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On some doubly transitive permutation groups of degree n and order $2^l(n-1)n$

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1. Introduction.

Doubly transitive permutation groups of degree n and order $2(n-1)n$ were determined by N. Ito ([**9**]). Some doubly transitive permutation groups of degree n and order $4(n-1)n$ were studied in [**10**].

The object of this paper is to prove the following result.

THEOREM. *Let Ω be the set of symbols* $1, 2, \cdots, n$. *Let $\mathfrak{G}$ be a doubly transitive group on Ω of order $2^l(n-1)n$ $(l>1)$ not containing a regular normal subgroup and let $\mathfrak{K}$ be the stabilizer of symbols* 1 *and* 2. *Assume that $\mathfrak{K}$ is cyclic. Then $\mathfrak{G}$ is isomorphic to one of the groups $PGL(2, *)$, $PSL(2, *)$, $PSU(3, 3^2)$ and $PSU(3, 5^2)$.*

We use the standard notation. $C_{\mathfrak{X}}(\mathfrak{T})$ denotes the centralizer of a subset $\mathfrak{T}$ in a group $\mathfrak{X}$ and $N_{\mathfrak{X}}(\mathfrak{T})$ stands for the normalizer of $\mathfrak{T}$ in $\mathfrak{X}$. $\langle S, T, \cdots\rangle$ denotes the subgroup of $\mathfrak{X}$ generated by elements $S, T, \cdots$ of $\mathfrak{X}$.

2. On the degree of the permutation group $\mathfrak{G}$.

1. Let $\mathfrak{H}$ be the stabilizer of the symbol 1. $\mathfrak{K}$ is of order 2^l and it is generated by a permutation K. Let us denote the unique involution $K^{2^{l-1}}$ of $\mathfrak{K}$ by τ. Since $\mathfrak{G}$ is doubly transitive on Ω it contains an involution I with the cyclic structure $(1\ 2)\cdots$. Then we have the following decomposition of $\mathfrak{G}$;

$$\mathfrak{G}=\mathfrak{H}+\mathfrak{H}I\mathfrak{H}.$$

Since I is contained in $N_{\mathfrak{G}}(\mathfrak{K})$, it induces an automorphism of $\mathfrak{K}$ and (i) $K^I=K$ or $K\tau$, (ii) $K^I=K^{-1}\tau$ or (iii) $K^I=K^{-1}$. (For the case $l=2$, (i) $K^I=K$ or (iii) $K^I=K^{-1}$.) If an element $H'IH$ of a coset $\mathfrak{H}IH$ of $\mathfrak{H}$ is an involution, then $IHH'I=(HH')^{-1}$ is contained in $\mathfrak{K}$. Hence, in the case (i) the coset $\mathfrak{H}IH$ contains just two involutions, namely $H^{-1}IH$ and $H^{-1}\tau IH$, in the case (ii) it contains just 2^{l-1} involutions, namely $H^{-1}K'IH$ for $K'\in\langle K^2\rangle$, and in the case

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(iii), it contains just 2^l involutions, namely $H^{-1}K'IH$ for $K' \in \mathfrak{K}$. Let $g(2)$ and $h(2)$ denote the numbers of involutions in $\mathfrak{G}$ and $\mathfrak{H}$, respectively. Then the following equality is obtained;

$$g(2) = h(2) + d(n-1)\,, \tag{2.1}$$

where $d = 2$, 2^{l-1} and 2^l for cases (i), (ii) and (iii), respectively.

2. For a set $\mathfrak{T}$ of permutations of $\mathfrak{G}$, the set of all symbols fixed by $\mathfrak{T}$ is denoted by $\mathfrak{J}(\mathfrak{T})$ and we denote the number of symbols in $\mathfrak{J}(\mathfrak{T})$ by $\alpha(T)$. Let K^{2l-j} denote the permutation of $\mathfrak{K}$ such that $\alpha(\tau) = \alpha(K^{2l-j}) > \alpha(K^{2l-j-1})$ and let $\mathfrak{K}_1$ be the subgroup of $\mathfrak{K}$ generated by K^{2l-j}. Then the order of $\mathfrak{K}_1$ is equal to 2^j. Let $\mathfrak{K}_1$ keep i $(i \geqq 2)$ symbols of Ω, say $1, 2, \cdots, i$, unchanged. It is trivial that $N_{\mathfrak{G}}(\mathfrak{K}_1) = C_{\mathfrak{G}}(\tau)$. Put $\mathfrak{J} = \mathfrak{J}(\mathfrak{K}_1) = \{1, 2, \cdots, i\}$. We denote the factor group $N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ by $\mathfrak{G}_1$. By a theorem of Witt ([**15**, Theorem 9.4]), $\mathfrak{G}_1$ can be considered as a doubly transitive permutation group on $\mathfrak{J}$. The stabilizer of symbols 1 and 2 in $\mathfrak{J}$ is the cyclic 2-group $\mathfrak{K}/\mathfrak{K}_1$. Thus the orders of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ and $\mathfrak{H} \cap N_{\mathfrak{G}}(\mathfrak{K}_1)$ are equal to $2^l i(i-1)$ and $2^l(i-1)$, respectively. Hence there exist $n(n-1)/i(i-1)$ involutions in $\mathfrak{G}$ each of which is conjugate to τ.

At first, let us assume that n is odd. Let $h^*(2)$ be the number of involutions in $\mathfrak{H}$ leaving only the symbol 1 fixed. Then from (2.1) and above argument the following equality is obtained;

$$h^*(2)n + n(n-1)/i(i-1) = (n-1)/(i-1) + h^*(2) + d(n-1)\,. \tag{2.2}$$

Since i is less than n, it follows from (2.2) that $h^*(2) < d$ and hence $n = i(\beta i - \beta + 1)$, where $\beta = d - h^*(2)$. Since n is odd, i must be odd.

Next let us assume that n is even. Let $g^*(2)$ be the number of involutions in $\mathfrak{G}$ leaving no symbol of Ω fixed. Then corresponding to (2.2) the following equality is obtained from (2.1);

$$g^*(2) + n(n-1)/i(i-1) = (n-1)/(i-1) + d(n-1)\,. \tag{2.3}$$

It is easily proved that $g^*(2)$ is a multiple of $n-1$ (see [8] or [9]). It follows from (2.3) that $g^*(2) < d(n-1)$. Thus we have $n = i(\beta i - \beta + 1)$, where $\beta = d - g^*(2)/(n-1)$. Since n is even, i must be even.

3. We prove the theorem by induction on the degree n. Let $SL(2, 8)$ denote the two-dimensional special linear group over the field $GF(8)$ of eight elements, and let σ be the automorphism of $GF(8)$ of order three such that $\sigma(x) = X^2$ for every element x of $GF(8)$. Then σ can be considered in a usual way an automorphism of $SL(2, 8)$. Let $SL^*(2, 8)$ be the splitting extension of $SL(2, 8)$ by the group $\langle \sigma \rangle$. Then $SL^*(2, 8)$ has doubly transitive permutation representation on the set of Sylow 3-subgroups and its degree is equal to 28. The stabilizer of two symbols leaves four Sylow 3-subgroups fixed and every

involution is conjugate (see [8]).

THEOREM 1 (N. Ito, [8]). *Let $\mathfrak{G}$ be a doubly transitive permutation group on Ω of order $2n(n-1)$ not containing a regular normal subgroup. Then $\mathfrak{G}$ is isomorphic to either $PSL(2, 5)$ or $SL^*(2, 8)$.*

If $\mathfrak{G}$ contains a regular normal subgroup, then its degree is equal to a power of a prime number. Thus, by Theorem 1, if $l=1$, then n is equal to 6, 28 or a power of a prime number.

3. The case n is odd.

1. Since $n=i(\beta i-\beta+1)$ is odd, i must be odd. The group $\mathfrak{G}_1=N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ is a doubly transitive permutation group on $\mathfrak{J}(\mathfrak{K}_1)$ and the stabilizer of symbols 1 and 2 is the subgroup $\mathfrak{K}/\mathfrak{K}_1$ of $\mathfrak{G}_1$ of order 2^{l-j}. By the inductive hypothesis, $\mathfrak{G}_1$ contains a regular normal subgroup and, in particular, i is equal to a power of an odd prime number, say p^m. Let $\mathfrak{P}$ be a Sylow p-subgroup of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ of order $i=p^m$. Since $\mathfrak{P}\mathfrak{K}_1/\mathfrak{K}_1$ is a regular normal subgroup of $\mathfrak{G}_1$, $\mathfrak{P}$ is elementary abelian and normal in $N_{\mathfrak{G}}(\mathfrak{K}_1)$. Let $\mathfrak{V}$ denote the subgroup $\mathfrak{H}\cap N_{\mathfrak{G}}(\mathfrak{K}_1)$. Then the order of $\mathfrak{V}$ is equal to $2^l(p^m-1)$.

2. Case $n=i^2=p^{2m}$. It can be proved in the same way as in [**9**, Case A] that there exists no group satisfying the conditions of the theorem in this case.

3. Case $n=p^m(\beta p^m-\beta+1)$ with $\beta>1$ and β, $\beta-1\not\equiv 0$ (mod. p). In this case it can be proved in the same way as in [**10**, § 2.5] that there is no group satisfying the conditions of the theorem in this case.

4. Case $n=p^m(\beta p^m-\beta+1)$ with $\beta>1$ and $\beta\equiv 0$ (mod. p). Since $\beta\geqq 3$, d must be greater than 2 and hence $\langle K, I\rangle$ is dihedral or semi-dihedral.

Consider the cyclic structure of K and it can be seen that $n-i=\beta p^m(p^m-1)$ is divisible by 2^l. Set $p=2^kq+1$, where $q(>0)$ is odd. Since $2^l\geqq\beta\geqq p$, β is not divisible by 2^{l-k} and therefore p^m-1 must be divisible by 2^{k+1}. Hence m is even.

At first assume that the order of $N_{\mathfrak{G}}(\mathfrak{K})$ is divisible by 2^{l+2}. Since $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ is a complete Frobenius group on $\mathfrak{J}(\mathfrak{K})$, any Sylow subgroup of a complement $\mathfrak{H}\cap N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ is cyclic or quaternion (ordinary or generalized). Hence there exists a subgroup $\mathfrak{S}$ of $N_{\mathfrak{G}}(\mathfrak{K})$ such that $\mathfrak{S}\supseteqq\langle I, K\rangle$ and $\mathfrak{S}/\mathfrak{K}$ is a cyclic group of order 4. $\mathfrak{S}$ contains S such that $S^2\equiv I(\mathfrak{K})$, S induces an automorphism of $\mathfrak{K}$ of order 4 and S^2 and I induce the same automorphism. But it is easily seen that, for any automorphism ζ of $\mathfrak{K}$ of order 4, $K^{\zeta^2}=\tau K$. This is a contradiction since $\langle K, I\rangle$ is dihedral or semi-dihedral.

Next assume that the order of $N_{\mathfrak{G}}(\mathfrak{K})$ is not divisible by 2^{l+2}. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ containing $\langle I, K\rangle$. Since m is even, the order of $\mathfrak{S}$ is greater than 2^{l+2}. By the assumption of the order of $N_{\mathfrak{G}}(\mathfrak{K})$, $\mathfrak{S} \cap N_{\mathfrak{G}}(\mathfrak{K}) = \langle K, I \rangle$ is a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K})$. Therefore $N_{\mathfrak{S}}(\langle K, I \rangle)$ is greater than $N_{\mathfrak{S}}(\mathfrak{K})$. Let S $(\neq 1)$ be a permutation of $N_{\mathfrak{S}}(\langle K, I\rangle)-\langle K, I \rangle$. Since K^S is contained in $\langle K, I \rangle$, we have $K^S = K'I$, where K' is a permutation of $\mathfrak{K}$. Hence, if $\langle K, I \rangle$ is dihedral, then $(K^S)^2 = 1$ and the order of K equals 2 and, if $\langle K, I \rangle$ is semi-dihedral, then $(K^S)^4 = 1$ and the order of K equals 4. This is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in this case.

5. Case $n = p^m(\beta p^m - \beta + 1)$ with $\beta - 1 = 0$ (mod. p).

At first we shall prove that the order of $C_{\mathfrak{G}}(\mathfrak{P})$ is equal to $2^{j'}p^{m+m'}y$, where $j' \geqq j$, $m' > 0$ and y is a factor of $\beta p^m - (\beta - 1)$ and not divisible by p. Assume that the order of $C_{\mathfrak{G}}(\mathfrak{P})$ is equal to $2^{j'}p^m$. Let $\mathfrak{K}'$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\mathfrak{P})$. Every element $(\neq 1)$ of $\mathfrak{P}$ leaves no symbol of Ω fixed. Then $\mathfrak{K}'$ must leave at least two symbols of Ω fixed. Therefore $\mathfrak{K}'$ is conjugate to a subgroup of $\mathfrak{K}$ containing $\mathfrak{K}_1$. Since $C_{\mathfrak{G}}(\mathfrak{P})$ is a direct product of $\mathfrak{K}'$ and $\mathfrak{P}$, $\mathfrak{K}'$ is normal in $N_{\mathfrak{G}}(\mathfrak{P})$. Since the order of $N_{\mathfrak{G}}(\mathfrak{K}')$ is a factor of the order of $N_{\mathfrak{G}}(\mathfrak{K}_1)$, the order of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ is greater than or equal to the order of $N_{\mathfrak{G}}(\mathfrak{P})$. This contradicts the order of $N_{\mathfrak{G}}(\mathfrak{P})$. Hence the order of $C_{\mathfrak{G}}(\mathfrak{P})$ is equal to $2^{j'}p^m y$, where y is odd and $y > 1$. Let q $(\neq 2, p)$ be a prime factor of the order of $C_{\mathfrak{G}}(\mathfrak{P})$ and let Q be a permutation of $C_{\mathfrak{G}}(\mathfrak{P})$ of order q. If q is a factor of $n-1$, then Q leaves just one symbol of Ω fixed and hence Q cannot be contained in $C_{\mathfrak{G}}(\mathfrak{P})$. Thus q is a factor of n and so is y. Next assume that y is not divisible by p. Let $\mathfrak{A}'$ be a normal p-complement in $C_{\mathfrak{G}}(\mathfrak{P})$. Since $\mathfrak{K}'$ is cyclic, $\mathfrak{A}'$ has a normal 2-complement $\mathfrak{Y}'$. Since $\mathfrak{Y}'$ is a normal Hall subgroup of $\mathfrak{A}'$, $\mathfrak{Y}'$ is normal even in $N_{\mathfrak{G}}(\mathfrak{P})$. Let Y' $(\neq 1)$ be a permutation of $\mathfrak{Y}'$. Then Y' does not leave any symbol of Ω fixed. If $\mathfrak{V} \cap G_{\mathfrak{G}}(Y')$ contains an involution τ', then τ' is conjugate to τ under $\mathfrak{G}$ and, since $C_{\mathfrak{G}}(\tau')$ contains Y', the order of $C_{\mathfrak{G}}(\tau')$ is divisible by the order of Y'. But since $C_{\mathfrak{G}}(\tau')$ is conjugate to $C_{\mathfrak{G}}(\tau) = N_{\mathfrak{G}}(\mathfrak{K}_1)$ and the order of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ and y are relatively prime, the order of $\mathfrak{V} \cap C_{\mathfrak{G}}(Y')$ is odd. Let q be a prime factor of the order of $\mathfrak{V} \cap C_{\mathfrak{G}}(Y')$ and let Q be a permutation of $\mathfrak{V} \cap C_{\mathfrak{G}}(Y')$ of order q. Then Q leaves at least one symbol of Ω fixed and hence it leaves at least two symbols of Ω fixed, which is a contradiction. Thus $\mathfrak{V} \cap C_{\mathfrak{G}}(Y') = (1)$. Hence we have the following relation;

$$y - 1 = |\mathfrak{Y}'| - 1 \geqq |\mathfrak{V}|,$$

$$\text{i. e., } y \geqq 2^l(p^m - 1) + 1 = 2^l p^m - (2^l - 1).$$

On the other hand y is a factor of $\beta p^{m-1} - (\beta - 1)p^{-1}$. This is a contradiction. Hence y is divisible by p.

Let us assume $p^{m'}<2^l$. Let $\mathfrak{A}$ be a normal 2-complement of $C_{\mathfrak{G}}\mathfrak{P}$. Then $\mathfrak{A}$ is normal in $N_{\mathfrak{G}}(\mathfrak{P})$. Let $\mathfrak{P}'$ be a Sylow p-subgroup of $\mathfrak{A}$. By the Frattini argument $N_{\mathfrak{G}}(\mathfrak{P})=\mathfrak{A}(N_{\mathfrak{G}}\mathfrak{P}'\cap N_{\mathfrak{G}}(\mathfrak{P}))$. Since the order of $\mathfrak{A}$ is odd, we may assume that $\mathfrak{K}$ is a subgroup of $N_{\mathfrak{G}}(\mathfrak{P}')\cap N_{\mathfrak{G}}(\mathfrak{P})$. Thus there exists a homomorphism π of $\mathfrak{K}$ into $\mathrm{Aut}\,\mathfrak{P}'/\mathfrak{P}$. If τ is contained in $\ker\pi$, then τ acts trivially on $\mathfrak{P}'/\mathfrak{P}$ and $\mathfrak{P}$. Therefore τ acts also trivially on $\mathfrak{P}'$ and $C_{\mathfrak{G}}\tau$ contains $\mathfrak{P}'$ ([4, Theorem 5.3.2]). Hence we have $\ker\pi=1$ and $\mathrm{Aut}\,\mathfrak{P}'/\mathfrak{P}$ contains a cyclic subgroup of order 2^l. But the order $(=p^{m'})$ of $\mathfrak{P}'/\mathfrak{P}$ is less than 2^l. This is a contradiction. If $m'\leqq m$, then $p^{m'}<2^l$. Thus we may assume $p^{m'}>2^l$. Then $m'>m$.

Assume $y>1$. Since $\mathfrak{A}$ is solvable, there exists a subgroup $\mathfrak{Y}$ of $\mathfrak{A}$ of order y. Now Y is a factor of $\beta-(\beta-1)p^{-m}$. By the Frattini argument it can be assumed that $\mathfrak{K}$ is a subgroup of $N_{\mathfrak{G}}(\mathfrak{Y})$. Thus there exists a homomorphism π' of $\mathfrak{K}$ into $\mathrm{Aut}\,\mathfrak{Y}$. Since the orders of $C_{\mathfrak{G}}(\tau)$ and $\mathfrak{Y}$ are relatively prime, any elements $(\neq 1)$ of $\mathfrak{Y}$ are not fixed by $\pi'(\tau)$. Therefore we have $y>2^l$. This is impossible and hence $y=1$. $\mathfrak{P}'$ is normal in $N_{\mathfrak{G}}(\mathfrak{P})$. Let P' $(\neq 1)$ be an element of $\mathfrak{P}'$. It can be seen that $\mathfrak{B}\cap C_{\mathfrak{G}}(\mathfrak{P}')$ is a subgroup of $\mathfrak{K}$. Hence we have the following relation;

$$p^{m+m'}-1=x(p^m-1)\,,\qquad x>1\,.$$

From this it is easily seen that m' is divisible by m.

If $\beta p^m-\beta+1$ is divisible by $p^{\delta m}$ $(\delta>1)$ exactly, then $\beta-1$ must be equal to $p^{\delta m}z+p^{(\delta-1)m}+\cdots+p^m$ $(z>1)$ or $p^{(\delta-1)m}+\cdots+p^m$. If $\beta-1$ is equal to $p^{\delta m}z+p^{(\delta-1)m}+\cdots+p^m$ $(z>1)$, then $2^l>p^{\delta m}$ $(\geqq p^{m'})$. Therefore we may assume $\beta=p^{(\delta-1)m}+\cdots+p^m+1=(p^{\delta m}-1)/(p^m-1)$ and $m'=\delta m$. $\mathfrak{P}'$ is a Sylow p-subgroup of $\mathfrak{G}$.

Next we shall prove that $m=1$ and K has only 2^l-cycles in its cyclic decomposition, i.e., $N_{\mathfrak{G}}(\mathfrak{K})=C_{\mathfrak{G}}(\tau)$ and $\mathfrak{K}\cap\mathfrak{K}^G=1$ or $\mathfrak{K}$ for every element G of $\mathfrak{G}$. From (2.2) it can be seen that the number of involutions with the cyclic structures $(1, 2)\cdots$ which are conjugate to τ is equal to β. If $\langle K, I\rangle$ is dihedral, then every involution in $I\mathfrak{K}$ is conjugate to I or IK and if $\langle K, I\rangle$ is semi-dihedral, then every involution in $I\mathfrak{K}$ is conjugate to I. Since all involutions with the cyclic structures $(1, 2)\cdots$ are contained in $I\mathfrak{K}$, β is equal to $d/2$ or d. Thus p^m+1 is a power of two and hence $m=1$. Therefore $\mathfrak{G}_1$ is a complete Frobenius group, $\mathfrak{Z}(\tau)=\mathfrak{Z}(K)$, $N_{\mathfrak{G}}(\mathfrak{K})=C_{\mathfrak{G}}(\tau)$ and $C_{\mathfrak{G}}(\mathfrak{K})$ contains $\mathfrak{P}$. Therefore the number of elements which leave only the symbol 1 fixed is equal to $2^l(n-1)-1-(2^l-1)(\beta i+1)$ and the number of elements which leave i symbols of Ω fixed is equal to $(2^l-1)(\beta i-\beta+1)(\beta i+1)$. Let G be an element of $\mathfrak{G}$ of order $2^{l'}p\,(l'\geqq 1)$. Then $\alpha(G)=0$ and $\alpha(G^p)=i$. Therefore the number of cyclic subgroups of $\mathfrak{G}$ of order $2^l p$ is equal to $(\beta i-\beta+1)(\beta i+1)$ and those groups are independent. Thus the number of elements of order $2^{l'}p(l'\geqq 1)$ which leave no symbol of Ω fixed is equal to $(2^l-1)(i-1)(\beta i-\beta+1)(\beta i+1)$. Therefore we have

$$|\mathfrak{G}|-(n(2^l(n-1)-1-(2^l-1)(\beta i+1))+(2^l-1)(\beta i-\beta+1)(\beta i+1)$$
$$+(2^l-1)(n-1)(\beta i-\beta+1)+1)=n-1\,.$$

Hence $\mathfrak{P}'$ is a regular normal subgroup of $\mathfrak{G}$.

Thus there exists no group satisfying the conditions of the theorem in this case.

4. The case n is even and $N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ contains a regular normal subgroup.

1. Since $n=i(\beta i-\beta+1)$ is even, i must be even. $\mathfrak{G}_1=N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ is a doubly transitive permutation group on $\mathfrak{J}(\mathfrak{K}_1)$ containing a regular normal subgroup. In particular, i is equal to a power of 2, say 2^m.

Let $\mathfrak{S}$ be the normal 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ containing $\mathfrak{K}_1$ such that $\mathfrak{S}/\mathfrak{K}_1$ is a regular normal subgroup of $\mathfrak{G}_1=N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$. Since the order of $\mathfrak{H}\cap N_{\mathfrak{G}}(\mathfrak{K}_1)$ is equal to $2^l(2^m-1)$, $\mathfrak{K}$ is a Sylow 2-subgroup of $\mathfrak{H}\cap N_{\mathfrak{G}}(\mathfrak{K}_1)$. Let $\mathfrak{V}$ be a normal 2-complement of $\mathfrak{H}\cap N_{\mathfrak{G}}(\mathfrak{K}_1)$. The group $\mathfrak{V}\mathfrak{S}/\mathfrak{K}_1$ is a complete Frobenius group on $\mathfrak{J}(\mathfrak{K}_1)$ with kernel $\mathfrak{S}/\mathfrak{K}_1$ and complement $\mathfrak{V}\mathfrak{K}_1/\mathfrak{K}_1$ ($\cong\mathfrak{V}$). Since $C_{\mathfrak{G}}(\mathfrak{K}_1)\cap\mathfrak{V}\mathfrak{S}$ is normal in $\mathfrak{V}\mathfrak{S}$, $C_{\mathfrak{G}}(\mathfrak{K}_1)\cap\mathfrak{V}\mathfrak{S}$ contains $\mathfrak{S}$ or is contained in $\mathfrak{S}$ ([**13**, 12.6.8]). If $\mathfrak{S}$ is greater than $C_{\mathfrak{G}}(\mathfrak{K}_1)\cap\mathfrak{V}\mathfrak{S}$, since the index of $\mathfrak{S}$ in $\mathfrak{V}\mathfrak{S}$ must be equal to a power of two, we have $m=1$. Hence $\mathfrak{G}$ is a Zassenhaus group. Thus we have that $\mathfrak{G}$ is isomorphic to either $PGL(2,2^l+1)$ or $PSL(2,2^{l+1}+1)$, where 2^l+1 and $2^{l+1}+1$ are powers of prime numbers for $PGL(2,2^l+1)$ and $PSL(2,2^{l+1}+1)$, respectively ([**1**], [**8**], [**14**] and [**18**]). Thus it will be assumed that $\mathfrak{S}$ is contained in $C_{\mathfrak{G}}(\mathfrak{K}_1)\cap\mathfrak{V}\mathfrak{S}$ and m is greater than one.

Since the index of $\mathfrak{V}\mathfrak{S}\cap C_{\mathfrak{G}}(\mathfrak{K}_1)$ in $\mathfrak{V}\mathfrak{S}$ is odd and the order of $\mathrm{Aut}\,\mathfrak{K}_1$ is equal to 2^{j-1}, $\mathfrak{V}\mathfrak{S}\cap C_{\mathfrak{G}}(\mathfrak{K}_1)$ is equal to $\mathfrak{V}\mathfrak{S}$. Hence $C_{\mathfrak{G}}(\mathfrak{K}_1)$ is equal to $N_{\mathfrak{G}}(\mathfrak{K}_1)$ since $N_{\mathfrak{G}}(\mathfrak{K}_1)=\mathfrak{K}\mathfrak{V}\mathfrak{S}$.

PROPOSITION 4.1. *Let $\mathfrak{G}$ be as in Theorem and let $\mathfrak{K}_1$ and $\mathfrak{G}_1$ as above. Assume that $\mathfrak{G}_1$ contains a regular normal subgroup and $N_{\mathfrak{G}}(\mathfrak{K}_1)$ is equal to $C_{\mathfrak{G}}(\mathfrak{K}_1)$. Let $\mathfrak{S}$ be as above. Then $\mathfrak{S}$ contains an involution ($\neq\tau$).*

PROOF. If $\mathfrak{K}_1$ is equal to $\mathfrak{K}$, then $\mathfrak{S}$ is a normal Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K})$ and hence it contains I. Therefore it can be assumed that $\mathfrak{K}_1$ is less than $\mathfrak{K}$ and $I\notin\mathfrak{S}$. Assume that τ is the unique involution in $\mathfrak{S}$. Since $\mathfrak{S}/\mathfrak{K}_1$ is an elementary abelian group of order 2^m and $m\geqq 2$, $\mathfrak{S}$ is a quaternion group (ordinary or generalized) and hence $m=2$ (and $i=4$). Thus we have $\alpha(K)=\cdots=\alpha(K^{2^{l-j-1}})=2<\alpha(K^{2^{l-j}})=4$. Since $\mathfrak{K}\mathfrak{S}$ is a Sylow 2-subgroup of

$N_{\mathfrak{G}}(\mathfrak{K}_1)$, it may be assumed that I is contained in the coset $K^{2^{l-j-1}}\mathfrak{S}$ and hence we have $IK^{2^{l-j-1}}=S$, where S is an element ($\notin K_1$) of $\mathfrak{S}$. Thus $(K^{2^{l-j-1}})^I=S^2K^{-2^{l-j-1}}$. Since $N_{\mathfrak{G}}(\mathfrak{K}_1)=C_{\mathfrak{G}}(\mathfrak{K}_1)$, we have $K^{2^{l-j}}=S^4K^{-2^{l-j}}$ and $S^4=K^{2^{l-j+1}}$. At first assume that $S^4=1$. Then $j=1$ and $(K^{2^{l-2}})^I=K^{-2^{l-2}}\tau=K^{2^{l-2}}$. This implies $d=2$. Hence $n=16$ or 28. Since $n-i$ and $i-\alpha(K)$ are divisible by 2^l and 2^{l-1}, respectively, the order of $\mathfrak{K}$ is equal to four. It can easily be seen that there exists no group satisfying the conditions of Proposition in these cases. Next assume that $S^4\neq 1$ (i.e., $j\neq 1$). Then $(K^{2^{l-j-1}})^I=K^{2^{l-j-1}}$ or $K^{2^{l-j-1}}\tau$ and hence $d=2$. This implies $n=16$ or 28. Since $n-i$ is divisible by 2^l and $j>1$, we have $n=28$, $l=3$ and $j=2$. By [**15**] $\mathfrak{G}$ must be isomorphic to $PSU(3,3^2)$. But a Sylow 2-subgroup of $PSU(3,3^2)$ is isomorphic to $Z_4\sim Z_2$ and it does not contain a quaternion group of order 16. This is a contradiction. Thus the proof is completed.

COROLLARY 4.2. *Let $\mathfrak{G}$, $\mathfrak{S}$ be as in Proposition 4.1. If d is equal to two, then $\mathfrak{S}$ contains an involution τ' such that it is conjugate to τ.*

PROOF. By Proposition, $\mathfrak{S}$ contains an involution $\eta(\neq\tau)$ with the cyclic structure $(1\ a)\cdots$, where a is a symbol of $\mathfrak{J}(\mathfrak{K}_1)$. Then $\eta\tau$ has also the cyclic structure $(1\ a)\cdots$. Hence since $\mathfrak{G}$ is doubly transitive, there exist two involutions with the cyclic structure $(1,b)$, where b is any symbol of Ω, such that those are conjugate to η or $\eta\tau$. If τ is neither conjugate to η nor $\eta\tau$, then $g^*(2)$ is greater than $(n-1)$. This contradicts the inequality $g^*(2)<d(n-1)$.

By the above proposition, since $N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ is doubly transitive, we may assume that I is contained in $\mathfrak{S}$. Since $\mathfrak{V}\mathfrak{S}/\mathfrak{K}_1$ is complete Frobenius group, all elements $(\neq 1)$ of $\mathfrak{S}/\mathfrak{K}_1$ are conjugate under $\mathfrak{V}\mathfrak{K}_1/\mathfrak{K}_1$. Thus every permutation $(\neq\mathfrak{K}_1)$ of $\mathfrak{S}$ can be represented in the form $V^{-1}IVK'$, where V and K' are permutations of $\mathfrak{V}$ and $\mathfrak{K}_1$, respectively.

2. Case $\mathfrak{K}_1=\mathfrak{K}$. In this case $\mathfrak{S}$ is a normal Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K})$. Let S be an element of order 2^l in $\mathfrak{S}$. Since S^2 is contained in $\mathfrak{K}$, $S^{2^{l-1}}$ is equal to τ. Assume that I is conjugate to τ. Since $C_{\mathfrak{G}}(\mathfrak{K})$ and $C_{\mathfrak{G}}(I)$ are conjugate and K is contained in $C_{\mathfrak{G}}(I)$, $K^{2^{l-1}}$ must be equal to I. This is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in this case.

3. Case $\mathfrak{K}\supsetneqq\mathfrak{K}_1\supsetneqq\langle\tau\rangle$. Since $\mathfrak{K}_1$ is greater than $\langle\tau\rangle$, a group $\langle K, I\rangle$ is neither dihedral nor semi-dihedral and therefore d is equal to two. By Corollary 4.2 it may be assumed that I is conjugate to τ.

LEMMA 4.3. *If $\mathfrak{K}_1$ is greater than $\langle\tau\rangle$ and less than $\mathfrak{K}$, then the order of $\mathfrak{K}_1$ is equal to four and I is not contained in $C_{\mathfrak{G}}(\mathfrak{K})$.*

PROOF. At first assume that the order of $\mathfrak{K}_1$ is greater than four. Let $\mathfrak{S}'$ be a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K}_1)$. Let S be an element of $\mathfrak{S}'$ of order

2^{l-1}. The index of $\mathfrak{S}$ in $\mathfrak{S}'$ is equal to 2^{l-j}. Therefore $S^{2^{l-j}}$ is contained in $\mathfrak{S}$ and, since $\mathfrak{S}/\mathfrak{K}_1$ is elementary abelian, $S^{2^{l-j+1}}$ is contained in $\mathfrak{K}_1$. Since j is greater than 2, $S^{2^{l-j+1}}$ is not identity element. Thus we have that $S^{2^{l-2}}$ is equal to τ. Since IKI is equal to K or $K\tau$, I is contained in $C_{\mathfrak{G}}(K^2)$ and hence K^2 is contained in $C_{\mathfrak{G}}(I)$. Since $N_{\mathfrak{G}}(\mathfrak{K}_1)=C_{\mathfrak{G}}(\tau)$ is conjugate to $C_{\mathfrak{G}}(I)$, we have that $(K^2)^{2^{l-2}}=\tau$ must be equal to I. This is a contradiction.

Next assume that I is contained in $C_{\mathfrak{G}}\mathfrak{K}$. Let $\mathfrak{S}'$ be as above. Let S be an element of $\mathfrak{S}'$ of order 2^l. Then $S^{2^{l-j}}$ is contained in $\mathfrak{S}$, $S^{2^{l-j+1}}$ is contained in K_1 and finally $S^{2^{l-1}}$ is equal to τ. Since K is contained in $C_{\mathfrak{G}}(I)$ and $C_{\mathfrak{G}}(I)$ is conjugate to $C_{\mathfrak{G}}(\tau)$, $K^{2^{l-1}}$ must be equal to I. This is a contradiction. Thus the proof is completed.

LEMMA 4.4. *Let $\mathfrak{K}_1$ be as in Lemma 4.3. Then the order of $\mathfrak{K}$ is equal to* 8.

PROOF. Assume that the order of $\mathfrak{K}$ is greater than 8. Then $\langle K^{2^{l-3}}, I\rangle$ is abelian since $d=2$ and $l>3$. Let η be an involution of $N_{\mathfrak{G}}(\langle K^{2^{l-3}}\rangle)$. Then $\langle K^{2^{l-3}}, \eta\rangle$ must be abelian, for if it is not abelian, then $\langle K^{2^{l-3}}, I\rangle$ is dihedral and hence $d\neq 2$.

At first we shall prove that a coset $K^{2^{l-3}}\mathfrak{S}$ does not contain an element of order 4. By Lemma 4.3 the order of $\mathfrak{K}_1$ is equal to 4. Let $K^{2^{l-3}}S$ be an element of order 4 in $K^{2^{l-3}}\mathfrak{S}$, where S is an element of $\mathfrak{S}$. Then S is not contained in $C_{\mathfrak{G}}(K^{2^{l-3}})$. Set $S=I^VK_1$, where K_1 and V are elements of $\mathfrak{K}_1$ and $\mathfrak{V}$, respectively. Then $K^{2^{l-3}}I^V$ must be of order 4. Thus it may be assumed that S is equal to I^V not contained in $C_{\mathfrak{G}}(K^{2^{l-3}})$, where V is an element of $\mathfrak{V}$. $(K^{2^{l-3}}S)^2$ is an element of $\mathfrak{S}$ and therefore is equal to τ, I^W or $I^W\tau$, where W is an element of $\mathfrak{V}$. If $(K^{2^{l-3}}S)^2=\tau$, then $(K^{2^{l-3}})^S=(K^{-2^{l-3}})\tau$ and hence $S\in N_{\mathfrak{G}}(\langle K^{2^{l-3}}\rangle)$. Thus $\langle K^{2^{l-3}}, S\rangle$ must be abelian. This is a contradiction. If $(K^{2^{l-3}}S)^2=I^W$ or $I^W\tau$, then $(K^{2^{l-3}})^S=K^{-2^{l-3}}I^W$ or $K^{-2^{l-3}}I^W\tau$, respectively. Hence

$$K^{2^{l-2}}=(K^{2^{l-2}})^S=(K^{-2^{l-3}}I^W)^2$$

and

$$(K^{-2^{l-3}})^{I^W}=K^{2^{l-2}}K^{2^{l-3}}.$$

Thus I^W is contained in $N_{\mathfrak{G}}(\langle K^{2^{l-3}}\rangle)$ and therefore $\langle I^W, K^{2^{l-3}}\rangle$ must be abelian. Hence $K^{2^{l-2}}K^{2^{l-3}}=K^{-2^{l-3}}$. Thus the order of $\mathfrak{K}$ must be equal to $l-1$. This is a contradiction.

Next let S be an element of order 2^{l-1} in $\mathfrak{K}\mathfrak{S}$, and let $\bar{S}$ be the image of S by the natural homomorphism of $\mathfrak{K}\mathfrak{S}$ onto $\mathfrak{K}\mathfrak{S}/\mathfrak{S}$. If the order of $\bar{S}$ is equal to 2^{l-2}, then $S^{2^{l-3}}$ is contained in a coset $K^{2^{l-3}}S$. This contradicts the first part in the proof. Hence we have that the order of $\bar{S}$ is less than 2^{l-2} and hence $S^{2^{l-3}}$ is contained in S. Therefore $S^{2^{l-2}}$ is equal to τ. Since $C_{\mathfrak{G}}(I)$ is conjugate to $N_{\mathfrak{G}}(\mathfrak{K}_1)$ and K^2 is contained in $C_{\mathfrak{G}}(I)$, $K^{2^{l-1}}=I$. This is a contradiction. Thus the proof is completed.

By two lemmas the orders of $\mathfrak{K}$ and $\mathfrak{K}_1$ are equal to 8 and 4, respectively. Clearly $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ is a complete Frobenius group on $\mathfrak{J}(\mathfrak{K})$. Apply the argument in §2 to $N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ and we obtain that $\alpha(\mathfrak{K})$ must be a power of two and $i=\alpha(\mathfrak{K})^2$. Thus a Frobenius kernel of $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ is a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$. Since, by Lemma 4.3, I is not contained in $C_G(K)$, a Sylow 2-subgroup of $N_G(K)$ is greater than $C_{\mathfrak{G}}(\mathfrak{K})$ ([**13**, 12.6.8]). Since the order of $N_{\mathfrak{G}}(\mathfrak{K})/C_{\mathfrak{G}}(\mathfrak{K})$ is a power of two, $\alpha(K)-1$ must be equal to one and hence $\alpha(K)=2$. Thus we have $i=4$ and $n=16$ or 28. Since $n-i$ must be divisible by the order of $\mathfrak{K}$, we have $n=28$. $\mathfrak{G}$ satisfies the conditions of the theorem in [**15**] and hence $\mathfrak{G}$ is isomorphic to $PSU(3, 3^2)$.

4. Case $\mathfrak{K}_1=\langle\tau\rangle$. We shall prove that $d=2$ or the order of $\mathfrak{K}$ is equal to four, $\langle K, I\rangle$ is dihedral and $i=4$. In this case every permutation $(\notin\mathfrak{K}_1)$ of $\mathfrak{S}$ can be represented uniquely in the form I^V or $I^V\tau$, where V is any permutation of $\mathfrak{V}$. Thus every permutation $(\neq 1)$ of $\mathfrak{S}$ is of order 2 and hence $\mathfrak{S}$ is elementary abelian. Set $\mathfrak{K}_2=\langle K^{2^{l-j'}}\rangle$, where $\alpha(\tau)>\alpha(K^{2^{l-2}})=\cdots=\alpha(K^{2^{l-j'}})>\alpha(K^{2^{l-j'-1}})$. Set $i'=\alpha(K_2)$. Then we may assume $\mathfrak{J}(\mathfrak{K}_2)=\{1, 2, \cdots, i'\}$. Apply the argument in §2 to $N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$, and we have $i=i'(\beta'i'-\beta'+1)$. Hence i' is equal to a power of two, say $2^{m'}$. By the inductive hypothesis $N_{\mathfrak{G}}(\mathfrak{K}_2)/\mathfrak{K}_2$ contains a regular normal subgroup. Let $\mathfrak{S}_2$ be a normal 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K}_2)$ containing $\mathfrak{K}_2$ such that $\mathfrak{S}_2/\mathfrak{K}_2$ is a regular normal subgroup of $N_{\mathfrak{G}}(\mathfrak{K}_2)/\mathfrak{K}_2$ and let $\mathfrak{V}_2$ be a 2-complement of $\mathfrak{H}\cap\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K}_2)$. Then $\mathfrak{V}_2\mathfrak{S}_2/\mathfrak{K}_2$ is a complete Frobenius group on $\mathfrak{J}(\mathfrak{K}_2)$. Thus $C_{\mathfrak{G}}(\mathfrak{K}_2)\cap\mathfrak{V}_2\mathfrak{S}_2$ contains $\mathfrak{S}_2$ or is less than $\mathfrak{S}_2$.

If $C_{\mathfrak{G}}(\mathfrak{K}_2)\cap\mathfrak{V}_2\mathfrak{S}_2$ is less than $\mathfrak{S}_2$, then I is not contained in $C_{\mathfrak{G}}(\mathfrak{K}_2)$ and, since the order of $\mathfrak{V}_2\mathfrak{S}_2/C_{\mathfrak{G}}(\mathfrak{K}_2)\cap\mathfrak{V}_2\mathfrak{S}_2$ is a power of two, m' must be equal to one. Thus $i'=2$ and $\mathfrak{K}_2=\mathfrak{K}$. On the one hand, it is trivial that $i-2$ must be divisible by 2^{l-1}. On the other hand, i is of a form $2(2\beta'-\beta'+1)$ where β' is less than or equal to 2^{l-1} and hence β' is odd. Therefore we have $l=2$, $\beta'=1$ and $i=4$.

If $C_{\mathfrak{G}}(\mathfrak{K}_2)\cap\mathfrak{V}_2\mathfrak{S}_2$ contains $\mathfrak{S}_2$, then $K^I=K$ or K_τ and hence $d=2$.

5. Case $|\mathfrak{K}|=4$, $\mathfrak{K}_1=\langle\tau\rangle$ and $K^I=K^{-1}$. Let $\mathfrak{K}_2$ and $\mathfrak{S}_2$ be as in §4.4. Since $\mathfrak{K}_2=\mathfrak{K}$, $\mathfrak{S}_2/\mathfrak{K}$ is a regular normal subgroup of $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ and $N_{\mathfrak{G}}(\mathfrak{K})=\mathfrak{K}+I\mathfrak{K}$. Since $\langle K, I\rangle$ is dihedral, involutions with the cyclic structure $(12)\cdots$ are I, IK, IK^2 and IK^3, and I and IK are conjugate to IK^2 and IK^3, respectively. Therefore $g^*(2)=0$ or $2(n-1)$.

If $g^*(2)=0$, then $n=4(4\cdot4-3)=4\cdot13$. Let $\mathfrak{P}_{13}$ be a Sylow 13-subgroup of $\mathfrak{G}$. Since every involution leaves four symbols of Ω fixed, the order of $C_{\mathfrak{G}}(\mathfrak{P}_{13})$ is equal to 13. Thus the index of $N_{\mathfrak{G}}(\mathfrak{P}_{13})$ in $\mathfrak{G}$ is a multiple of $17\cdot4$. This contradicts the Sylow's theorem.

If $g^*(2)=2(n-1)$, then $n=4(2\cdot4-1)=4\cdot7$. Let η be an involution leaving no symbol of Ω fixed. Then, since $g^*(2)=2(n-1)$, $G_{\mathfrak{G}}\eta$ must be equal to $2n$. Let $\mathfrak{P}_7$ be a Sylow 7-subgroup of $\mathfrak{G}$ contained in $C_{\mathfrak{G}}\eta$. Using Sylow's theorem $\mathfrak{P}_7$ is normal in $C_{\mathfrak{G}}\eta$. Hence the order of $N_{\mathfrak{G}}(\mathfrak{P}_7)$ is a multiple of $8\cdot 7$. This contradicts the Sylow's theorem.

Thus there exists no group satisfying the conditions of the theorem in this case.

6. Case $\mathfrak{K}_1=\langle\tau\rangle$, $d=2$ and $n=i^2$. In this case a normal subgroup $\mathfrak{S}$ of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ is an elementary abelian 2-group. We shall prove several lemmas.

LEMMA 4.5. *$\mathfrak{S}$ contains every involution of $N_{\mathfrak{G}}(\mathfrak{K}_1)$.*

PROOF. Set $\mathfrak{S}'=\mathfrak{K}\mathfrak{S}$. Then $\mathfrak{S}'$ be a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K}_1)$. If a coset $K^{2l-2}\mathfrak{S}$ does not contain an involution, then the proof is complete. Let $K^{2l-2}S$ be an involution in a coset $K^{2l-2}\mathfrak{S}$, where S is a permutation of $\mathfrak{S}$. Then $(K^{2l-2})^S=K^{-2l-2}$. Therefore, since S is an involution, d must be greater than two. This is a contradiction.

LEMMA 4.6. *Let G be an element of $\mathfrak{G}$. Then $\mathfrak{S}^G\cap\mathfrak{S}=1$ or $\mathfrak{S}$.*

PROOF. Let τ' be an involution of $\mathfrak{S}^G\cap\mathfrak{S}$. If τ' is conjugate to τ, then, since $C_{\mathfrak{G}}(\tau')$ contains $\mathfrak{S}^G$ and $\mathfrak{S}$, $\mathfrak{S}^G$ coincide with $\mathfrak{S}$ by Lemma 4.5. Thus an involution of $\mathfrak{S}$ which is conjugate to τ in $\mathfrak{G}$ is conjugate to τ in $N_{\mathfrak{G}}(\mathfrak{S})$. By Corollary 4.2, I or $I\tau$ is conjugate to τ in G. On the other hand, I or $I\tau$ is not conjugate to τ in $\mathfrak{G}$, since $g^*(2)=n-1$. Hence the number of involutions of $\mathfrak{S}$ each of which is conjugate to τ is equal to i and the number of involutions of $\mathfrak{S}$ each of which leaves no symbol of Ω fixed is equal to $i-1$. Hence the order of $N_{\mathfrak{G}}(\mathfrak{S})$ is equal to $2^l i^2(i-1)$ and the following relation is obtained;

$$n-1=g^*(2)\leqq(i-1)[\mathfrak{G}:N_{\mathfrak{G}}(\mathfrak{S})]=n-1\,.$$

Thus $\mathfrak{S}^G\cap\mathfrak{S}=1$ or $\mathfrak{S}$.

LEMMA 4.7. *Let η and ζ be different involutions. If $\alpha(\eta)=\alpha(\zeta)=0$, then $\alpha(\eta\zeta)=0$.*

PROOF. Let a be a symbol of $\mathfrak{J}(\eta\zeta)$. Let $(a, b)\cdots$ and $(b, c')\cdots$ be the cyclic structure of η and ζ, respectively. Then $a=c'$. Since $g^*(2)=n-1$, there exists just one involution leaving no symbol of Ω fixed with the cyclic structure $(a, b)\cdots$ and hence $\eta=\zeta$.

COROLLARY 4.8. *A set $\mathfrak{S}_1$ consisting of all involutions of $\mathfrak{S}$ each of which is not conjugate to τ and identity element is a characteristic subgroup of $\mathfrak{S}$. In particular $N_{\mathfrak{G}}(\mathfrak{S}_1)=N_{\mathfrak{G}}(\mathfrak{S})$.*

By Corollary 4.8, there exists just $i+1$ subgroups $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_{i+1}$ which are conjugate in $\mathfrak{G}$ and $\mathfrak{S}_s\cap\mathfrak{S}_t=1$ for $s\neq t$.

LEMMA 4.9. *Let τ' be an involution of $N_{\mathfrak{G}}(\mathfrak{S})$. If τ' is conjugate to τ, then τ' is contained in $\mathfrak{S}$.*

PROOF. Set $\tau^G=\tau'$. Since the order of $\mathfrak{S}$ is even, it is trivial that there exists an element ζ of $\mathfrak{S}$ with $\zeta\tau'=\zeta$. $\mathfrak{S}^G$ is normal in $C_{\mathfrak{G}}(\tau')$ and it contains ζ and τ' by Lemma 4.5. Thus $\mathfrak{S}\cap\mathfrak{S}^G$ contains ζ and hence $\mathfrak{S}=\mathfrak{S}^G$ by Lemma 4.6. Finally τ' is an element of $\mathfrak{S}$.

LEMMA 4.10. *Let η be an involution which is not contained in $\mathfrak{S}$. If $\alpha(\eta)=0$, then $\alpha(\tau\eta)=0$ and the order of $\tau(\eta)$ is equal to 2^r with $r>1$.*

PROOF. Assume $\alpha(\tau\eta)\neq 0$. Let a be a symbol of $\mathfrak{J}(\tau\eta)$. It is trivial that a is not a symbol of $\mathfrak{J}(\tau)$. Thus let $(a, b)\cdots$ and $(b, c')\cdots$ be the cyclic structures of τ and η, respectively. Then $a=c'$ and $\tau\eta\tau=(a, b)\cdots$. Since $g^*(2)=n-1$, there exists just one involution with the cyclic structure $(a, b)\cdots$ such that it leaves no symbol of Ω fixed. Thus we have $\tau\eta\tau=\eta$. Therefore η must be contained in $\mathfrak{S}$ and hence $\alpha(\tau\eta)=0$. Next assume that the order of $\tau\eta$ is not equal to 2^r. Let p be an odd prime factor of the order of $\tau\eta$ and let pq be the order of $\tau\eta$. Then the order of $(\tau\eta)^q$ is equal to p and hence $\alpha((\tau\eta)^q)=1$. Therefore $\alpha(\tau\eta)=1$. Thus the order of $\tau\eta$ is equal to a power of two.

LEMMA 4.11. *Let η be an involution which is not conjugate to τ. Then η is contained in $N_{\mathfrak{G}}(\mathfrak{S})$.*

PROOF. Let us assume that η is not contained in $\mathfrak{S}$. By Lemma 4.10, the order of $\tau\eta$ is equal to 2^r with $r>1$. Thus $\tau(\tau\eta)^{2^r}=\tau$. Set $\gamma_{\tau,\eta}(s)=\tau(\tau\eta)^{2^s}$ $=\tau^{\underbrace{\tau^{\cdot^{\cdot^{\cdot^{\tau\eta}}}}}_{s}}$. Then $\gamma_{\tau,\eta}(r-1)$ is contained in $C_{\mathfrak{G}}(\tau)$ and hence by Lemma 4.5, it is contained in $\mathfrak{S}$. Since $\gamma_{\tau,\eta}(r-1)=\tau^{\gamma_{\tau,\eta}(r-2)}$, $\gamma_{\tau,\eta}(r-2)$ is contained in $N_{\mathfrak{G}}(\mathfrak{S})$ by Lemma 4.6. By Lemma 4.9 it is contained in $\mathfrak{S}$. Continuing in the similar way, it can be shown that $\gamma_{\tau,\eta}(1)=\tau^\eta$ is contained in S. By Lemma 4.6, η is contained in $N_{\mathfrak{G}}(\mathfrak{S})$.

By Lemma 4.11, $N_{\mathfrak{G}}(\mathfrak{S})=N_{\mathfrak{G}}(\mathfrak{S}_1)$ contains $\mathfrak{S}_t$ $(2\leqq t\leqq i+1)$. Similarly $N_{\mathfrak{G}}(\mathfrak{S}_t)$ contains $\mathfrak{S}_1$. Therefore $\mathfrak{S}_1\mathfrak{S}_t$ is the direct product $\mathfrak{S}_1\times\mathfrak{S}_t$. In the similar way it can be proved that every element of $\mathfrak{S}_t$ is commutative with any element of $\mathfrak{S}_{t'}$ $(1\leqq t,\ t'\leqq i+1)$. Thus $\mathfrak{N}=\mathfrak{S}_1\cup\cdots\cup\mathfrak{S}_{i+1}$ is a group. Hence $\mathfrak{N}$ is a regular normal subgroup of $\mathfrak{G}$.

Thus there exists no group satisfying the conditions of the theorem in this case.

7. Case $\mathfrak{K}_1=\langle\tau\rangle$, $d=2$ and $n=i(2i-1)$. In this case $g^*(2)=0$. Hence every involution is conjugate to τ. The order of $\mathfrak{G}$ is equal to $2^{l+m}(2^{m+1}-1)(2^{m+1}+1)(2^m-1)$.

Set $\mathfrak{S}'=\mathfrak{K}\mathfrak{S}$. Since $\mathfrak{S}'/\mathfrak{S}$ is a cyclic Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{S})/\mathfrak{S}$, $N_{\mathfrak{G}}(\mathfrak{S})/\mathfrak{S}$ is solvable and hence $N_{\mathfrak{G}}(\mathfrak{S})$ is solvable. We shall prove that the order of $N_{\mathfrak{G}}(\mathfrak{S})$ is equal to $2^{l+m}(2^m-1)(2^{m+1}-1)$. Remark that Lemma 4.5 is also true for this case. Let $\tau'=\tau^G$ be an element of $\mathfrak{S}$, where G is an element of $\mathfrak{G}$. The same argument as in the proof of Lemma 4.6 shows that G is contained in $N_{\mathfrak{G}}(\mathfrak{S})$. Thus every element $(\neq 1)$ of $\mathfrak{S}$ is conjugate to τ under $N_{\mathfrak{G}}(\mathfrak{S})$.

Hence the index of $C_{\mathfrak{G}}(\tau)$ in $N_{\mathfrak{G}}(\mathfrak{S})$ is equal to $2^{m+1}-1$.

Let $\mathfrak{V}$ be a normal 2-complement of $\mathfrak{H}\cap N_{\mathfrak{G}}(\mathfrak{K}_1)$. Since $N_{\mathfrak{G}}(\mathfrak{S})$ is solvable, there exists a Hall subgroup $\mathfrak{A}$ of order $(2^m-1)(2^{m+1}-1)$ of $N_{\mathfrak{G}}(\mathfrak{S})$ containing $\mathfrak{V}$. Since $\mathfrak{S}\mathfrak{V}/\mathfrak{K}_1$ is a complete Frobenius group of degree 2^m, all Sylow subgroups of $\mathfrak{V}$ are cyclic. Let r be the least prime factor of the order of $\mathfrak{V}$. Let $\mathfrak{R}$ be a Sylow r-subgroup of $\mathfrak{V}$. Then $\mathfrak{R}$ is cyclic and leaves only the symbol 1 fixed. Hence $N_{\mathfrak{G}}(\mathfrak{R})$ is contained in $\mathfrak{H}$. Let $\mathfrak{K}'$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\mathfrak{R})$. Since $\mathfrak{K}$ is a Sylow 2-subgroup of $\mathfrak{H}$ and $C_{\mathfrak{G}}(\mathfrak{R})$ is a subgroup of $\mathfrak{H}$, $\mathfrak{K}'$ is conjugate to a subgroup of $\mathfrak{K}$. Thus it may be assumed that $\mathfrak{K}'$ is a subgroup of $\mathfrak{K}$. Using Sylow's theorem, we obtain that $N_{\mathfrak{G}}(\mathfrak{R})=C_{\mathfrak{G}}(\mathfrak{R})(N_{\mathfrak{G}}(\mathfrak{R})\cap N_{\mathfrak{G}}(\mathfrak{K}'))=C_{\mathfrak{G}}(\mathfrak{R})(N_{\mathfrak{G}}(\mathfrak{R})\cap N_{\mathfrak{G}}(\mathfrak{K}_1))=C_{\mathfrak{G}}(\mathfrak{R})(N_{\mathfrak{G}}(\mathfrak{R})\cap\mathfrak{V}\mathfrak{K})$ since $\mathfrak{K}_1$ is a subgroup of $\mathfrak{K}'$. Let CVK' be an element of $N_{\mathfrak{G}}(\mathfrak{R})$ of odd order u, where C, V and K' are elements of $C_{\mathfrak{G}}(\mathfrak{R})$, $\mathfrak{V}$ and $\mathfrak{K}$, respectively. Then $(CVK')^u=C'(VK')^u$, where C' is an element of $C_G(R)$, and $(VK')^u=C'^{-1}$. Set $s=|(VK')^u|/|K'|$, where $|(VK')^u|$ and $|K'|$ are orders of $(VK')^u$ and K', respectively. Then s is an odd integer and $(VK')^{us}$ is contained in a Sylow 2-subgroup of $C_{\mathfrak{G}}(\mathfrak{R})$ and hence so is VK'. In particular CVK' is an element of $C_{\mathfrak{G}}(\mathfrak{R})$. Hence we obtain that $N_{\mathfrak{G}}(\mathfrak{R})\cap\mathfrak{A}=C_{\mathfrak{G}}(\mathfrak{R})(N_{\mathfrak{G}}(\mathfrak{R})\cap\mathfrak{V}\mathfrak{K}')\cap\mathfrak{A}=C_{\mathfrak{G}}(\mathfrak{R})(N_{\mathfrak{G}}(\mathfrak{R})\cap\mathfrak{V})\cap\mathfrak{A}=C_{\mathfrak{G}}(\mathfrak{R})\cap\mathfrak{A}$. By the splitting theorem of Burnside $\mathfrak{A}$ has the normal r-complement. Continuing in the similar way, it can be shown that $\mathfrak{A}$ has the normal subgroup $\mathfrak{B}$ of order $2^{m+1}-1$, which is a complement of $\mathfrak{V}$. Every permutation $(\neq 1)$ of $\mathfrak{B}$ leaves no symbol of Ω fixed and hence it is not commutative with any permutation $(\neq 1)$ of $\mathfrak{V}$. Let B be a permutation of $\mathfrak{B}$ of a prime order, say q. Then all the permutations are conjugate to either B or B^{-1} under $\mathfrak{V}$. This implies that $\mathfrak{B}$ is an elementary abelian q-group of order q^s. Then it follows that $2^{m+1}-1=q^s$. Hence $s=1$ and $\mathfrak{B}$ is cyclic of order q. $\mathfrak{V}$ is also cyclic.

Let the order of $N_{\mathfrak{G}}(\mathfrak{B})$ be equal to $\frac{1}{2}x(q-1)q$. If the order of $C_{\mathfrak{G}}(\mathfrak{B})$ is even, then there exists an involution τ' in $C_{\mathfrak{G}}(\mathfrak{B})$ which is conjugate to τ and such that $C_G(\tau')$ contains $\mathfrak{B}$. But the orders of $C_{\mathfrak{G}}(\tau)$ and $\mathfrak{B}$ are relatively prime. Hence, since $C_{\mathfrak{G}}(\tau')$ is conjugate to $C_{\mathfrak{G}}(\tau)$, the order of $C_{\mathfrak{G}}(\mathfrak{B})$ is odd. Therefore, since the order of the automorphism group of $\mathfrak{B}$ is equal to $q-1=2^{m+1}-2$, the order of $N_{\mathfrak{G}}(\mathfrak{B})$ is not divisible by four.

Using Sylow's theorem we obtain the following congruence;

$$2^{l-1}(q+1)(q+2)/x\equiv 1 \pmod{q}.$$

This implies that $2^{l-1}(q+1)(q+2)=x(yq+1)$, where y is positive since x is less than $2^{l-1}(q+1)(q+2)$. Then we have that $x=zq+2^l$, where $2^{l-1}\geqq z\geqq 0$. It can be proved that z must be equal to 0 or 2^{l-1}. If $z=0$, then the order of $N_{\mathfrak{G}}(\mathfrak{B})$ is equal to $2^l q\frac{1}{2}(q-1)$ and hence, since $l>1$, it is divisible by four. If $z=2^{l-1}$, then the order of $N_{\mathfrak{G}}(\mathfrak{B})$ is equal to $2^{l-1}(q+2)\frac{1}{2}q(q-1)$. Let Y be a permutation ($\neq 1$) of odd prime order dividing $(q+2)\frac{1}{2}(q-1)$ which is contained in $N_{\mathfrak{G}}(\mathfrak{B})$. Since Y leaves just one symbol of Ω fixed, Y is not contained in $C_{\mathfrak{G}}(\mathfrak{B})$. Hence we obtain the following;

$$q-1 \geqq |N_{\mathfrak{G}}(\mathfrak{B})/C_{\mathfrak{G}}(\mathfrak{B})| > \frac{1}{2}(q+2)(q-1).$$

But this is impossible.

Thus there exists no group satisfying the conditions of the theorem in this case.

5. The case n is even and $N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ does not contain a regular normal subgroup.

1. Since $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ is a complete Frobenius group and hence it contains a regular normal subgroup, $\mathfrak{K}_1$ is a proper subgroup of $\mathfrak{K}$.

2. Case $\mathfrak{K}_1=\langle\tau\rangle$ and $2^l\leqq 8$. By inductive hypothesis, if $2^l=4$, then $\mathfrak{G}_1=N_{\mathfrak{G}}(\mathfrak{K}_1)/\mathfrak{K}_1$ is isomorphic to either $PSL(2, 5)$ or $SL^*(2, 8)$ and, if $2^l=8$, then $\mathfrak{G}_1$ is isomorphic either $PGL(2, 5)$ or $PSL(2, 9)$.

At first assume that $d=2$. If $2^l=8$, then $i=6$ or 10. Hence $n-i=\beta i(i-1)$ ($\beta=1$ or 2) is not divisible by 8. But $n-i$ must be divisible by the order of $\mathfrak{K}$. This is a contradiction. If $\mathfrak{G}_1$ is isomorphic to $PSL(2, 5)$, then $i=6$ and, since $n-i$ must be divisible by 4, n is equal to $6(2\cdot 6-1)=6\cdot 11$. Let $\mathfrak{P}_{11}$ be a Sylow 11-subgroup of $\mathfrak{G}$. It is trivial that, since $g^*(2)=0$ and the order of $N_{\mathfrak{G}}(\mathfrak{K}_1)$ is equal to $6\cdot 5\cdot 4$, the order of $C_{\mathfrak{G}}(\mathfrak{P}_{11})$ is odd. Since the order of $C_{\mathfrak{G}}(\mathfrak{P}_{11})$ and $n-1$ are relatively prime, the order of $C_{\mathfrak{G}}(\mathfrak{P}_{11})$ is equal to 11 or 33. The index of $C_{\mathfrak{G}}(\mathfrak{P}_{11})$ in $N_{\mathfrak{G}}(\mathfrak{P}_{11})$ is a factor of 10. Thus this contradicts the Sylow's theorem.

If $\mathfrak{G}_1$ is isomorphic to $SL^*(2, 8)$, then $i=28$. Since every involution of $\mathfrak{G}_1$ leaves just four symbols of $\mathfrak{J}(\mathfrak{K}_1)$, we obtain that $\alpha(I)\neq 0$. Therefore, since every involution of $\mathfrak{G}$ is conjugate to a permutation with the cyclic structure $(12)\cdots$, we have that $g^*(2)=0$ and hence $n=i(2i-1)$. Thus the order of $\mathfrak{H}$ is equal to $4\cdot 3^4\cdot 19$. Since $\mathfrak{K}$ is cyclic, $\mathfrak{H}$ has a normal 2-complement $\mathfrak{Q}$ of order $3^4\cdot 19$. Let $\mathfrak{P}_{19}$ be Sylow 19-subgroup of $\mathfrak{Q}$. By Sylow's theorem $\mathfrak{P}_{19}$ is normal in $\mathfrak{Q}$. $\mathfrak{P}_{19}$ is normal even in $\mathfrak{H}$. Since the order of the automorphism group of $\mathfrak{P}_{19}$ is equal to 18, τ must be contained in $C_{\mathfrak{H}}(\mathfrak{P}_{19})$. This is a contradiction.

Next we shall consider the case $d\neq 2$. If $2^l=4$, then $\langle K, I\rangle$ is dihedral. If $\mathfrak{G}_1$ is isomorphic to $PSL(2, 5)$, then $i=6$ and, since $n-i=i\beta(i-1)$ must be divisible by 4, $\beta=2$ or 4. Therefore $\langle K, I\rangle$ is a Sylow 2-subgroup of $\mathfrak{G}$. By [**4**, Theorem 7.7.3] $C_{\mathfrak{G}}(\tau)$ has a normal 2-complement and hence $C_{\mathfrak{G}}(\tau)$ is solvable.

Thus $\mathfrak{G}_1 = C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ must be solvable and this is a contradiction. If $\mathfrak{G}_1$ is isomorphic to $SL^*(2, 8)$, then, since for every involution η of $SL^*(2, 8)\alpha(\eta) = 4$, $\alpha(\mathfrak{K}) = 4$. Hence the order of $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ is equal to $4\cdot 3$. Since I is not contained in $C_{\mathfrak{G}}(\mathfrak{K})$ and $N_{\mathfrak{G}}(\mathfrak{K})/\mathfrak{K}$ is a complete Frobenius group, $C_{\mathfrak{G}}(\mathfrak{K})$ is contained in a Sylow 2-subgroup. Thus the order of $N_{\mathfrak{G}}(\mathfrak{K})/C_{\mathfrak{G}}(\mathfrak{K})$ is divisible by 3. This is a contradiction.

If $2^l = 8$, then $i = 6$ or 10. Since $n - i = \beta i(i-1)$ must be divisible by 8, β is equal to 4 or 8. If $\langle K, I\rangle$ is dihedral, then $\langle K, I\rangle$ is a Sylow 2-subgroup of $\mathfrak{G}$. Thus $C_{\mathfrak{G}}(\tau)$ is solvable and also $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is solvable. Hence $\langle K, I\rangle$ must be semi-dihedral and $d = 4$. Since $g^*(2) = 0$ and $\mathfrak{G}_1$ is a Zassenhaus group, all involutions are conjugate and a permutation leaving at least three symbols of Ω fixed is an involution. Thus $\mathfrak{G}$ satisfies the conditions in [**12**]. Hence by [**6**] and [**12**] $\mathfrak{G}$ is isomorphic to either $PSU(3, 5^2)$ or one of the groups of Ree type (see [**16**]). Since a Sylow 2-subgroup of a group of Ree type is elementary abelian of order 8, G is isomorphic to $PSU(3, 5^2)$.

3. Case $\mathfrak{K}_1 = \langle\tau\rangle$ and $2^l > 8$. $\mathfrak{G}_1$ is isomorphic to one of the groups $PSU(3, 3^2)$, $PSU(3, 5^2)$, $PGL(2, *)$ and $PSL(2, *)$. Then i is not divisible by 8. Since $n - i = \beta i(i-1)$ is divisible by 2^l, β is divisible by 4. Thus we have that $d > 2$ and hence $\langle K, I\rangle$ is dihedral or semi-dihedral and in particular $\langle K, I\rangle/\langle\tau\rangle$ is dihedral. Therefore $\mathfrak{G}_1$ is isomorphic to either $PGL(2, *)$ or $PSL(2, *)$ and i is divisible by 2 exactly. Thus we have that $\beta = 2^{l-1}$ or 2^l. Thus $\langle K, I\rangle$ is a Sylow 2-subgroup of $\mathfrak{G}$. If $\langle K, I\rangle$ is dihedral, then $C_{\mathfrak{G}}(\tau)$ is solvable and hence $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is solvable. If $\langle K, I\rangle$ is semi-dihedral, then $\beta = 2^{l-1}$ and $g^*(2) = 0$. Again by [**6**] and [**12**], G must be isomorphic to either $PSU(3, 5^2)$ or one of the groups of Ree type. This is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in this case.

4. Case $\mathfrak{K}_1 > \langle\tau\rangle$. Since $\mathfrak{K}_1$ is a proper subgroup of $\mathfrak{K}$, the order of $\mathfrak{K}$ is greater than 4. At first assume that $d = 2$. By inductive hypothesis i is not divisible by 8. Since $n - i = \beta i(i-1)$ is divisible by 2^l, $\beta = 2$, $2^l = 8$ and i is divisible by 4. Thus we obtain that $\mathfrak{G}_1$ is isomorphic to $SL^*(2, 8)$ and $n = 2^2\cdot 7\cdot 5\cdot 11$. If we consider a Sylow 19-subgroup of $\mathfrak{H}$, likewise in 5.2, we can obtain a contradiction.

Next we assume that $d > 2$. Then $\langle K, I\rangle/\mathfrak{K}_1$ is dihedral. Hence $\mathfrak{G}_1$ is isomorphic to either $PGL(2, *)$ or $PSL(2, *)$. Since $n - i$ is divisible by 2^l, we have that $\beta = 2^l$ or 2^{l-1}. Therefore $\langle K, I\rangle$ is a Sylow 2-subgroup of $\mathfrak{G}$. If $\langle K, I\rangle$ is dihedral, then $C_{\mathfrak{G}}(\tau)$ is solvable and hence $C_{\mathfrak{G}}(\tau)/\mathfrak{K}_1$ must be solvable. Thus $\langle K, I\rangle$ is semi-dihedral. Set $\mathfrak{G}_0 = C_G(\tau)/\langle\tau\rangle$ $(= N_{\mathfrak{G}}(\mathfrak{K}_1)/\langle\tau\rangle)$. Then, since $\langle K, I\rangle/\mathfrak{K}_1$ is a Sylow 2-subgroup of $\mathfrak{G}_0$ and a dihedral group. Let $\eta = K^{2l-2}\langle\tau\rangle$ be the involution in the center of $\langle K, I\rangle/\langle\tau\rangle$. It can be easily proved that η is contained in the center of $\mathfrak{G}_0$. Thus, by [4, Theorem 7.7.3], $\mathfrak{G}_0$ has a normal 2-complement and hence $\mathfrak{G}_0$ is solvable. Hence $\mathfrak{G}_1$ must be solvable. This is a contradiction.

Thus there exists no group satisfying the conditions of the theorem in this case.

Thus Theorem is proved.

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