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Homogeneous hypersurfaces in spaces of constant curvature

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Introduction.

S. Kobayashi proved [4] that a connected compact homogeneous Riemannian manifold of dimension n is isometric to the sphere if it is isometrically imbedded in the Euclidean space E^{n+1} of dimension $n+1$. T. Nagano and the present author proved [5] that a connected homogeneous Riemannian manifold M of dimension n is isometric to the Riemannian product of a sphere and a Euclidean space if M is isometrically imbedded in the Euclidean space E^{n+1} and the rank of the second fundamental form (which is called the type number in this paper) is not equal to 2 at some point.

One of the purposes of the present paper is to consider the case which was not treated in [5], that is, the case of the type number being equal to 2.

In this paper we consider the isometric immersion of a connected homogeneous Riemannian manifold of dimension n not only in a Euclidean space E^{n+1}, but also in a hyperbolic space H^{n+1} and we determine all the types of M.

Let $S^m(K)$ denote an m-dimensional sphere of radius $1/K$ for a positive constant K and $H^m(K)$ denote an m-dimensional hyperbolic space of negative curvature K for a negative constant K.

The underlying manifold of $H^m(K)$ is that of a Euclidean space E^m and the Riemannian metric of $H^m(K)$ is given by

$$ds^2=\sum_i (dx_i)^2+\frac{K}{1-K\sum_i (x_i)^2}(\sum_i x_i dx_i)^2 .$$

The main theorems are the following:

THEOREM A. *If a connected homogeneous Riemannian manifold M of dimension n admits an isometric immersion f in a Enclidean space E^{n+1}, M is isometric to $S^m\times E^{n-m}$ $(0\leqq m\leqq n)$. If the type number of f is greater than* 1 *at a point, f is an imbedding.*

THEOREM B. *If a connected homogeneous Riemannian manifold M of dimension n admits an isometric immersion f in a hyperbolic space $H^{n+1}(K)$ of curvature $K(<0)$, the type number $t(p)$ of f is either constantly equal to n or*

$t(p)\leqq 1$ at each point p of M and we have the following:

I) *If $t(p)\leqq 1$, M is isometric to $H^n(K)$.*

II) *If $t(p)=n$ and the immersion is totally umbilical, f is an imbedding and M is isometric to $S^n(K_1)$ with $K_1>0$ or $H^n(K_1)$ with $0\geqq K_1>K$, where $H^n(0)=E^n$.*

III) *If $t(p)=n$ and the immersion is not totally umbilical, the immersed hypersurface $f(M)$ is isometric to $S^m(K_1)\times H^{n-m}(K_2)$ $(0<m<n)$, where K_1 and K_2 satisfy the relation $\frac{1}{K_1}+\frac{1}{K_2}=\frac{1}{K}$. If $m\neq 1$, f is an imbedding.*

In § 1, we summarize the fundamental formulas and theorems in the theory of hypersurfaces in a space of constant curvature for later use. Theorem 1.3 suggests us that it is convenient to devide the proofs of the main theorems into three cases according to the type number of the immersion; in the first case the type number is greater than 2, in the second case it is equal to 2, and in the last case it is smaller than 2.

The proofs of the main theorems are given in §§ 2~4 parallely.

§ 2 is the first case. In this case the universal covering manifold $\tilde{M}$ of M is, roughly speaking, a Riemannian product of two manifolds of constant curvature and $\tilde{M}$ has a natural imbedding f_0 into $H^{n+1}(K)$ $(K\leqq 0)$ and by Theorem 1.3 we see that f_0 differs with $f\circ\varphi$ in an isometry of $H^{n+1}(K)$ (φ is a covering map). So we can conclude that f is an imbedding and $\tilde{M}=M$.

In § 3 we consider the second case. At first we prove that in a hyperbolic space there exist no homogeneous hypersurface whose type number is equal to 2 (Lemma 3.5). In a Euclidean space we see that M has the involutive distributions D_1 and D_2 and the integral manifolds of D_1 and D_2 are orthogonal with each other. The integral manifolds of D_2 are contained in parallel planes of dimension 3 and isometric to S^2 and those of D_1 are the parallel planes of dimension $n-2$.

§ 4 is the case of the type number being smaller than 2. In this case, M is a space of constant curvature with the same curvature as the ambient space, if the ambient space is $H^{n+1}(K)$ $(K<0)$.

§ 1. Preliminaries.

Throughout this paper, we denote by M a connected Riemannian manifold of dimension $n(n>2)$ and by $F(M)$ the bundle of the orthogonal frames of M. $F(M)$ is a principal fibre bundle over M with group $O(n)$. The projection of $F(M)$ onto M is denoted by π. The right translation of $F(M)$ by $a=(a_{ij})$*) $\in O(n)$ is denoted by R_a; for a frame $u=(p; e_1, \cdots, e_n)$ at $p\in M$,

*) Throughout this paper, the indices i, j, k, l run over the range $1, \cdots, n$ and the indices A, B run over the range $1, \cdots, n+1$.

$R_a u = ua$ is a frame $(p; \sum_i a_{i1}e_i, \cdots, \sum_i a_{in}e_i)$ at $p \in M$.

The canonical forms $\omega_1, \cdots, \omega_n$ of $F(M)$ are the linear differential forms on $F(M)$ which are defined by the following equations:

$$d\pi(X) = \sum_i \omega_i(X)e_i \tag{1.1}$$

where X is a tangent vector to $F(M)$ at $u=(p; e_i, \cdots, e_n)$ and $d\pi$ is a differential map of π.

The Riemannian connection forms (or simply connection forms) ω_{ij} on $F(M)$ are the linear differential forms on $F(M)$ which are uniquely determined by the following conditions:

$$\omega_{ij} + \omega_{ji} = 0 \quad \text{and} \quad d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0\,. \tag{1.2}$$

The curvature forms Ω_{ij} of the connection are given by

$$\Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} \tag{1.3}$$

Let $I(M)$ denote the group of all isometries of M and $G = I_0(M)$ the identity component of $I(M)$. $I(M)$ acts on $F(M)$ in a natural manner: for $u=(p, e_i, \cdots, e_n) \in F(M)$ and $g \in I(M)$, $g(u)$ is, by definition, a frame $(g(p), dg(e_i), \cdots, dg(e_n))$ at a point $g(p)$. Then $g \in I(M)$ commutes with the projection and the right translation R_a $(a \in O(n))$:

$$\pi \circ g = g \circ \pi \quad \text{and} \quad g \circ R_a = R_a \circ g\,. \tag{1.4}$$

It is easily seen that the action of $I(M)$ leaves the differential forms ω_i, ω_{ij}, Ω_{ij} invariant: for $g \in I(M)$ we have

$$g^*\omega_i = \omega_i\,, \quad g^*\omega_{ij} = \omega_{ij}\,, \quad \text{and} \quad g^*\Omega_{ij} = \Omega_{ij}\,. \tag{1.5}$$

If $I(M)$ acts on M transitively (then $G = I_0(M)$ acts also transitively on M), M is said to be a homogeneous Riemannian manifold. If M is assumed to be homogeneous, M is identified with the factor space G/H of G by the isotropy group H at a fixed point $o \in M$, and G is considered as a principal fibre bundle over M with group H. If we fix a frame u_0 at o, the orbit $G(u_0)$ of u_0 under the action of G on $F(M)$ is a subbundle of $F(M)$ which is isomorphic to the bundle G over $M = G/H$. If we identify G with $G(u_0)$ by this isomorphism, the restriction of the differential forms ω_i, ω_{ij} and Ω_{ij} to G are the left-invariant differential forms on G.

Let V denote, throughout this paper, one of $S^{n+1}(K)$, E^{n+1} and $H^{n+1}(K)$. Let $F(V)$ (resp. $F_0(V)$) denote the bundle of the frames (resp. oriented frames) of V and the differential forms θ_A, θ_{AB} and Θ_{AB} denote the canonical forms, the connection forms and the curvature forms respectively, then in the case of the manifold of constant curvature K the curvature forms Θ_{AB} are written as

$$\Theta_{AB}=K\theta_A\wedge\theta_B\,. \tag{1.6}$$

Fixing a frame $v_0\in F(V)$, the group $I(V)$ of all isometries of V can be identified with $F(V)$ and the identity component $I_0(V)$ is identified with $F_0(V)$ if $v_0\in F_0(V)$.

Let $f: M\to V$ be an isometric immersion of M into V, f induces a bundle isomorphism $\tilde{f}$ of $F(M)$ into $F_0(V)$. For a frame $u=(p; e_1, \cdots, e_n)\in F(M)$, there exists a unique tangent vector e_{n+1} to $F_0(V)$ at $f(p)$ such that $(f(p), df(e_1), \cdots, df(e_n), e_{n+1})$ is a frame in $F_0(V)$. $\tilde{f}(u)$ is, by definition, this frame. $\tilde{f}$ satisfies the following:

$$\pi\circ\tilde{f}=\tilde{f}\circ\pi \qquad \tilde{f}\circ R_a=R_{\sigma(a)}\circ\tilde{f}\,, \tag{1.7}$$

where σ is an isomorphism of $O(n)$ into $SO(n+1)$ defined by

$$\sigma(a)=\begin{pmatrix} a & 0 \\ 0 & \det a\end{pmatrix} \qquad \text{for } a\in O(n)\,. \tag{1.8}$$

Also f gives the following relations between the canonical forms on $F(M)$ and on $F_0(V)$ and also between the connection forms:

$$\tilde{f}^*\theta_i=\omega_i\,, \qquad f^*\theta_{n+1}=0\,, \qquad \tilde{f}^*\theta_{ij}=\omega_{ij}\,. \tag{1.9}$$

Thus $\tilde{f}^*\theta_A$ and $\tilde{f}^*\theta_{ij}$ do not depend on the immersion f, but the induced forms $\tilde{f}^*\theta_{n+1i}=-\tilde{f}^*\theta_{in+1}$ relate closely to the immersion.

THEOREM 1.1. *Let f and f' be two isometric immersions of M into V. If $\tilde{f}^*\theta_{n+1i}=\pm\tilde{f}'^*\theta_{n+1i}$, there exists a unique isometry g of V such that $g\circ f=f'$.*

PROOF. Let a_0 denote an element of $O(n+1)$ such that

$$a_0=\begin{pmatrix} 1 & & 0 & \\ & \ddots & & \\ & & 1 & \\ 0 & & & \varepsilon\end{pmatrix}$$

where ε is equal to 1 or -1 according as $\tilde{f}^*\theta_{n+1i}=\tilde{f}'^*\theta_{n+1i}$ or $\tilde{f}^*\theta_{n+1i}=-\tilde{f}'^*\theta_{n+1i}$. Identifying $F(V)$ with $I(V)$ there exists a map $\eta: F(M)\to I(V)$ such that

$$\tilde{f}'(u)=\eta(u)\tilde{f}(u)a_0 \qquad \text{for } u\in F(M)\,.$$

For an arbitrary tangent vector X to $F(M)$ at u, we have

$$d\tilde{f}'(X)=dL_{\eta(u)}dR_{a_0}d\tilde{f}(X)+Y\,,$$

where $Y=dR_{\tilde{f}(u)a_0}d\eta(X)$ and $L_{\eta(u)}$ (resp. $R_{\tilde{f}(u)a_0}$) is the left translation by $\eta(u)$ (resp. right translation by $\tilde{f}(u)a_0$) on $I(V)$. Using (1.9) and the left-invariance of θ_A, θ_{AB}, we have

$$\theta_i(Y)=\theta_i(d\tilde{f}'(X))-\theta_i(dR_{a_0}d\tilde{f}(X))=\theta_i(X)-\theta_i(X)=0$$

$$\begin{aligned}\theta_{n+1}(Y)&=\theta_{n+1}(d\tilde{f}'(X))-\theta_{n+1}(dR_{a_0}d\tilde{f}(X))\\&=\theta_{n+1}(d\tilde{f}'(X))-\theta_{n+1}(d\tilde{f}(X))=0\\\theta_{ij}(Y)&=\theta_{ij}(d\tilde{f}'(X))-\theta_{ij}(dR_{a_0}d\tilde{f}(X))=\theta_{ij}(X)-\theta_{ij}(X)=0\,.\end{aligned}$$

And also by the assumption we have

$$\begin{aligned}\theta_{n+1i}(Y)&=\theta_{n+1i}(d\tilde{f}'(X))-\theta_{n+1i}(dR_{a_0}d\tilde{f}(X))\\&=\theta_{n+1i}(d\tilde{f}'(X))-\theta_{n+1i}(d\tilde{f}(X))=0\,.\end{aligned}$$

Hence we have $\theta_A(Y)=0$ and $\theta_{AB}(Y)=0$ which imply that $Y=0$ and thus $d\eta(X)=0$. So η is a constant map on a connected component of $F(M)$. If $F(M)$ is not connected, there exists an element $a\in O(n)$ such that u and ua are not contained in the same connected component for any frame $u\in F(M)$. Since $\sigma(a)a_0=a_0\sigma(a)$, we have

$$\eta(ua)\tilde{f}(ua)a_0=\tilde{f}'(ua)=\tilde{f}'(u)\sigma(a)=\eta(u)\tilde{f}(u)a_0\sigma(a)=\eta(u)\tilde{f}(ua)a_0\,.$$

Hence we see that $\eta(ua)=\eta(u)$ and η is constant on the whole $F(M)$. Denoting $g=\eta(u)$, we can easily see that $g\circ f=f'$.

Conversely if $g\in I(V)$ satisfies $g\circ f=f'$, g must satisfy $g\tilde{f}(u)a_0=\tilde{f}'(u)$ for any $u\in F(M)$. Therefore g is unique. Q. E. D.

If we put $\phi_i=\tilde{f}^*\theta_{n+1i}$, by the exterior differentiation of the second equation of (1.9) we have

$$\sum_i \phi_i\wedge\omega_i=0$$

which implies that ϕ_i is written as

$$\phi_i=\sum_j H_{ij}\omega_j\,,\qquad H_{ij}=H_{ji}\,. \tag{1.10}$$

Also from the structure equations of the connection forms θ_{AB}, we have

$$d\phi_i+\sum_j \omega_{ij}\wedge\phi_j=0 \tag{1.11}$$

$$\Omega_{ij}=K\omega_i\wedge\omega_j+\phi_i\wedge\phi_j\,. \tag{1.12}$$

For a point $p\in M$ and a frame $u\in F(M)$ at p, the type number $t(p)$ of an isometric immersion f of M into V is, by definition, the rank of the matrix $(H_{ij}(u))$, or the number of the linearly independent forms in $\omega_1,\cdots,\omega_n$ at u; it is independent of the choice of the frame u at p.

LEMMA 1.2. *Let f and f' be two isometric immersions of M into V and $t(p)$ and $t'(p)$ be the type number of f and f' respectively. Then $t(p)\neq t(p')$ if and only if $t(p)=1$ and $t'(p)=0$ or $t(p)=0$ and $t'(p)=1$. Moreover if $t(p)=t'(p)\geqq 3$, we have $\tilde{f}^*\theta_{n+1i}=\pm\tilde{f}'^*\theta_{n+1i}$ at the frame u at p.*

This lemma has been proved by E. Cartan in the case $K=0$, but the same proof is applied to the case $K<0$. (See [1]).

From this lemma we have the following Theorem.

THEOREM 1.3. *If M is a connected homogeneous Riemannian manifold and f is an isometric immersion of M into V, the type number $t(p)$ of f is either constant on M or $t(p)\leqq 1$ at each point p of M. If there exists a point p_0 such that $t(p_0)\geqq 3$, $\phi_i=f^*\theta_{n+1i}$ is left invariant under the action of $I_0(M)$ and f is a unique isometric immersion of M into V up to the isometry of V.*

PROOF. For an isometry $g\in I_0(M)$, $f_g=f\circ g$ is also an isometric immersion of M into V and we have

$$\phi_i'=\tilde{f}_g^*\theta_{n+1i}=g^*\tilde{f}^*\theta_{n+1i}=g^*\phi_i\,. \tag{1.13}$$

Hence the type number $t_g(p)$ of f_g at p is equal to $t(g(p))$:

$$t_g(p)=t(g(p))\,. \tag{1.14}$$

Assume that there exists a point $p_0\in M$ such that $t(p_0)\geqq 2$, then from Lemma 1.2 we have

$$t_g(p_0)=t(p_0)\,. \tag{1.15}$$

Therefore from (1.14) and (1.15) we have

$$t(g(p_0))=t_g(p_0)=t(p_0)\,. \tag{1.16}$$

Since M is assumed to be homogeneous, (1.16) means that $t(p)$ is constant.

If we assume that $t(p_0)\geqq 3$, then we have $t(p)\geqq 3$ at each point of M and by the first part of Lemma 1.2 we have $t_g(p)=t(p)\geqq 3$. Then by the second part of Lemma 1.2 we have $\tilde{f}_g^*\theta_{n+1i}=\pm\tilde{f}^*\theta_{n+1i}=\pm\phi_i$. Hence from (1.13) we obtain $g^*\phi_i=\pm\phi_i$. Since g is assumed to be contained in $I_0(M)$, we have $g^*\phi_i=\phi_i$. If f' is another isometric immersion of M into V, by Lemma 1.2 we see that $\tilde{f}'^*\theta_{n+1i}=\pm\tilde{f}^*\theta_{n+1i}$ and thus by Theorem 1.1 there exists an isometry $\varphi\in I(V)$ such that $\varphi\circ f=f'$. Q. E. D.

§ 2. The case $t(p)\geqq 3$.

In the remainder of this paper we shall assume that the curvature K of V is non positive and M is a connected homogeneous Riemannian manifold immersed isometrically into V by a map f. The connected isometry group $G=I_0(M)$ of M is identified with the orbit $G(u_0)$ of the fixed frame u_0 at a point $o\in M$ which is suitably chosen and the restrictions of the differential forms ω_i, ω_{ij}, Ω_{ij} and $\phi_i=\tilde{f}^*\theta_{n+1i}$ to $G(u_0)$ are written by the same notations.

In this section, moreover, we assume that the type number of f is greater than 2 at $o\in M$. Then by Proposition 1.3 the differential forms ϕ_i are invariant by the isometries in G and consequently the differential forms ω_i,

ω_{ij}, Ω_{ij} and ϕ_i are considered as the left-invariant forms on $G=G(u_0)$. By the suitable choice of u_0 we can assume that the differential forms ϕ_i are written on $G(u_0)$ as

$$\phi_i=\lambda_i\omega_i \tag{2.1}$$

where $\lambda_1, \cdots, \lambda_n$ are the principal curvatures of the immersion f and are constant because ϕ_i and ω_i are left-invariant.

THEOREM 2.1. *Assume that the immersion f is totally umbilical, i.e. $\lambda_1=\cdots=\lambda_n=\lambda$ and put $K_1=K+\lambda^2$. Then f is an imbedding and M is isometric to $S^n(K_1)$, E^n or $H^n(K_1)$ according as $K_1>0$, $K_1=0$ or $K_1<0$.*

PROOF. From the assumption and (1.12) we see that

$$\Omega_{ij}=(K+\lambda^2)\omega_i\wedge\omega_j$$

i.e. M is a manifold of constant curvature $K_1=K+\lambda^2$. The universal covering manifold $\tilde{M}$ of M is $S^n(K_1)$, E^n or $H^n(K_1)$ according as $K_1>0$, $K_1=0$ or $K_1<0$. In any case we have the isometric imbedding f_0 of M into $V=H^{n+1}(K)$, here $H^{n+1}(0)$ stand for E^{n+1}: if $K_1>0$, $f_0(p)=(x_1, \cdots, x_{n+1})\in H^{n+1}(K)$ for $p=(x_1, \cdots, x_{n+1})\in S^n(K_1)$ where $(x_1)^2+\cdots+(x_{n+1})^2=1/K_1$; if $K_1=0$, $f_0(p)=(x_1, \cdots, x_n, \frac{K}{2}\sum_i(x_i)^2)\in H^{n+1}(K)$ for $p=(x_1, \cdots, x_n)\in E^n$; if $K_1<0$, $f_0(p)=(x_1, \cdots, x_n, \lambda/\sqrt{KK_1})\in H^{n+1}(K)$ for $p=(x_1, \cdots, x_n)\in H^n(K_1)$. Let φ be a covering map of M onto M, $f\circ\varphi$ is also an isometric immersion of M into V and, since φ is locally isometry, the type number of $f\circ\varphi$ is equal to that of f. Therefore by Theorem 1.3, there exists an isometry $g\in I(V)$ such that $g\circ f_0=f\circ\varphi$. As $g\circ f_0$ is imbedding, f must be imbedding, φ is one to one and $M=\tilde{M}$ which complete the proof. Q.E.D.

In the remainder of this section we assume that the immersion is not totally umbilical, i.e. $\lambda_1, \cdots, \lambda_n$ are not all equal. Then by the result of E. Cartan ([2], [3]) we can assume that $\lambda_1=\cdots=\lambda_m=\lambda$, and $\lambda_{m+1}=\cdots=\lambda_n=\mu$, $(0<m<n)$. In the remainder of this section we agree about the ranges of the indices that

$$1\leqq p,\ q\leqq m \quad\text{and}\quad m+1\leqq\alpha,\ \beta,\ \leqq n\,.$$

LEMMA 2.2. *M is locally isometric to the Riemannian product of the manifolds of constant curvature $K+\lambda^2$ and $K+\mu^2$ and λ, μ satisfy*

$$\lambda\mu=-K\,. \tag{2.2}$$

PROOF. We can write (2.1) as

$$\phi_p=\lambda\omega_p \quad\text{and}\quad \phi_\alpha=\mu\omega_\alpha\,. \tag{2.3}$$

By the exterior differentiation of (2.3) and taking account of (1.2) and (1.1) we obtain

$$\sum_{\alpha} \omega_{p\alpha} \wedge \omega_{\alpha} = 0 , \qquad \sum_{p} \omega_{\alpha p} \wedge \omega_{p} = 0 .$$

Hence $\omega_{p\alpha} = -\omega_{\alpha p}$ must vanish and then from (1.12) we have

$$\Omega_{pq} = (K+\lambda^2)\omega_p \wedge \omega_q , \qquad \Omega_{\alpha\beta} = (K+\mu^2)\omega_\alpha \wedge \omega_\beta , \tag{2.4}$$

and

$$\Omega_{p\alpha} = (K+\lambda\mu)\omega_p \wedge \omega_\alpha . \tag{2.5}$$

On the other hand from (1.3) we have

$$\Omega_{p\alpha} = d\omega_{p\alpha} + \sum_{q} \omega_{pq} \wedge \omega_{q\alpha} + \sum_{\beta} \omega_{p\beta} \wedge \omega_{\beta\alpha} = 0 .$$

Therefore from (2.5) we obtain (2.2) and from (2.4) we see that M is locally isometric to the Riemannian product of the manifold of constant curvature $K+\lambda^2$ and $K+\lambda^2$. Q. E. D.

THEOREM 2.3. *If $K=0$, M is isometric to $S^m(K_1)\times E^{n-m}$, where m is the type number of f and f is an imbedding.*

PROOF. From (2.2) we see that one of λ, μ is equal to 0. We assume that $\mu=0$ and $\lambda\neq 0$, then by the definition of the type number, m is equal to the type number of f. If we put $K_1=\lambda^2>0$, by Lemma 2.2 the universal covering manifold $\tilde{M}$ is isometric to $S^m(K_1)\times E^{n-m}$ which has a natural isometric imbedding in E^{n+1}. Then by the similar consideration in the proof of Theorem 2.1, we can conclude that f is an imbedding and M is isometric to $S^m(K_1)\times E^{n-m}$. Q. E. D.

THEOREM 2.4. *If $K<0$, the type number of f is equal to n and the image $f(M)$ is isometric to $S^m(K_1)\times H^{n-m}(K_2)$ where K_1 and K_2 satisfy*

$$\frac{1}{K_1} + \frac{1}{K_2} = \frac{1}{K} . \tag{2.6}$$

If $m\neq 1$, f is an imbedding.

PROOF. By Lemma 2.2, without loss of generality, we can assume that $\lambda > \sqrt{-K} > \mu > 0$. Thus the type number is equal to n.

If we put $K_1=K+\lambda^2>0$ and $K_2=K+\mu^2<0$, from (2.2) we see that

$$K^2=\lambda^2\mu^2=(K-K_1)(K-K_2)=K^2-(K_1+K_2)K+K_1K_2$$

from which we obtain (2.6).

By lemma 2.2, the universal covering manifold $\tilde{M}$ is isometric to $S^m(K_1)\times H^{n-m}(K_2)$ if $m>1$ or $E\times H^{n-1}(K_2)$ if $m=1$.

If $m\neq 1$, $\tilde{M}=S^m(K_1)\times H^{n-m}(K_2)$ has an isometric imbedding f_0 into $V=H^{n+1}(K)$: for $p=(y_1, \cdots, y_{m+1})\in S^m(K_1)$ and $q=(z_1, \cdots, z_{n-m})\in H^{n-m}(K_2)$, $f_0(p, q)=(y_1, \cdots, y_{m+1}, z_1, \cdots, z_{n-m})\in H^{n+1}(K)$. Then similarly in the proof of Theorem 2.1 we see that f is an imbedding and $M=\tilde{M}$.

If $m=1$, $\tilde{M}=E\times H^{n-m}(K_2)$ has also an isometric immersion f_0 into V: for $p=(y)$ and $q=(z_1, \cdots, z_{n-1})\in H^{n-1}(K_2)$, $f_0(p, q)=\left(\frac{1}{\sqrt{k_1}}\cos\sqrt{K_1}\,y, \frac{1}{\sqrt{K_1}}\sin\sqrt{K_1}\,y, z_1, \cdots, z_{n-1}\right)\in H^{n+1}(K)$ and the image $f_0(M)$ is isometric to $S^1(K_1)\times H^{n-1}(K_2)$. On the other hand by Theorem 1.3, there exists an isometry g of V such that $g\circ f_0=f\circ\varphi$, where φ is the covering map of M onto M. Hence $f(M)$ is isometric to $f_0(M)$ and therefore isometric to $S^1(K_1)\times H^{n-1}(K_2)$.

§ 3. The case $t(p)=2$.

This section is devoted to the following theorem.

THEOREM 3.1. *If a connected homogeneous Riemannian manifold M admits an isometric immersion f into V and the type number of f is equal to 2 at some point, then $V=E^{n+1}$, M is isometric to $S^2(K_1)\times E^{n-2}$ and f is an imbedding.*

We break the proof of the theorem up into the series of lemmas. In this section we agree that the Greek indices α, β, γ run over the range $3, 4, \cdots, n$.

LEMMA 3.1.

1) *$H_{ik}H_{jl}-H_{il}H_{jk}$ is constant on $G(u_0)$.*

2) *If we choose u_0 so that $\phi_\alpha=0$ ($\alpha=3, \cdots, n$) at u_0, then ϕ_α vanish identically on $G(u_0)$ and we have*

$$(3.1)\qquad \begin{cases} \phi_1=H_{11}\omega_1+H_{12}\omega_2 \\ \phi_2=H_{12}\omega_1+H_{22}\omega_2 \end{cases}$$

PROOF. From (1.12) we have

$$(3.2)\qquad \Omega_{ij}=\frac{1}{2}\sum_{k,l}\{K(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+(H_{ik}H_{jl}-H_{il}H_{jk})\}\omega_k\wedge\omega_l\,.$$

Since Ω_{ij} and ω_k are the left-invariant forms on $G(u_0)=G$ the coefficients of the right hand side of (3.2) are constant and hence $H_{ik}H_{jl}-H_{il}H_{jk}$ is constant.

If we put $\bar{\phi}_i=\sum_j H_{ij}(u_0)\omega_j$ on $G(u_0)$, by the results just proved we have

$$(3.3)\qquad \bar{\phi}_i\wedge\bar{\phi}_j=\phi_i\wedge\phi_j \qquad (i, j=1, \cdots, n)\,.$$

If we assume that $\phi_\alpha=0$ at u_0, then $H_{\alpha j}(u_0)=0$, accordingly $\bar{\phi}_\alpha=0$ on $G(u_0)$. Also we obtain

$$\begin{cases} \bar{\phi}_1=H_{11}(u_0)\omega_1+H_{12}(u_0)\omega_2 \\ \bar{\phi}_2=H_{12}(u_0)\omega_1+H_{22}(u_0)\omega_2\,. \end{cases}$$

Put $K_1=H_{11}H_{22}-(H_{12})^2$. Since K_1 is constant and does not vanish, ϕ_1 and ϕ_2 are linearly independent. From (3.3) we have

Thus $\phi_\alpha=0$ on $G(u_0)$, so $H_{\alpha j}=0$ on $G(u_0)$. Q. E. D.

In the remainder of this section we fix a frame u_0 at which $\phi_\alpha=0$.

LEMMA 3.2.

$$\begin{cases} \omega_{\alpha 1}=a_\alpha\omega_1+b_\alpha\omega_2 \\ \omega_{\alpha 2}=c_\alpha\omega_1-a_\alpha\omega_2 \end{cases} \tag{3.4}$$

where a_α, b_α, c_α are constant.

PROOF. By the exterior differentiation of $\phi_\alpha=0$, we have

$$\omega_{\alpha 1}\wedge\phi_1+\omega_{\alpha 2}\wedge\phi_2=0\,. \tag{3.5}$$

From this we see that $\omega_{\alpha 1}$ and $\omega_{\alpha 2}$ are the linear combinations of ϕ_1 and ϕ_2 and hence from (3.1) the linear combinations of ω_1 and ω_2. $\omega_{\alpha 1}$, $\omega_{\alpha 2}$, ω_1 and ω_2 are left-invariant on $G=G(u_0)$, the coefficients are constant. We put

$$\begin{aligned} \omega_{\alpha 1}&=a_\alpha\omega_1+b_\alpha\omega_2 \\ \omega_{\alpha 2}&=c_\alpha\omega_1+d_\alpha\omega_2\,. \end{aligned} \tag{3.6}$$

By the Bianchi's identity we have

$$d\Omega_{12}=\sum_\alpha(\Omega_{1\alpha}\wedge\omega_{\alpha 2}-\omega_{1\alpha}\wedge\Omega_{\alpha 2})\,. \tag{3.7}$$

Substituting $\Omega_{12}=(K+K_1)\omega_1\wedge\omega_2$, $\Omega_{1\alpha}=K\omega_1\wedge\omega_\alpha$, $\Omega_{\alpha 2}=K\omega_\alpha\wedge\omega_2$ into (3.7) and taking account of (3.6) we have,

$$\sum_\alpha K_1(a_\alpha+d_\alpha)\omega_1\wedge\omega_\alpha\wedge\omega_2=0\,. \tag{3.8}$$

In (3.8) K_1 can not be zero, we obtain $a_\alpha+d_\alpha=0$. Thus the Lemma 3.2 is proved.

We denote by Γ the following matrix:

$$\Gamma=\begin{pmatrix} a_3\,b_3\,c_3 \\ \cdots\cdots\cdots \\ a_n\,b_n\,c_n \end{pmatrix}.$$

LEMMA 3.3. *The linear homogeneous equations*

$$a_\alpha x+b_\alpha y+c_\alpha z=0 \qquad (\alpha=3,4,\cdots,n) \tag{3.9}$$

has the solution $x=2H_{12}$, $y=-H_{11}$, $z=H_{22}$ and hence the rank of the matrix Γ can not exceed 2.

PROOF. Substituting (3.1) and (3.6) into (3.5) we have easily

$$2a_\alpha H_{12}-b_\alpha H_{11}+c_\alpha H_{22}=0\,. \tag{3.10}$$

Since $(2H_{12}, -H_{11}, H_{22})\neq(0,0,0)$, the rank of Γ is not greater than 2.

LEMMA 3.4. *The rank of Γ can not be equal to 2.*

PROOF. We assume that the rank of Γ is equal to 2. Then the system of the solution of (3.9) is a 1-dimensional vector space and by Lemma 3.3 $(2H_{12}, -H_{11}, H_{22})$ is a solution of (3.9) we have a function ρ on $G(u_0)$ such that $H_{ij}=\rho H_{ij}(u_0)$.

On the other hand since $K_1=H_{11}H_{22}-(H_{12})^2$ is constant, we see $\rho=\pm 1$. By the continuity of H_{ij}, ρ must be 1. Thus H_{ij} is constant.

By the exterior differentiation of (3.1) we have

$$\{(H_{11}-H_{22})\omega_{12}+\sum_{\alpha}(b_\alpha H_{11}-a_\alpha H_{12})\omega_\alpha\}\wedge\omega_2 \tag{3.11}$$
$$=\{2H_{12}\omega_{12}-\sum_{\alpha}(a_\alpha H_{11}+c_\alpha H_{12})\omega_\alpha\}\wedge\omega_1 .$$

$$\{(H_{11}-H_{22})\omega_{12}+\sum_{\alpha}(a_\alpha H_{12}+c_\alpha H_{22})\omega_\alpha\}\wedge\omega_1 \tag{3.12}$$
$$=\{-2H_{12}\omega_{12}-\sum_{\alpha}(b_\alpha H_{12}-a_\alpha H_{22})\omega_\alpha\}\wedge\omega_2 .$$

Making an exterior product of (3.11) and ω_2, we obtain

$$\{2H_{12}\omega_{12}-\sum_{\alpha}(a_\alpha H_{11}+c_\alpha H_{12})\omega_\alpha\}\wedge\omega_1\wedge\omega_2=0 . \tag{3.13}$$

Also from (3.12) we obtain

$$\{-2H_{12}\omega_{12}-\sum_{\alpha}(b_\alpha H_{12}-a_\alpha H_{22})\omega_\alpha\}\wedge\omega_1\wedge\omega_2=0 . \tag{3.14}$$

Adding (3.13) and (3.14) we have

$$a_\alpha(H_{11}-H_{22})+b_\alpha H_{12}+c_\alpha H_{12}=0 .$$

Therefore $(H_{11}-H_{22}, H_{12}, H_{12})$ is a solution of (3.9) and there exists a real number ε such that

$$H_{11}-H_{22}=2\varepsilon H_{12},\quad H_{12}=-\varepsilon H_{11},\quad H_{12}=\varepsilon H_{22} .$$

or

$$\begin{cases} H_{11}-H_{22}-2\varepsilon H_{12}=0 \\ \varepsilon H_{11} \qquad\quad +H_{12}=0 \\ \qquad \varepsilon H_{22} \quad -H_{12}=0 . \end{cases}$$

Then since $(H_{11}, H_{22}, H_{12})\neq(0, 0, 0)$ we have $\varepsilon(1+\varepsilon^2)=0$ which imply that $\varepsilon=0$ and $H_{11}=H_{22}$, $H_{12}=0$. If we put $A=H_{11}=H_{22}\neq 0$, from (3.11) and (3.12) we have

$$A\sum b_\alpha\omega_\alpha\wedge\omega_2=-A\sum a_\alpha\omega_\alpha\wedge\omega_1 ,$$
$$A\sum c_\alpha\omega_\alpha\wedge\omega_1=A\sum a_\alpha\omega_\alpha\wedge\omega_2 .$$

From this we obtain $a_\alpha=b_\alpha=c_\alpha=0$ which contradicts to the assumption of the rank of Γ. Therefore the rank of Γ is not equal to 2.

LEMMA 3.5. *The matrix Γ is a zero matrix and the curvature of V is zero.*

PROOF. If we assume that Γ is not zero, then by above lemmas, we see that the rank of Γ is just 1. Then the matrices $\begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & -a_\alpha \end{pmatrix}$ can be changed to the form $\begin{pmatrix} 0 & b'_\alpha \\ c'_\alpha & 0 \end{pmatrix}$ by the orthogonal matrix of degree 2, and by the assumption of the rank of Γ, these matrices can be changed all together by one orthogonal matrix. This means that by the suitable change of u_0, we can assume that $a_\alpha = 0$ and

$$\omega_{\alpha 1} = b_\alpha \omega_2 , \qquad \omega_{\alpha 2} = c_\alpha \omega_1$$

without changing the properties of $\phi_\alpha = 0$ at u_0. By the exterior differentiation of these equation, we have

$$\{(b_\alpha + c_\alpha)\omega_{12} - b_\alpha \sum_\beta c_\beta \omega_\beta - K\omega_\alpha\} \wedge \omega_1 = -\sum_\beta b_\beta \omega_{\alpha\beta} \wedge \omega_2 \tag{3.15}$$

$$\{(b_\alpha + c_\alpha)\omega_{12} + c_\alpha \sum_\beta b_\beta \omega_\beta + K\omega_\alpha\} \wedge \omega_2 = \sum_\beta c_\beta \omega_{\alpha\beta} \wedge \omega_1 . \tag{3.16}$$

If $n \geqq 4$, then (3.15) shows us that $\sum_\beta b_\beta \omega_{\alpha\beta}$ is a linear combination of ω_1 and ω_2 and we put

$$\sum_\beta b_\beta \omega_{\alpha\beta} = \lambda_\alpha \omega_1 + \mu_\alpha \omega_2$$

where λ_α, μ_α are constant. Then we have

$$\begin{aligned} \sum_\beta b_\beta \Omega_{\alpha\beta} &= d(\sum_\beta b_\beta \omega_{\alpha\beta}) + \sum_\beta b_\beta \omega_{\alpha 1} \wedge \omega_{1\beta} + \sum_\beta b_\beta \omega_{\alpha 2} \wedge \omega_{2\beta} \\ &\quad + \sum_\gamma \omega_{\alpha\gamma} \wedge (\sum_\beta b_\beta \omega_{\gamma\beta}) \\ &\equiv \lambda_\alpha d\omega_1 + \mu_\alpha d\omega_2 \qquad (\mathrm{mod}\ \omega_1, \omega_2) \\ &\equiv 0 \qquad (\mathrm{mod}\ \omega_1, \omega_2) \end{aligned}$$

because of $\omega_{\alpha 1} \equiv \omega_{\alpha 2} \equiv \sum_\beta b_\beta \omega_{\gamma\beta} \equiv d\omega_1 \equiv d\omega_2 \equiv 0\ (\mathrm{mod}\ \omega_1, \omega_2)$. On the other hand

$$\sum_\beta b_\beta \Omega_{\alpha\beta} = \sum_\beta K b_\beta \omega_\alpha \wedge \omega_\beta .$$

Accordingly we obtain $Kb_\beta = 0$ ($\beta = 3, \cdots, n$). Similarly we obtain $Kc_\beta = 0$. Multiplying K to (3.15) we have

$$K^2 \omega_\alpha \wedge \omega_1 = 0 .$$

Thus we obtain $K = 0$. Then making exterior product of (3.15) and ω_2, we have

$$\{(b_\alpha + c_\alpha)\omega_{12} - b_\alpha \sum_\beta c_\beta \omega_\beta\} \wedge \omega_1 \wedge \omega_2 = 0 . \tag{3.17}$$

Similarly we have from (3.16)

$$\{(b_\alpha + c_\alpha)\omega_{12} + c_\alpha \sum_\beta b_\beta \omega_\beta\} \wedge \omega_1 \wedge \omega_2 = 0 . \tag{3.18}$$

Subtracting (3.17) from (3.18) we have

$$\sum_{\beta}(b_\alpha c_\beta + c_\alpha b_\beta)\omega_\beta \wedge \omega_1 \wedge \omega_2 = 0 .$$

Therefore we obtain

$$b_\alpha c_\beta + c_\alpha b_\beta = 0 . \tag{3.19}$$

On the other hand since the rank of Γ is 1, we have

$$b_\alpha c_\beta - c_\alpha b_\beta = 0 . \tag{3.20}$$

From (3.19) and (3.20) we obtain

$$b_\alpha c_\beta = 0 \qquad (\alpha, \beta = 3, 4, \cdots, n) . \tag{3.21}$$

Because of the rank of Γ, one of b_α or c_α is not zero. If, for instance, one of b_α is not zero, then from (3.21) $c_\alpha = 0$ $(\alpha = 3, \cdots, n)$ and from (3.15) and (3.16) we have

$$b_\alpha \omega_{12} \wedge \omega_1 = -\sum b_\beta \omega_{\alpha\beta} \wedge \omega_2 \tag{3.22}$$

$$b_\alpha \omega_{12} \wedge \omega_2 = 0 \qquad \text{or} \qquad \omega_{12} \wedge \omega_2 = 0 . \tag{3.23}$$

Multiplying b_α to (3.22) and making a sum, we have

$$(\sum b_\alpha^2)\omega_{12} \wedge \omega_1 = 0 .$$

Since $\sum b_\alpha^2 \neq 0$, we obtain

$$\omega_{12} \wedge \omega_1 = 0 . \tag{3.24}$$

Then (3.23) and (3.24) show that

$$\omega_{12} = 0 .$$

From this and $\omega_{\alpha 2} = 0$ we have

$$0 = d\omega_{12} = \Omega_{12} + \sum_{\alpha} \omega_{1\alpha} \wedge \omega_{\alpha 2} = K_1 \omega_1 \wedge \omega_2 .$$

Then we have $K_1 = 0$ and this is a contradiction for the hypothesis of the type number.

If $n = 3$, (3.15) and (3.16) are reduced to

$$\{(b_3 + c_3)\omega_{12} - (b_3 c_3 + K)\omega_3\} \wedge \omega_1 = 0$$

$$\{(b_3 + c_3)\omega_{12} + (b_3 c_3 + K)\omega_3\} \wedge \omega_2 = 0 .$$

From these equations we can easily obtain

$$b_3 c_3 + K = 0 \qquad \text{and} \qquad (b_3 + c_3)\omega_{12} = 0 .$$

Since $K \leqq 0$ and the rank of the matrix Γ is 1, we have $b_3 c_3 = -K \geqq 0$ and $b_3 + c_3 \neq 0$, and therefore ω_{12} must be vanished. Then as in the case of $n \geqq 4$ we have

$$0=d\omega_{12}=\Omega_{12}+\omega_{13}\wedge\omega_{32}=(K_1+K+b_3c_3)\omega_1\wedge\omega_2=K_1\omega_1\wedge\omega_2\,.$$

Thus we have also $K_1=0$ which is a contradiction.

Consequently the matrix Γ must be 0 and $\omega_{\alpha 1}=\omega_{\alpha 2}=0$.

Then from (1.3) and (1.12) we have

$$K\omega_\alpha\wedge\omega_1=\Omega_{\alpha 1}=d\omega_{\alpha 1}+\omega_{\alpha 2}\wedge\omega_{21}+\sum_\beta\omega_{\alpha\beta}\wedge\omega_{\beta 1}=0\,.$$

Therefore $K=0$, which completes the proof of the lemma. Q. E. D.

REMARK. When $n\geqq 4$, Lemma 3.5 is also true for the case of V of positive constant curvature.

By Lemma 3.5, we see that by the suitable choice of u_0, the differential forms ϕ_α, $\omega_{\alpha 1}$, $\omega_{\alpha 2}$ vanish identically on $G(u_0)$. If we denote the structure group of $G(u_0)$ by Hu_0 which is isomorphic to the isotropy group H at $o=\pi(u_0)$, then for any $a=(a_{ij})\in Hu_0$ we have

$$0=R_a^*\phi_\alpha=\det a\sum_j a_{j\alpha}\phi_j=\det a\,(a_{1\alpha}\phi_1+a_{2\alpha}\phi_2)\,.$$

Therefore a is the matrix of the form

$$a=\begin{pmatrix}a' & 0\\ 0 & a''\end{pmatrix}\quad\text{where } a'\in O(2),\quad a''\in O(n-2)\,. \tag{3.26}$$

For a point $p\in M$ the subspaces $D_1(p)$ and $D_2(p)$ of the tangent space of M at p are defined by

$$D_1(p)=\{d\pi(X)\,|\,X\in T_u(G(u_0)),\ \omega_1(X)=\omega_2(X)=0\}$$

$$D_2(p)=\{d\pi(X)\,|\,X\in T_u(G(u_0)),\ \omega_\alpha(X)=0,\ \alpha=3,\cdots,n\}$$

where $T_u(G(u_0))$ is a tangent space of $G(u_0)$ at u and u is a frame in $G(u_0)$ at p. $D_1(p)$ and $D_2(p)$ are independent of the choice of u, because the elements of the structure group Hu_0 have the form of (3.26). D_1 and D_2 are the distributions on M of dimension $n-2$ and 2 respectively and mutually orthogonal. They are completely integrable because of $\omega_{\alpha 1}=\omega_{\alpha 2}=0$. If we denote by $M_1(p)$ and $M_2(p)$ the maximal integral manifolds through p of D_1 and D_2, respectively, $M_1(p)$ and $M_2(p)$ are the complete totally geodesic submanifolds.

If we identify G with $G(u_0)$ so that the identity element of G corresponds to u_0, then Lie algebra $\mathfrak{g}$ of G is identified with $T_{u_0}(G(u_0))$. The subspace $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ defined by

$$\mathfrak{g}_1=\{X\in\mathfrak{g}=T_{u_0}(G(u_0))\,|\,\omega_1(X)=\omega_2(X)=\omega_{12}(X)=0\}$$

$$\mathfrak{g}_2=\{X\in\mathfrak{g}\,|\,\omega_\alpha(X)=0,\ \omega_{\alpha\beta}(X)=0\}$$

are the ideals of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2$ (direct). Denote by G_1 and G_2 the connected subgroup with Lie algebra $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we can easily see that

$$M_1(o)=G/H_1 \qquad M_2(o)=G_2/H_2 \qquad (o=\pi(u_0))$$

where $H_1=H\cap G_1$ and $H_2=H\cap G_2$, and $M_1(p)$ and $M_2(p)$ are the orbits of p by the group gG_1g^{-1} and gG_2g^{-1} where $g(o)=p$, $g\in G$, or $M_1(p)=g(M_1(o))$ and $M_2(p)=g(M_2(o))$. Also the correspondence $(g_1H_1, g_2H_2)\to g_1g_2H$ of G_1/H_1 G_2/H_2 to G/H is an isometry.

For $u\in G(u_0)$ we denote $\tilde{f}(u)=(f(p), e_1, \cdots, e_n, e_{n+1})$, the differential of $e_1, \cdots, e_{n+1}$ are

$$\begin{cases} de_1=\omega_{21}e_2+\phi_1 e_{n+1} \\ de_2=\omega_{12}e_1+\phi_2 e_{n+1} \\ de_{n+1}=-\phi_1 e_1-\phi_2 e_2 \end{cases} \tag{3.27}$$

and

$$de_\alpha=\sum_\beta \omega_{\beta\alpha}e_\beta\,. \tag{3.28}$$

From (3.38) we see that the tangent spaces of $f(M_1(p))$ are parallel in E^{n+1} and thus $f(M_1(p))$ is an $(n-2)$-dimensional plane in E^{n+1} which is parallel to the fixed $(n-2)$-dimensional plane E^{n-2} in E^{n+1}. Also from (3.27) we see that $f(M_2(p))$ is contained in a 3-dimensional plane which is parallel to the fixed plane E^3 orthogonal to E^{n-2}. Then $f(M_2(p))$ is a complete connected surface of constant curvature K_1 in 3-dimensional Euclidean space. It is well known that a complete connected surface of constant curvature in E^3 is a sphere, so $f(M_2(p))$ is a sphere of curvature K_1. Therefore $M_1(p)$ is isometric to $S^2(K_1)$ and $f(M)$ is isometric to $S^2(K_1)\times E^{n-2}$. Then M is also isometric to $S^2(K_1)\times E^{n-2}$ and f is an imbedding. This completes the proof of Theorem 3.1.

§ 4. The case $t(p)\leqq 1$.

In case $t(p)\leqq 1$, since all $\phi_1, \cdots, \phi_n$ are linearly dependent with each other, we have

$$\phi_i\wedge\phi_j=0\,.$$

Thus the curvature form Ω_{ij} is

$$\Omega_{ij}=K\omega_i\wedge\omega_j\,.$$

So M is a manifold of constant curvature K.

If $K<0$, the homogeneous Riemannian manifold of constant curvature $K<0$ is the hyperbolic space $H^n(K)$ by the theorem of Wolf ([**6**] p. 88).

If $K=0$, also by the theorem of Wolf, M is isometric to $T^m\times E^{n-m}$ where T^m is an m-dimensional flat torus. Then by the same consideration as the proof of Lemma 3.2 in [**5**] we see that $m=1$ or 0. Thus M is isometric to

$S^1 \times E^{n-1}$ or E^n.

We have the theorem:

THEOREM 4.1. *If a connected homogeneous Riemannian manifold* M *admits an isometric immersion* f *into* V, *and the type number of* f *is* 0 *or* 1 *at a point, then if* $V = H^{n+1}(K)$ $(K<0)$, M *is isometric to* $H^n(K)$ *and if* $V = E^n$, M *is isometric to* E^n *or* $S^1 \times E^{n-1}$.

Theorems 2.1, 2.3, 3.1 and 4.1 complete the proof of the Theorems A and B in the introduction.

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