J. Math. Soc. Japan
Vol. 22, No. 3, 1970

On the totally geodesic submanifolds in locally symmetric spaces

By Hisao NAKAGAWA and Katsuhiro SHIOHAMA

(Received Sept. 24, 1969)

Let there be given an n-dimensional ($n \geqq 2$) complete and connected Riemannian manifold M of class C^{∞} whose sectional curvature $K(P)$ with respect to any plane section P is non-negative. The manifold structures of M with an r-dimensional totally geodesic submanifold V have been studied in two lines of investigation. One is concerning with the striking Toponogov's result, i. e., the basic theorem on triangles. Making use of the basic theorem on triangles, he proved [**6**] that if a complete Riemannian manifold of non-negative curvature admits a straight line, M is isometric to $V \times R$ where V is a totally geodesic hypersurface, and with the aid of the totally convex sets Cheeger-Gromoll [**1**] showed the existence of a compact totally geodesic submanifold S_M which is also a totally convex set in a complete and non-compact Riemannian manifold, and the second named author [**5**] investigated the isometric structures of a complete and non-compact Riemannian manifold of non-negative curvature with a compact totally geodesic hypersurface. The other has been made by T. Frankel [**3**], who gave the restrictions on the totally geodesic imbeddings in M of positive curvature and dimension $n \leqq 2r$, by showing the natural homomorphism of the fundamental groups $\pi_1(V) \rightarrow \pi_1(M)$ is surjective. In this paper we do not ask about the imbeddability of V as a compact totally geodesic submanifold of M. It seems, however, to the authors that it is essentially important that M is not of positive curvature but of non-negative curvature, as shown the following example that the product manifold $S^r \times S^{n-r}$ of r-dimensional sphere S^r and $(n-r)$-dimensional sphere S^{n-r} is of non-negative curvature. Thus, by taking account of the well known de Rham's theorem, it might be interesting to study precisely the manifold structure of a complete locally symmetric space of non-negative curvature and with a totally geodesic submanifold. Note the assumption "M is of positive curvature" gives the guaranty that the second variation formula with respect to every 1-parameter variation whose variation vector field is parallel along the geodesic is of strictly negative.

In § 1, we give definitions and notations in later use. In § 2, we shall deal with the variations of arc lengths with respect to the 1-parameter variation of a geodesic. As against the first and the second variations, we calculate straightforwardly the k-th, variation ($k \geqq 3$) of the arc length, which plays an important role for the proofs of the results obtained in § 3. In the last section, we apply the k-th, variation formula and investigate the manifold structure of a locally symmetric space of non-negative curvature.

§ 1. Preliminaries.

Let M be an n-dimensional complete and connected Riemannian manifold of class C^∞ and g the induced Riemannian metric tensor of M. Throughout this paper, we assume that the sectional curvature on M is non-negative with respect to the Riemannian metric g. ∇_X is the covariant differentiation with respect to a vector field X. Let a geodesic $\Gamma=\{\gamma(s)\}$ be always parametrized by arc length and $\gamma'(s)$ a tangent vector to Γ at $\gamma(s)$. For the tangent space, denoted by M_p, of M at a point p, $P=P(X, Y)$ is the plane section spanned by any two linearly independent vectors X and Y belonging to M_p. We denote by $K(P)=K(X, Y)$ the sectional curvature corresponding to a plane section $P=P(X, Y)$ which is given by $K(X, Y)=-g(R(X, Y)X, Y)/\{g(X, X)g(Y, Y)-g(X, Y)^2\}$, where R is the Riemannian curvature tensor on M. We denote also by G_p, G_Γ and G_M the set of all plane sections at p, those of all plane sections $P(\gamma'(s), Y(s))$ at a point $\gamma(s)$ on any geodesic $\Gamma=\{\gamma(s)\}$, $Y(s)$ being an arbitrary vector field along Γ and the union of all sets G_p for any point p, respectively. For any two points p, q in M, let $d(p, q)$ be the distance between p and q with respect to the metric tensor g. For any two disjoint compact subset A and B in M, we denote by $\Gamma(A, B)$ the set of all minimal geodesic segments each of which starts from a point $p\in A$ and ends at $q\in B$ such that $d(p, q)=d(A, B)$. We also denote by $\Gamma(A, \infty)$ the set of all rays from A to ∞. A submanifold N is by definition a Riemannian manifold which is a subset in M (as a set theoretical) and the inclusion map $\iota: N\to M$ is an isometric imbedding. A hypersurface V of M is a Riemannian submanifold whose inclusion map $\iota: V\to M$ is an isometric imbedding and $\dim V=\dim M-1$. For a submanifold N of M, a cut point $\gamma(a)$ to N along a geodesic segment Γ is by definition the minimal point to N along Γ whose starting point $\gamma(0)$ is in N and starting direction is normal to it at the starting point, i.e., $\Gamma|[0, t]\in\Gamma(\gamma(t), N)$ holds for $0<t\leqq a$ and $\Gamma|[0, t]\notin\Gamma(\gamma(t), N)$ for $t>a$. The cut locus $C(N)$ is by definition the set of all cut points to N along every geodesic starting from N and normal to it at the starting point. We also denote by $F(N)$ the first focal locus of N.

A soul of M is by definition a compact totally convex set in M which is also a compact totally geodesic submanifold without boundary. The existence of a soul in a complete and non-compact Riemannian manifold of non-negative curvature is announced in [**1**].

§ 2. The k-th variation formula of the arc length.

In this section, we fix two points p and q of M and a geodesic segment $\Gamma=\{\gamma(s)\}$ $(0\leqq s\leqq l)$ starting from $p=\gamma(0)$ and ending at $q=\gamma(l)$. Given a smooth vector field V along Γ, we consider a 1-parameter variation α of Γ such that $\alpha:[0, l]\times(-\varepsilon, \varepsilon)\rightarrow M$ defined by

$$\alpha(s, t)=\exp_{\gamma(s)}t\cdot V(s)\,, \tag{2.1}$$

where $\exp_{\gamma(s)}$ is the exponential mapping of $M_{\gamma(s)}$ into M. V is called the variation vector field along Γ associated with the variation α. For each fixed s, we denote by α_s a coordinate curve $s=$ constant given by $\alpha_s(t)=\alpha(s, t)$, and for each fixed t, we denote also by α_t another coordinate curve $t=$ constant, called the variation curve, given by $\alpha_t(s)=\alpha(s, t)$. Then we have two vector fields V and T along the smooth mapping α defined by $V=\alpha_*(\partial/\partial t)$ and $T=\alpha_*(\partial/\partial s)$, respectively. Thus we find

$$V(s, 0)=V(s)\,,\quad T(s, 0)=\gamma'(s)\qquad s\in[0, l]\,. \tag{2.2}$$

Since $\partial/\partial t$ and $\partial/\partial s$ are basis vector fields for $[0, l]\times(-\varepsilon, \varepsilon)$, it follows from $[\partial/\partial t, \partial/\partial s]=0$ that

$$\nabla_{\frac{\partial}{\partial t}}T=\nabla_{\frac{\partial}{\partial s}}V\,, \tag{2.3}$$

where we identify the connection along α with the usual ∇. We denote by $L(t)$ the length of the variation curve α_t. It is given by

$$L(t)=\int_0^l g(T(s, t), T(s, t))^{\frac{1}{2}}ds\,.$$

As is well known, if the variation vector field V is orthogonal to the tangent vector $\gamma'(s)$ of Γ at each point, then the first and the second variation formulas of the arc length $L(t)$ are given by

$$\begin{aligned} &L'(0)=0\,,\\ &L''(0)=\int_0^l[g(\nabla_{\frac{\partial}{\partial s}}V, \nabla_{\frac{\partial}{\partial s}}V)+g(R(V, \gamma')V, \gamma')](s, 0)ds\,. \end{aligned} \tag{2.4}$$

We shall consider the k-th variation of L. In the following we only consider the variation vector field V associated with the variation α being orthogonal to the geodesic Γ. Then we obtain

PROPOSITION 2.1. *For a 1-parameter variation α of Γ defined by* (2.1) *satisfying* $g(V, \gamma')=0$, *we have*

$$
(2.5)\qquad
\begin{aligned}
L^{(k)}(0) &= \int_0^l \left(\frac{\partial}{\partial t}\right)^{k-1} g(\nabla_{\frac{\partial}{\partial t}} T, T)|(s, 0)ds + \cdots \\
&= \int_0^l \left(\frac{\partial}{\partial t}\right)^{k-2} [g(\nabla_{\frac{\partial}{\partial s}} V, \nabla_{\frac{\partial}{\partial s}} V) + g(R(V, T)V, T)](s, 0)ds \\
&\qquad + \cdots, \qquad (k \geqq 3)
\end{aligned}
$$

where the remainder is an integral of the sum of terms consisting only of the factors $\frac{\partial}{\partial t} g(\nabla_{\frac{\partial}{\partial s}} V, T)$, $\left(\frac{\partial}{\partial t}\right)^2 g(\nabla_{\frac{\partial}{\partial s}} V, T), \cdots, \left(\frac{\partial}{\partial t}\right)^{k-2} g(\nabla_{\frac{\partial}{\partial s}} V, T)$.

PROOF. At first we shall show that the following equation

$$
(2.6)\qquad \int_0^l f(s, 0) g(\nabla_{\gamma'} V, \gamma')ds = 0
$$

holds for any function $f: [0, l] \times (-\varepsilon, \varepsilon) \to R$ of class C^1. In fact, the left hand side is rewritten as follows:

$$
\int_0^l f(s, 0) g(\nabla_{\gamma'} V, \gamma')ds = [f(s, 0) \cdot g(V, \gamma')]_0^l - \int_0^l \frac{\partial}{\partial s} f(s, 0) \cdot g(V, \gamma')ds,
$$

because Γ is a geodesic, that is, $\nabla_{\gamma'}\gamma' = 0$. Under the assumption that the variation vector field V is orthogonal to Γ, (2.6) holds clearly.

Now, making use of the Leibniz theorem and differentiating the function $t \to L(t)$, we get the following

$$
\begin{aligned}
L^{(k)}(0) &= \int_0^l \left(\frac{\partial}{\partial t}\right)^{k-1} g(\nabla_{\frac{\partial}{\partial s}} V, V)ds + \sum_{j=1}^{k-2} \int_0^l {}_{k-1}C_j \left(\frac{\partial}{\partial t}\right)^{k-1-j} g(\nabla_{\frac{\partial}{\partial s}} V, V) \\
&\quad \cdot \left(\frac{\partial}{\partial t}\right)^j g(T, T)^{-\frac{1}{2}} ds + \int_0^l g(\nabla_{\gamma'} V, \gamma') \cdot \left(\frac{\partial}{\partial t}\right)^{k-1} g(T, T)^{-\frac{1}{2}} ds.
\end{aligned}
$$

It follows from the equation (2.6) that the last member of the right hand side in the relation above vanishes identically and the function $\left(\frac{\partial}{\partial t}\right)^j g(T, T)^{-\frac{1}{2}}$ may be regarded as the polynomial of $\frac{\partial}{\partial t} g(\nabla_{\frac{\partial}{\partial s}} V, T)$, $\left(\frac{\partial}{\partial t}\right)^2 g(\nabla_{\frac{\partial}{\partial s}} V, T), \cdots, \left(\frac{\partial}{\partial t}\right)^{j-1} g(\nabla_{\frac{\partial}{\partial s}} V, T)$. This fact means that the k-th variation of L is expressed as the equation (2.5).

REMARK. We give explicitly the third, the fourth and the fifth variations of L:

$$
L'''(0) = \int_0^l \left[\frac{\partial}{\partial t} g(R(V, T)V, T) + 2g(R(V, \gamma')V, \nabla_{\gamma'} V)\right](s, 0)ds,
$$

$$
\begin{aligned}
L^{(4)}(0) = \int_0^l \Big[&\left(\frac{\partial}{\partial t}\right)^2 g(R(V, T)V, T) + 2g(\nabla_{\frac{\partial}{\partial t}} R(V, T)V, \nabla_{\gamma'} V) \\
&+ 2g(R(V, \gamma')V, R(V, \gamma')V) - 3\{g(\nabla_{\gamma'} V, \nabla_{\gamma'} V)
\end{aligned}
$$

$$+g(R(V, \gamma')V, \gamma')\}^2\Big](s, 0)ds\,,$$

$$L^{(5)}(0)=\int_0^l\Big[\Big(\frac{\partial}{\partial t}\Big)^3 g(R(V, T)V, T)+2g(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}R(V, T)V, \nabla_{\gamma'}V)$$

$$+6g(\nabla_{\frac{\partial}{\partial t}}R(V, T)V, R(V, \gamma')V)-10\{g(\nabla_{\gamma'}V, \nabla_{\gamma'}V)+g(R(V, \gamma')V, \gamma')\}$$

$$\cdot\Big\{\frac{\partial}{\partial t}g(R(V, T)V, T)+2g(R(V, T)V, \nabla_{\gamma'}V)\Big\}\Big](s, 0)ds\,.$$

Now, let us consider the case where the variation vector field V is parallel along Γ. Calculating straightforwardly, we get

$$\Big(\frac{\partial}{\partial t}\Big)^m g(\nabla_{\frac{\partial}{\partial t}}T, T)(s, 0)=g((\nabla_{\frac{\partial}{\partial t}})^{m-1}R(V, T)V, T)$$

$$+\sum_{i=2}^{m-1}{}_mC_i\, g((\nabla_{\frac{\partial}{\partial t}})^{m-i-1}R(V, T)V, (\nabla_{\frac{\partial}{\partial t}})^{i-2}R(V, T)V)(s, 0)\,,$$

$$\Big(\frac{\partial}{\partial t}\Big)^m g(\nabla_{\frac{\partial}{\partial t}}T, \nabla_{\frac{\partial}{\partial t}}T)(s, 0)$$

$$=\sum_{i=1}^{m-1}{}_mC_i\, g((\nabla_{\frac{\partial}{\partial t}})^{m-i-1}R(V, T)V, (\nabla_{\frac{\partial}{\partial t}})^{i-1}R(V, T)V)(s, 0)$$

for $m\geqq 3$. Combining together with these expressions and a consequence of Proposition 2.1, we obtain

PROPOSITION 2.2. *If, for a* 1-*parameter variation* α *of* Γ *defined by* (2.1), *the variation vector field* V *is orthogonal to* Γ *and parallel along it, then we get*

$$L^{(k)}(0)=\int_0^l\Big(\frac{\partial}{\partial t}\Big)^{k-2}g(R(V, T)V, T)(s, 0)ds+\cdots\,, \tag{2.7}$$

where the remainder depends only on vectors $R(V, T)V$, $\nabla_{\frac{\partial}{\partial t}}R(V, T)V, \cdots,$ $(\nabla_{\frac{\partial}{\partial t}})^{k-4}R(V, T)V$.

§ 3. An application of the k-th variation for locally symmetric spaces.

As an application of the k-th variation formula obtained in § 2, we shall consider M being a locally symmetric space of non-negative curvature. T. Frankel studied some relations of the fundamental groups of a complete Riemannian manifold M of strictly positive curvature and its compact totally geodesic hypersurface N [3]. One of the results obtained in [3] is the natural homomorphism of fundamental groups $\pi_1(N)\rightarrow\pi_1(M)$ is surjective. But in general $\pi_1(N)\rightarrow\pi_1(M)$ is not surjective if M is of non-negative curvature. For instance, consider M being a torus and N a closed geodesic. The purpose of this section is to investigate some isometric structures of a complete locally symmetric space M of non-negative curvature which admits a compact totally geodesic hypersurface N where the natural homomorphism of fundamental groups $\pi_1(N)\to\pi_1(M)$ is not surjective.

Let M be a connected and complete locally symmetric space of non-negative curvature and W, V be a compact totally geodesic hypersurface and a submanifold of M, respectively.

First of all, we shall consider M being non-compact.

THEOREM 3.1. *Let M be a complete and non-compact locally symmetric space of non-negative curvature with compact totally geodesic hypersurface W. Then M is isometric to $W\times R$ and W is a soul of M if $C(W)=\emptyset$, or otherwise $C(W)=W^*$ is a soul of M, and W is the double covering of W^*. Furthermore M is isometric to $W\times R/f$, where f is the isometric involution of fixed point free defined by the family of rays $\{\Lambda_x;\Lambda_x\in\Gamma(W,\infty),\lambda_x(0)=x\in W\}$.*

The proof is concluded in Theorems A and B in [5]. They state that if a compact totally geodesic hypersurface N in a complete and non-compact Riemannian manifold M of non-negative curvature has the property $F(N)=\emptyset$, then M is isometric to either $N\times R$ (if $C(N)=\emptyset$) or otherwise, $C(N)=N^*$ is a soul of M. Moreover, every geodesic starting from $x^*\in N^*$ and normal to N^* at the starting point x^* is a ray from N^* to ∞ which strikes N with right angle. And there is a unique ray $\Lambda_x=\{\lambda_x(t)\}(0\leqq t<\infty)$ from N to ∞ through each point $x=\lambda_x(0)\in N$ which is contained in some ray from N^* to ∞. Let $f: N\times R\to N\times R$ be defined by $f(x,t)=(\lambda_x(-2l),-t)$ for all $x\in N$ and all $t\in R$, where we put $l=d(N,N^*)$. Then M is isometric to $N\times R/f$ and the mapping $\pi: N\to N^*$ defined by $\pi(x)=\lambda_x(-l)$ is a local isometry which can be considered as a covering mapping. Therefore N is the double covering of N^*.

If M is locally symmetric, we have $K(P)=0$ for any $P\in G_{\Lambda_x}$ and any $\Lambda_x\in\Gamma(V,\infty)$, $x\in V$. Then $F(V)=\emptyset$ is automatically satisfied. Therefore the proof is a consequence of Theorems A and B in [5].

COROLLARY. *Let V be a totally geodesic submanifold in a complete and non-compact locally symmetric space M with non-negative curvature. Then, V is contained entirely in some W_t, where $W_t=\{x\in M; d(W,x)=t\}$ (or $W_t^*=\{x\in M; d(W^*,x)=t\}$) is a compact totally geodesic hypersurface. Hence $V\cap W\neq\emptyset$ implies that V is a totally geodesic submanifold of W and $V\cap W=\emptyset$ implies that V is a totally geodesic submanifold of W^* or otherwise there is an isometric imbedding $\iota: V\to W$.*

PROOF. First of all we suppose that $M=W\times R/f$. In other word, $C(W)=W^*$ is a compact totally geodesic hypersurface and $W_t^*=\{x\in M; d(x,W^*)=t\}$ is isometric to W for each $t>0$. Since V is compact, there is a number t_0 such that $V\cap W_{t_0}^*=\emptyset$. Take two points $p\in V$, $q\in W_{t_0}^*$ such that $d(p,q)$

$=d(V, W_{t_0}^*)$. There is a uniquely determined shortest geodesic $\Gamma \in \Gamma(q, p)$, $\Gamma=\{\gamma(t)\}(0 \leqq t \leqq l)$, $\gamma(0)=q$, $\gamma(l)=p$ such that $\Gamma|(-\infty, 0]$ is a ray from $W_{t_0}^*$ to ∞. We shall find $V \subset W_{t_0-l}^*$. In fact, the tangent space $(W_{t_0-l}^*)_p$ at p contains V_q and recall that V and $W_{t_0-l}^*$ are totally geodesic submanifold and hypersurface, respectively.

The proof in the case $M=W \times R$ is covered essentially in the one stated above. Q. E. D.

Next, we shall restrict our attention to a compact locally symmetric space M of non-negative curvature which admits a compact totally geodesic submanifold V and a compact totally geodesic hypersurface W. If M is of positive curvature, W and V must intersect each other by virtue of the second variation formula (see Frankel [2]). There is no guaranty that $W \cap V \neq \emptyset$ under our assumption.

But if $W \cap V=\emptyset$, taking two points $y \in V$, $x \in W$ such that $d(x, y)=d(V, W)$ and a geodesic $\Gamma \in \Gamma(y, x)$, we get $K(\gamma'(s), X(s))=0$ for all $s \in [0, d(y, x)]$ and all parallel vector field X along Γ satisfying $X(0) \in V_y$, where we use the 1-parameter variation $\alpha(s, t):=\exp_{\gamma(s)} tX(s)$ and the variation formulas $L^{(k)}(0)$ stated in § 2. Because the function $t \rightarrow L(\alpha_t)$ is analytic and $L^{(k)}(0)=0$ for all $k=1, 2, \cdots$, we must have $L(t)=L(0)=d(y, x)$ for all $t \in(-\varepsilon, \varepsilon)$. Therefore, we get a neighborhood $U_x \subset V$ of x, every point of which has the constant distance $d(V, W)$ to W. By compactness of V, we have $d(z, W)=d(V, W)$ for any point $z \in V$ and $K(\gamma_z'(s), X_z(s))=0$ for any $\Gamma_z \in \Gamma(z, W)$ and any parallel vector field X_z along Γ_z satisfying $X_z(0) \in V_z$. Summing up these facts, we obtain

THEOREM 3.2. *Let W and V be a compact totally geodesic hypersurface and a submanifold of a compact locally symmetric space M of non-negative curvature. Then, we have $W \cap V \neq \emptyset$ or otherwise, we have $d(y, W)=d(V, W)$ for any point $y \in V$ and $K(\gamma_y'(s), X_y(s))=0$ for any $\Gamma_y \in \Gamma(y, W)$ and any parallel vector field along Γ_y such that $X_y(0) \in V_y$.*

Let us consider V, W being hypersurfaces. From Theorem 3.2, we have $K(\gamma_x'(s), X(s))=0$ for all geodesic Γ_x starting at $x \in V$ and normal to it at $\gamma_x(0)=x$ and all parallel vector field X along Γ_x such that $X(0) \in V_x$ if $V \cap W=\emptyset$. This fact implies that $F(V)=\emptyset$ and the compactness of M implies $C(V) \neq \emptyset$.

PROPOSITION 3.3. *$C(V)$ and $C(W)$ are compact totally geodesic hypersurfaces of M. Moreover $C(V)$ (resp. $C(W)$) is locally isometric to V (resp. W).*

PROOF. Let $p \in C(V)$ and $x \in V$ be points such that $d(p, x)=d(C(V), V)$, and $\Gamma \in \Gamma(x, p)$ be a shortest geodesic. Since p is not a focal point to x along Γ, there exist a point $x_1 \in V$ and a shortest geodesic $\Gamma_1 \in \Gamma(x_1, p)$ satisfying $\gamma_1'(l)=-\gamma'(l)$ and $x_1 \neq x$, where we put $l=d(C(V), V)$. Putting Z the unit normal vector field defined in a small neighborhood $U_x \subset V$ of x in such a way that $Z(x)=\gamma'(0)$, we may consider that the mapping φ defined by $\varphi(y)=\exp_y l\cdot Z(y)$, $y\in U_x$ is a diffeomorphism of U_x onto $\varphi(U_x)$. Making use of Warner's metric comparison theorem [7], we see that $L(c)=L(\varphi\circ c)$ holds for arbitrary piecewise smooth curve c in U_x. Therefore, φ being an isometry of U_x onto $\varphi(U_x)$, $\varphi(U_x)$ is a piece of totally geodesic hypersurface which is isometric to U_x. We can also construct the unit normal vector field Z_1 defined in a neighborhood $U_{x_1}\subset V$ of x_1 such that $Z_1(x_1)=\gamma_1'(0)$ and the mapping φ_1 defined by $\varphi_1(y_1)=\exp_{y_1}l\cdot Z_1(y_1)$. Because we have the same argument as φ and U_x, there exist $U_x'\subset U_x$ and $U_{x_1}'\subset U_{x_1}$ such that $\varphi(U_x')$ coincides with $\varphi_1(U_{x_1}')$. In fact, both $\varphi(U_x)$ and $\varphi_1(U_{x_1})$ must have the common tangent space at p and are totally geodesic. Then we find that for each point $y\in\varphi^{-1}(U_x')$, we have $\gamma_y(2l)\in\varphi_1^{-1}(\varphi(y))$. In fact, by virtue of Ōmori's Proposition (3.4 Proposition, [4]), $\gamma_y'(l)\neq-\gamma_{y_1}'(l)$, $y_1\equiv\varphi_1^{-1}(\varphi(y))$ implies the existence of Jacobi field Y along Γ_y such that $g(Y,\gamma_y')=0$, $Y(0)\in V_y$, $Y(0)\neq 0$ and $Y(l)=0$. But this contradicts Theorem 3.2. Hence we see that $\varphi(U_x')\subset C(V)$.

We shall next prove that $C(V)$ is a compact totally geodesic hypersurface. Let $p_0\in M$ be a point such that $\lim_{k\to\infty}p_k=p_0$, $p_k\in\varphi(U_x)$. There is a sequence of points $x_k\in U_x$ satisfying $\varphi(x_k)=p_k$ and the geodesic Γ_k starting from x_k and normal to V at the starting point. Then, $\gamma_k(2l)\in U_1$ holds for each $k=1, 2, \cdots$. Hence there is a geodesic Γ_0 starting from $x_0=\lim_{k\to\infty}x_k$ and normal to V at the starting point which satisfies $\gamma_0(2l)\in V$. This fact implies that $\gamma_0(l)\in C(V)$ and there exist neighborhoods U_{x_0}, $U_{\gamma_0(2l)}$ and mappings φ_{x_0}, $\varphi_{\gamma_0(2l)}$ as well as U_x and φ. Γ_0 containing no focal point to x_0, we have the same argument as U_x, φ. Since V is compact, we can choose a family of points $\{x_i\}$ and corresponding neighborhoods $\{U_i\}$ in such a way that V can be covered by $\bigcup_{i=1}^k U_i$, and we also see that $C(V)\subset\bigcup_{i=1}^k\varphi_i(U_i)$, which shows that $C(V)$ is a compact totally geodesic hypersurface. Q. E. D.

In the following we shall classify the structures of M with disjoint compact totally geodesic hypersurfaces V and W. Recall that we have $K(P)=0$ for any plane section $P\in G_{\Gamma_x}$ and every geodesic Γ_x starting from a point x of V or W with normal direction to V or W at the point. And hence $F(V)=F(W)=\emptyset$ is automatically satisfied. Since V and W are in the same position, we restrict our attention to V.

THEOREM 3.4. *Assume that there is a unit normal vector field N which is defined globally over V and V divides M into two connected components. Then $C(V)$ consists of two connected components, each of which is a compact totally geodesic hypersurface, say $C_1(V)$ and $C_2(V)$, and V is the double covering of them. Moreover $V_t=\{\exp_x tN(x)\,;\,x\in V\}$ is isometric to V for each $t\in[0, d(V, C(V)))$. Especially M is isometric to a Klein bottle if* $\dim M=2$.

PROOF. Let Γ_x be the geodesic defined by $\gamma_x(0)=x\in V$ and $\gamma_x'(0)=N(x)$. Then $K(P)=0$ holds for any plane section P satisfying $P\in G_{\Gamma_x}$, which implies that the mapping $\varphi_t\colon V\to V_t$ defined by $\varphi_t(x):=\gamma_x(t)$ is a local isometry. Putting $C_1(V)=\{\gamma_x(l(x))\,;\,\gamma_x(l(x))\in C(V)\}$, we see from Proposition 3.3 that $l(x)=l$ (constant) and $\gamma_x(2l)\in V$ for any $x\in V$. Hence the mapping $\varphi_{2l}\colon V\to V$ is an isometric involution of fixed point free and $C_1(V)$ is isometric to V/φ_{2l}. Then $\varphi_l\colon V\to C_1(V)$ can be considered as a covering projection. For any fixed $t\in(0,l)$, V_t is a compact totally geodesic hypersurface by virtue of Warner's metric comparison theorem and the property of φ_t implies that V_t is compact.

Q. E. D.

REMARK. Let $\varphi_l'\colon V\to C_2(V)$ be defined as well as $\varphi_l\colon V\to C_1(V)$. Then for a point $x\in V$, $\varphi_{2l'}'\circ\varphi_{2l}(x)\neq x$ holds in general. There is no guaranty for $\varphi_{2l'}'\circ\varphi_{2l}(x)=x$ holding. We note that for any point $p\in C(V)$, there are just two minimal geodesics Γ_1, Γ_2 in $\Gamma(p,V)$ and its tangent vectors satisfy $\gamma_1'(l)=-N(\gamma_1(l))$, $\gamma_2'(l)=-N(\gamma_2(l))$ under the assumption of Theorem 3.4. And Γ_1, Γ_2 lie entirely in the component containing $C_1(V)$.

THEOREM 3.5. *Assume that there is a unit normal vector field N which is defined globally over V and V does not divide M. Then, $C(V)$ is isometric to V. Especially M is isometric to a flat torus if* $\dim M=2$.

PROOF. We have seen that $C(V)$ is a compact totally geodesic hypersurface and $\Gamma(p,V)$ consists of just two geodesics for every point $p\in C(V)$. Suppose that there is a point $p\in C(V)$ in such a way that the geodesics Γ_1, $\Gamma_2\in\Gamma(p,V)$ satisfy both $\gamma_1'(l)=-N(\gamma_1(l))$ and $\gamma_2'(l)=-N(\gamma_2(l))$, where $l=d(V,C(V))$. Then we see that $C(V)$ contains the set $\{\gamma_x(l)\,;\,x\in V,\gamma_x'(0)=N(x)\}$ which is a compact totally geodesic hypersurface and coincides with $C_1(\boldsymbol{V})$ stated in Theorem 3.4. Then V divides M into two connected components. Therefore we must have $\gamma_1'(l)=N(\gamma_1(l))$ and $\gamma_2'(l)=-N(\gamma_2(l))$ for any point $p\in C(V)$ and Γ_1, $\Gamma_2\in\Gamma(p,V)$. This fact implies that $\varphi_l\colon V\to C(V)$ defined by $\varphi_l(x):=\exp_x l\cdot N(x)$ is injective. Hence $C(V)$ is isometric to V.

Q. E. D.

REMARK. There is no guaranty that $\gamma_x(2l)=\gamma_x(0)$ holds for every point $x\in V$; in other words, M is isometric to $V\times S^1$. In fact, consider a flat torus whose covering fold is given as a parallelogram with the same side length. Then $\gamma_x(2l)=\gamma_x(0)$ holds for every point $x\in V$ (V is a closed geodesic) if and only if one of its angles is equal to right angle. When one of its angles is equal to $\pi/3$, $\gamma_x(4l)=\gamma_x(0)$ holds for every $x\in V$.

Now, we assume that there is no unit normal vector field which is defined globally over V. Then the set $V_t^*=\{x\in M\,;\,d(x,V)=t\}$ is a connected and compact totally geodesic hypersurface for each $t\in(0,d(V,C(V)))$. For a fixed $t_0\in(0,d(V,C(V)))$ we see that $V_{t_0}^*$ divides M into two connected components, that is to say, one is the set $\{x \in M; d(x, V) < t_0\}$ and the other is $\{x \in M; d(x, V) > t_0\}$ each of which has boundary as $V_{t_0}^*$. It is easily seen that $C(V_{t_0}^*) \supset V$ and $C(V_{t_0}^*)$ has two connected components. Therefore $V_{t_0}^*$ satisfies the hypothesis of Theorem 3.4 and hence we have the same argument. Summing up these facts, we obtain

THEOREM 3.6. *Assume that there is no unit normal vector field which is defined globally over* V. *Then* $V_{t_0}^*$ *is the double covering of* V *for each* $t \in (0, d(V, C(V)))$. *Moreover* $C(V)$ *is isometric to* V.

It follows from the generalized Gauss-Bonnet theorem that $\chi(M)=0$ if even dimensional, oriented and compact locally symmetric space M has two compact totally geodesic hypersurfaces V and W such that $V \cap W = \emptyset$.

Now, the natural homomorphism of fundamental groups $\pi_1(V) \rightarrow \pi_1(M)$ is surjective if the inverse image $\pi^{-1}(V)$ of V under the projection map $\pi: \tilde{M} \rightarrow M$ is connected, where $\tilde{M}$ is the universal covering manifold of M [**3**]. The discussion in this section shows that $\pi_1(V) \rightarrow \pi_1(M)$ is not necessarily surjective under our situation that M is a complete locally symmetric space of non-negative curvature. Therefore let us consider $\pi_1(V) \rightarrow \pi_1(M)$ being not surjective. If M is isometric to $V \times R$, we easily see that $\pi_1(V) \rightarrow \pi_1(M)$ is surjective. Developing the same argument as in [**3**] we obtain that $K(P)=0$ for any plane section $P \in G_{\Gamma_x}$ and any geodesic Γ_x, $x \in V$ whose starting point is x and normal to V at x. Then we get

THEOREM 3.7. *Let* M *be a complete locally symmetric space of non-negative curvature and* V *be a compact totally geodesic hypersurface. Suppose that the natural homomorphism of fundamental groups* $\pi_1(V) \rightarrow \pi_1(M)$ *is not surjective. Then we have*

(1) *if* M *is non-compact,* M *is isometric to* $V \times R/f$ *where* f *is the isometric involution of fixed point free defined in Theorem* 3.1,

(2) *if* M *is compact,* M *has the isometric structures stated in Theorem* 3.4 *or* 3.5.

REMARK. If $\dim M=2$, M is isometric to an open Möbius band if M is non-compact, and M is isometric to either a Klein bottle or a flat torus if M is compact.

Tokyo University of Agriculture and Technology
Tokyo Institute of Technology

References

[1] J. Cheeger and D. Gromoll, The structure of complete manifolds of non-negative curvature, Bull. Amer. Math. Soc., **74** (1968), 1147-1150.
[2] T. Frankel, Manifolds with positive curvature, Pacific J. Math., **11** (1961), 165-174.
[3] T. Frankel, On the fundamental group of a compact minimal submanifold, Ann.

of Math., **33** (1966), 68–73.

[4] H. Ōmori, A class of riemannian metric on a manifold, J. Differential Geometry, **2** (1968), 233-252.

[5] K. Shiohama, On complete non-compact riemannian manifolds with certain properties, to appear.

[6] V. A. Toponogov, Riemannian space which contain straight lines, Amer. Math. Soc. Transl. Ser., **37** (1964), 287–290.

[7] F. W. Warner, Extension of the Rauch comparison theorem to submanifolds, Trans. Amer. Math. Soc., **122** (1960), 341–356.