, then $V^{\sigma}(0)=\emptyset$ for some $\sigma$. This means that $t\neq 0$ when $(x, t)\in V^{\sigma}$. If we set $v^{\sigma}(x, t)=t^{-1}$, then $v^{\sigma}\in\mathcal{O}(V^{\sigma})$ and $(\partial/\partial x)v^{\sigma}=0$ while $v^{\sigma}$ can not be continued analytically to a neighborhood of (0, 0).

## § 5. Proof of Theorem B.

(B)$\Rightarrow$($b$). Suppose that the condition (B) holds in a neighborhood of (0, 0). It follows from ($\beta$) that either $\xi(x', t)$ has at most one zero or else vanishes identically. Hence for every $\sigma$ and every $t$, $V^{\sigma}(t)$ is simply-connected or empty. We have by Lemma 2 $(\partial/\partial x)\mathcal{O}(V^{\sigma})=\mathcal{O}(V^{\sigma})$, so that $p\mathcal{O}(U^{\sigma})=\mathcal{O}(U^{\sigma})$.

$(b)\Rightarrow$(B). Let $k_0=k_p(0, 0)$ be odd. It follows from (4.1) that for sufficiently small real $t$ and $x'$

$$|\xi(x', t)+t+dx'^{k_0+1}|\leqq(|t|+|d|x'^{k_0+1})/2\,.$$

To fix ideas, suppose that $d>0$. Then we have

$$\xi(x', t)\leqq(-t-dx'^{k_0+1})/2$$

when $t\geqq 0$. If we set $S(t)=\{x\in X;(-t-dx'^{k_0+1})/2<x''<0\}$ for $t\geqq 0$, then $S(t)\subset V^{\sigma}(t)$, $\sigma=2$.

Set $f(x, y)=1/xy$ and $g(x, t)=1/x\psi(x, t)$. $f(x, y)$ is a defining function of $(2\pi i)^2\delta(x, y)$. We now claim that the equation $(\partial/\partial x)v=g$ has no solution $v\in\mathcal{O}(V^2)$. Suppose on the contrary that there were a solution $v\in\mathcal{O}(V^2)$ of $(\partial/\partial x)v=g$. Choose two points $x_1$ and $x_2$ in $S(0)$ so that $x_1'<0$ and $x_2'>0$. For $t>0$ let $\Gamma(t)$ be a path from $x_1$ to $x_2$ lying in $S(t)$. Then we have

$$\int_{\Gamma(t)}g(x, t)dx=v(x_2, t)-v(x_1, t)\,.$$

As $t\downarrow 0$ the right hand side tends to $v(x_2, 0)-v(x_1, 0)$.

On the other hand, $\psi(x, t)$ does not vanish when $x''>(-t-dx'^{k_0+1})/2$. Let $\Gamma_1(t)$ be a circle with center at $x=0$ which is oriented counter-clockwise and contained in the region $\{x\in X; x''>(-t-dx'^{k_0+1})/2\}$. Let $\Gamma_2$ be a path from $x_1$ to $x_2$ lying in the region $\{x\in X; x''>-dx'^{k_0+1}/2\}$. Then we have

$$\int_{\Gamma(t)}g(x, t)dx=\int_{\Gamma_1(t)}g(x, t)dx+\int_{\Gamma_2}g(x, t)dx\,.$$

The first integral on the right hand side is equal to $2\pi i/\psi(0, t)=2\pi/t$. As $t\downarrow 0$ the second integral tends to the integral of $g(x, 0)$ along $\Gamma_2$. Hence the left hand side tends to $\infty$. This gives a contradiction.

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