J. Math. Soc. Japan
Vol. 22, No. 2, 1970

Orbits of one-parameter groups II

(Linear group case)

By Morikuni GOTO[1)]

(Received Feb. 17, 1969)

§ 1. Introduction.

Let $\boldsymbol{R}$ denote the field of real numbers. We denote by $\mathcal{K}$ the factor group of the additive group of $\boldsymbol{R}$ modulo the subgroup composed of integers. A compact connected one-dimensional Lie group is called a *circle*. A circle is topologically isomorphic with $\mathcal{K}$. A direct product Lie group of a finite number of circles will be called a *toral group*. By a torus we shall mean the underlying analytic manifold of a toral group.

We can classify one-parameter subgroups of Lie groups topologically into three types: (1) a *closed straight line*, which is topologically isomorphic with the additive group of $\boldsymbol{R}$; (2) a circle; and (3) a non-closed one-parameter subgroup. When a one-parameter subgroup $\mathcal{X}$ is non-closed, the closure $\overline{\mathcal{X}}$ is a toral group of dimension at least two.

We let $M(n, \boldsymbol{R})$ denote the Lie algebra of all n by n matrices with real entries, and $\mathcal{GL}(n, \boldsymbol{R})$ the *general linear group*, the group of all invertible matrices in $M(n, \boldsymbol{R})$. In this paper we shall generalize the foregoing topological classification of one-parameter subgroups of $\mathcal{GL}(n, \boldsymbol{R})$ to the following form:

THEOREM 1. *Let $\mathcal{L}$ be a closed connected subgroup, and let $\mathcal{X}$ be a one-parameter subgroup of $\mathcal{GL}(n, \boldsymbol{R})$. Then an orbit of $\mathcal{X}$ in the left coset space $\mathcal{GL}(n, \boldsymbol{R})/\mathcal{L}$ is either locally compact and homeomorphic with a point, $\boldsymbol{R}$ or $\mathcal{K}$, or there exists an analytic submanifold $\mathcal{M}$ in $\mathcal{GL}(n, \boldsymbol{R})/\mathcal{L}$, which is a torus, such that the orbit can be regarded as an everywhere dense one-parameter subgroup with respect to the toral group structure of $\mathcal{M}$.*

We note here that although a locally compact one-parameter subgroup is closed (and vice versa), a locally compact orbit is not necessarily closed. Also it is to be noted that in general it is impossible to find a toral subgroup $\mathcal{T}$ of $\mathcal{GL}(n, \boldsymbol{R})$ such that an orbit of $\mathcal{T}$ coincides with the torus $\mathcal{M}$ in Theorem 1.

When $\mathcal{L}$ is a (not necessarily connected) algebraic subgroup in $\mathcal{GL}(n, \boldsymbol{R})$,

---

1) Research supported in part by NSF Grant GP 4503.

we can get an analogous theorem, essentially as an easier part of the proof of Theorem 1. Moreover, in this case we can get a connection between the topology of an orbit and the notion of *play*, which was introduced by the author in a previous paper.[2]

Let $\mathcal{G}$ be a group, and let $\mathcal{A}$ and $\mathcal{B}$ be subgroups of $\mathcal{G}$. By the *play*[3] of $\mathcal{A}$ in $\mathcal{B}$, denoted by $\mathcal{P}(\mathcal{A}, \mathcal{B})$, we mean the intersection of all $a\mathcal{B}a^{-1}$ for a in $\mathcal{A}$. By definition, $\mathcal{P}(\mathcal{A}, \mathcal{B})$ is a subgroup, and $\mathcal{A}$ normalizes $\mathcal{P}(\mathcal{A}, \mathcal{B})$. Hence $\mathcal{Q}(\mathcal{A}, \mathcal{B}) = \mathcal{A} \cdot \mathcal{P}(\mathcal{A}, \mathcal{B})$ is a subgroup of $\mathcal{G}$. The group $\mathcal{Q}(\mathcal{A}, \mathcal{B})$ will be called the *extended play* of $\mathcal{A}$ in $\mathcal{B}$.

Now we can state our theorem:

THEOREM 2. *Let $\mathcal{L}$ be an algebraic subgroup, and let $\mathcal{X}$ be a one-parameter subgroup, of $\mathcal{GL}(n, \boldsymbol{R})$. Then the orbit $\mathcal{X}\mathcal{L}/\mathcal{L}$ is locally compact if and only if the extended play $\mathcal{Q} = \mathcal{Q}(\mathcal{X}, \mathcal{L})$ is closed. When the orbit $\mathcal{X}\mathcal{L}/\mathcal{L}$ is not locally compact, we can find a toral subgroup $\mathcal{T}$ of the closure $\bar{\mathcal{Q}}$ of $\mathcal{Q}$ such that $\mathcal{X}\mathcal{L}/\mathcal{L}$ is everywhere dense in $\mathcal{T}\mathcal{L}/\mathcal{L}$.*

An example in § 6 will show that it is impossible to generalize Theorem 2 to the case of non-algebraic $\mathcal{L}$.

This paper is organized into six sections. Because our proofs are based mainly on the "category argument" and on the "orbit theorem" of algebraic groups, we first develop machinery on locally compact groups and on algebraic groups in § 2 and § 3, respectively. We apply the results in § 4 to obtain Theorem 1. The proof of Theorem 2 is given in § 5. Finally in § 6 we give examples of orbits and their closures.

§ 2. Locally compact groups.[4]

Let $\mathcal{M}$ be a locally compact Hausdorff space, and let $\mathcal{S}$ be a subset of $\mathcal{M}$. If $\mathcal{S}$ is open or closed in $\mathcal{M}$, then $\mathcal{S}$ is locally compact (with respect to the relative topology). Conversely if $\mathcal{S}$ is locally compact, then $\mathcal{S}$ is an intersection of a closed set and an open set, i. e. $\mathcal{S}$ is open in the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$.

Now let $\mathcal{G}$ be a locally compact group, and let $\mathcal{L}$ be a subgroup of $\mathcal{G}$. If $\mathcal{L}$ is closed, then of course it is locally compact. Conversely if $\mathcal{L}$ is locally compact, then because the closure $\bar{\mathcal{L}}$ is also a subgroup of $\mathcal{G}$, $\mathcal{L}$ is an open subgroup of $\bar{\mathcal{L}}$, and since an open subgroup of a topological group is closed, we have that $\mathcal{L}$ is closed. Thus, *a subgroup of a locally compact group is locally compact if and only if it is closed.*

---

2) Goto [**3**].

3) In Goto [**3**] the notion of "*play*" was introduced infinitesimally, i. e. the play in [**3**] is the Lie algebra of the play in this paper.

4) Refer Montgomery and Zippin [**4**].

For a pair of subsets $\mathcal{A}$ and $\mathcal{B}$ of a group $\mathcal{G}$, we adopt the notation: $\mathcal{A}\cdot\mathcal{B}=\mathcal{A}\mathcal{B}=\{ab\,;\,a\in\mathcal{A},\,b\in\mathcal{B}\}$.

(2.1) *Let $\mathcal{G}$ be a locally compact group, and let $\mathcal{L}$ be a closed subgroup of $\mathcal{G}$. We denote by $\mathcal{G}/\mathcal{L}$ the left coset space. Let $\mathcal{H}$ be a subset of $\mathcal{G}$. Then the set $\mathcal{H}\mathcal{L}$ is locally compact (or closed) in $\mathcal{G}$ if and only if $\mathcal{H}\mathcal{L}/\mathcal{L}$ is locally compact (or closed) in $\mathcal{G}/\mathcal{L}$.*

PROOF. Because the projection $\mathcal{G}\ni g\mapsto\pi(g)=g\mathcal{L}\in\mathcal{G}/\mathcal{L}$ is continuous and open, the set $\mathcal{H}\mathcal{L}/\mathcal{L}$ is closed or open, according as $\pi^{-1}(\mathcal{H}\mathcal{L}/\mathcal{L})=\mathcal{H}\mathcal{L}$ is closed or open. A necessary and sufficient condition for $\mathcal{H}\mathcal{L}$ to be locally compact is that $\mathcal{H}\mathcal{L}$ is open in the closure $\overline{\mathcal{H}\mathcal{L}}$. On the other hand, the closure of $\mathcal{H}\mathcal{L}/\mathcal{L}$ in $\mathcal{G}/\mathcal{L}$ coincides with $\overline{\mathcal{H}\mathcal{L}}/\mathcal{L}$, and $\mathcal{H}\mathcal{L}/\mathcal{L}$ is open in $\overline{\mathcal{H}\mathcal{L}}/\mathcal{L}$ if and only if $\mathcal{H}\mathcal{L}$ is open in $\overline{\mathcal{H}\mathcal{L}}$. Q.E.D.

(2.2) *Let $\mathcal{G}$ be a locally compact group, and let $\mathcal{A}$ and $\mathcal{B}$ be locally compact groups with countable bases. Let α and β be continuous one-one homomorphisms from $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{G}$, respectively. Suppose that $\alpha(\mathcal{A})\beta(\mathcal{B})$ is locally compact. Then the mapping ρ:*

$$\mathcal{A}\times\mathcal{B}\ni(a,b)\mapsto\rho(a,b)=\alpha(a)^{-1}\beta(b)$$

is (continuous and) *open. More precisely, setting*

$$C=\alpha(\mathcal{A})\cap\beta(\mathcal{B})\qquad\text{and}\qquad\mathcal{D}=\{(\alpha^{-1}(c),\beta^{-1}(c))\,;\,c\in C\}\,,$$

we have a homeomorphism $\tilde{\rho}$, induced by ρ, from the right coset space $\mathcal{D}\backslash(\mathcal{A}\times\mathcal{B})$ onto $\alpha(\mathcal{A})\cdot\beta(\mathcal{B})$.

(2.3) *In* (2.2) *we suppose moreover that C is compact. Then for a closed subset $\mathcal{A}_1$ of $\mathcal{A}$ and a closed subset $\mathcal{B}_1$ of $\mathcal{B}$, the set $\alpha(\mathcal{A}_1)\beta(\mathcal{B}_1)$ is closed in $\alpha(\mathcal{A})\beta(\mathcal{B})$, and so it is locally compact.*

PROOF OF (2.2). Let a and a' be elements of $\mathcal{A}$, and let b and b' be elements of $\mathcal{B}$. If $\rho(a,b)=\rho(a',b')$ then we have that $(a',b')\in\mathcal{D}(a,b)$ and vice versa. Hence $\mathcal{D}$ is a closed subgroup of $\mathcal{A}\times\mathcal{B}$ and ρ induces a continuous one-one mapping $\tilde{\rho}$ from the right coset space $\mathcal{D}\backslash(\mathcal{A}\times\mathcal{B})$ onto $\alpha(\mathcal{A})\beta(\mathcal{B})$.

Let $\mathcal{U}_A$ and $\mathcal{U}_B$ be compact symmetric neighborhoods of the identities of $\mathcal{A}$ and $\mathcal{B}$, respectively. We take compact symmetric neighborhoods $\mathcal{V}_A$ and $\mathcal{V}_B$ of the identities of $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{V}_A^2\subset\mathcal{U}_A$ and $\mathcal{V}_B^2\subset\mathcal{U}_B$. Then $\mathcal{V}_A\times\mathcal{V}_B=\mathcal{V}$ is a compact neighborhood of the identity of $\mathcal{A}\times\mathcal{B}$, and the image $\rho(\mathcal{V})$ is compact. Because $\mathcal{A}\times\mathcal{B}$ has a countable base, we can find an at most countable subset $\{(a_1,b_1),(a_2,b_2),\cdots\}$ of $\mathcal{A}\times\mathcal{B}$ such that $\mathcal{A}\times\mathcal{B}=\bigcup_{k=1}^{\infty}\mathcal{V}(a_k,b_k)$, from which it follows that $\alpha(\mathcal{A})\beta(\mathcal{B})=\bigcup_{k=1}^{\infty}\alpha(a_k)^{-1}\rho(\mathcal{V})\beta(b_k)$. Since $\alpha(\mathcal{A})\beta(\mathcal{B})$ is locally compact we can find a number k such that $\alpha(a_k)^{-1}\rho(\mathcal{V})\beta(a_k)$ contains an interior point.

The direct product group $\mathcal{A}\times\mathcal{B}$ is acting as a transformation group on the space $\mathcal{G}$ by

$$(\mathcal{A}\times\mathcal{B})\times\mathcal{G}\ni((a, b), g)\mapsto\alpha(a)^{-1}g\beta(b)\in\mathcal{G},$$

and $\alpha(\mathcal{A})\beta(\mathcal{B})$ is one of the orbits. Hence the space $\alpha(\mathcal{A})\beta(\mathcal{B})$ is homogeneous with respect to the action of $\mathcal{A}\times\mathcal{B}$, and in particular we see that $\rho(\mathcal{V})$ contains an interior point, say $\rho(a_0, b_0)$. Denoting $\mathcal{U}_A\times\mathcal{U}_B=\mathcal{U}$ we have that $\mathcal{V}(a_0^{-1}, b_0^{-1})\subset\mathcal{U}$. Since $\rho(\mathcal{V}(a_0^{-1}, b_0^{-1}))$ contains the identity as an interior point, $\rho(\mathcal{U})$ is a neighborhood of the identity. This obviously implies that ρ is an open mapping. Q. E. D.

PROOF OF (2.3). Let us first prove that $\mathcal{D}$ is compact. Since $\mathcal{D}$ is a locally compact group with a countable base, and

$$\mathcal{D}\ni(\alpha^{-1}(c), \beta^{-1}(c))\mapsto\beta^{-1}(c)\mapsto c\in C$$

gives a continuous one-one homomorphism from $\mathcal{D}$ onto a (locally) compact group, $\mathcal{D}$ is homeomorphic with C.

Now because $\mathcal{A}_1^{-1}$ and $\mathcal{B}_1$ are closed subsets of $\mathcal{A}$ and $\mathcal{B}$ respectively, we have that $\mathcal{A}_1^{-1}\times\mathcal{B}_1$ is closed in $\mathcal{A}\times\mathcal{B}$. By the compactness of $\mathcal{D}$, $\mathcal{D}(\mathcal{A}_1^{-1}\times\mathcal{B}_1)$ is closed, and from which it follows that $\rho(\mathcal{D}(\mathcal{A}_1^{-1}\times\mathcal{B}_1))=\alpha(\mathcal{A}_1)\beta(\mathcal{B}_1)$ is closed in $\alpha(\mathcal{A})\beta(\mathcal{B})$. Q. E. D.

§ 3. Algebraic groups in $\mathcal{GL}(n, \boldsymbol{R})$.[5]

Throughout this paper, for a topological group $\mathcal{G}$, $\mathcal{G}^0$ will denote the connected component of $\mathcal{G}$ containing the identity; also, *by a linear group we shall mean an analytic subgroup of* $\mathcal{GL}(n, \boldsymbol{R})$ *for a suitable* n.

Let $\mathcal{G}$ be an algebraic group in $\mathcal{GL}(n, \boldsymbol{R})$. Then $\mathcal{G}$ is a closed subgroup, and $\mathcal{G}^0$ is of finite index in $\mathcal{G}$. For a linear group $\mathcal{H}$ we denote by $[\mathcal{H}]$ the *algebraic hull* of $\mathcal{H}$, which is the smallest algebraic group containing $\mathcal{H}$.

A subalgebra of $M(n, \boldsymbol{R})$ is said to be *algebraic* if it is a Lie algebra of an algebraic group. Let us call an element X of $M(n, \boldsymbol{R})$ *algebraic* if the one-dimensional Lie algebra $\boldsymbol{R}X$ is algebraic. For a subalgebra H of $M(n, \boldsymbol{R})$, we denote by $[H]$ the algebraic hull of H, which is the smallest algebraic Lie algebra containing H. If H is the Lie algebra of a linear group $\mathcal{H}$, then $[H]$ is the Lie algebra of $[\mathcal{H}]$.

Let $\mathcal{G}$ be a linear group, and let $\mathcal{N}$ be the normalizer of $\mathcal{G}$. Denoting $Ad(g)A=gAg^{-1}$ for $g\in\mathcal{GL}(n, \boldsymbol{R})$ and $A\in M(n, \boldsymbol{R})$, we have that $\mathcal{N}=\{g\in\mathcal{GL}(n, \boldsymbol{R});\ Ad(g)G=G\}$, and so $\mathcal{N}$ is an algebraic group. This implies in particular

(3.1) *A linear group* $\mathcal{G}$ *is a normal subgroup of its own algebraic hull* $[\mathcal{G}]$.

5) Throughout this section, the reader may refer Chevalley [**1**].

Let X be in $M(n, \boldsymbol{R})$. The algebraic hull $[\boldsymbol{R}X]$ of $\boldsymbol{R}X$ may also be denoted simply by $[X]$. $[X]$ is the set of replicas of X and forms an abelian Lie algebra, and it is possible to find a basis of $[X]$ composed of algebraic elements. From this fact we can obtain the following

(3.2) *Let X be in $M(n, \boldsymbol{R})$, and let B be a (not necessarily algebraic) subalgebra of $[X]$. Then we can find an algebraic Lie algebra A such that*

$$[X] = A \oplus B \qquad \text{(direct sum)}.$$

Now the following theorem is sometimes quoted as the orbit theorem:

(Orbit Theorem). *Let $\mathcal{A}$ and $\mathcal{B}$ be algebraic groups in $\mathcal{GL}(n, \boldsymbol{R})$. Then the product $\mathcal{A}\mathcal{B}$ is Zariski-open in the Zariski closure of $\mathcal{A}\mathcal{B}$. If, in particular, $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$, then $\mathcal{A}\mathcal{B}$ is an algebraic group.*

For our purposes we need only the following direct consequence of the orbit theorem:

(3.3) *Let $\mathcal{A}$ and $\mathcal{B}$ be algebraic groups in $\mathcal{GL}(n, \boldsymbol{R})$. Then the product $\mathcal{A}\mathcal{B}$ is locally compact, and so is $\mathcal{A}^0\mathcal{B}^0$.*

§ 4. Proof of Theorem 1.

Let $\mathcal{L}$ be a closed connected subgroup, and let $\mathcal{X} = \exp \boldsymbol{R}X$ be a one-parameter subgroup, of $\mathcal{GL}(n, \boldsymbol{R})$. Let g be an element of $\mathcal{GL}(n, \boldsymbol{R})$. The orbit of $\mathcal{X}$ passing through the point $g\mathcal{L}$ is $\mathcal{X}g\mathcal{L}/\mathcal{L}$. Since the right translation by g^{-1} maps $\mathcal{X}g\mathcal{L}$ onto $\mathcal{X}(g\mathcal{L}g^{-1})$ and $g\mathcal{L}g^{-1}$ is a closed connected subgroup with $\mathcal{L}$, in order to prove Theorem 1 we may assume that $g = I$, the identity matrix, without loss of generality.

Let L denote the Lie algebra of $\mathcal{L}$. We first exclude the trivial case when $X \in L$. We set $[X] \cap L = R$. Then by (3.2) we can find an algebraic subalgebra S such that $[X] = R \oplus S$. Hence there exist an element X_1 in R and an element X_2 in S with $X = X_1 + X_2$. Since X_1 and X_2 are commutative, we have that

$$\exp \lambda X \cdot \mathcal{L} = \exp \lambda X_2 \cdot \exp \lambda X_1 \cdot \mathcal{L} = \exp \lambda X_2 \cdot \mathcal{L} \qquad \text{for} \quad \lambda \in \boldsymbol{R}.$$

Moreover, that $X_2 \in S$ implies $[X_2] \cap L = \{0\}$. Hence after this we may assume that $[X] \cap L = \{0\}$, without loss of generality.

If X is in $[L]$, then the orbit $\mathcal{X}\mathcal{L}/\mathcal{L}$ is a one-parameter subgroup of the Lie group $[\mathcal{L}]/\mathcal{L}$, by (3.1), and so our theorem is obvious in this case. We set $[X] \cap [L] = B$, and take an algebraic Lie algebra A with $[X] = A \oplus B$, by (3.2). By the foregoing remark, we may assume that $A \neq \{0\}$ after this.

Let us suppose that $B = \{0\}$. By (3.3) the set $[\mathcal{X}][\mathcal{L}]$ is locally compact, and moreover that $B = \{0\}$ implies that $[\mathcal{X}] \cap [\mathcal{L}]$ is a finite group, as a zero-dimensional algebraic group. On the other hand, $\bar{\mathcal{X}}$ is closed in $[\mathcal{X}]$ and $\mathcal{L}$

is closed in $[\mathcal{L}]$. Hence $\bar{\mathcal{X}}\mathcal{L}$ is locally compact by (2.3). Therefore after this let us consider only the case when $B \neq \{0\}$. We decompose X into the form $X=Y+Z$, where $Y \in A$ and $Z \in B$. Obviously, $[Y]=A$ and $[Z]=B$.

Thus we are going to prove Theorem 1 under the following assumptions:

$$\begin{cases} X=Y+Z\,, \quad [X]=[Y]\oplus[Z]\,, \\ [Z]=[X]\cap[L]\,, \quad [Z]\cap L=\{0\}\,, \\ Y\neq 0\,, \quad Z\neq 0\,. \end{cases}$$

We set

$$\mathcal{Y}=\exp \boldsymbol{R}Y\,, \quad \mathcal{Z}=\exp \boldsymbol{R}Z\,, \quad [\mathcal{Y}]=\mathcal{A}\,, \quad [\mathcal{X}]\cap[\mathcal{L}]=\mathcal{B}\,.$$

Since $\mathcal{A}$ and $[\mathcal{L}]$ are algebraic groups, the set $\mathcal{A}[\mathcal{L}]$ is locally compact by (3.3), and that $[Y]\cap[L]=\{0\}$ implies that $\mathcal{A}\cap[\mathcal{L}]$ is a finite group. Hence by (2.3) $\bar{\mathcal{Y}}\cdot\overline{\mathcal{Z}\mathcal{L}}$ is a locally compact set. Now we have two cases depending on the closedness of the linear group $\mathcal{Z}\mathcal{L}$.

Case 1. *The linear group $\mathcal{Z}\mathcal{L}$ is closed.*

We shall first prove that $\mathcal{Z}$ is closed in this case. Because the closure $\bar{\mathcal{Z}}$ is contained in $[\mathcal{Z}]$ and $[Z]\cap L=\{0\}$, it is obvious.

Since $\mathcal{Z}$ is in $[\mathcal{L}]$, $\bar{\mathcal{Y}}$ is in $\mathcal{A}$, and $\mathcal{A}\cap[\mathcal{L}]$ is finite, by using (2.3) again, we see that the abelian group $\bar{\mathcal{Y}}\mathcal{Z}$ is closed. Next let us apply (2.2) to the locally compact groups $\bar{\mathcal{Y}}\mathcal{Z}$ and $\mathcal{L}$. We set

$$\bar{\mathcal{Y}}\mathcal{Z}\cap\mathcal{L}=\mathcal{C}\,.$$

Then $\mathcal{C}$ is a discrete subgroup. Since $\mathcal{X}$ is in $\bar{\mathcal{Y}}\mathcal{Z}$, the closure $\overline{\mathcal{X}\mathcal{L}}$ in $\bar{\mathcal{Y}}\mathcal{Z}\mathcal{L}$ is given by $\overline{\mathcal{C}\mathcal{X}\mathcal{L}}$.

If $\mathcal{C}\mathcal{X}$ is a closed subgroup of $\bar{\mathcal{Y}}\mathcal{Z}$, then $\mathcal{X}\mathcal{L}=\mathcal{C}\mathcal{X}\mathcal{L}$ is locally compact. Otherwise, using the isomorphism theorem (as analytic manifolds):

$$\overline{\mathcal{C}\mathcal{X}\mathcal{L}}/\mathcal{L}\simeq\overline{\mathcal{C}\mathcal{X}}/\mathcal{C}\,,$$

we see that the orbit $\mathcal{X}\mathcal{L}/\mathcal{L}\simeq\mathcal{C}\mathcal{X}/\mathcal{C}$ is an everywhere dense one-parameter subgroup of the toral group $\overline{\mathcal{C}\mathcal{X}}/\mathcal{C}$.

Case 2. *The linear group $\mathcal{Z}\mathcal{L}$ is non-closed.*

We first prove the following Lemma:

LEMMA. *Let $\mathcal{G}$ be a Lie group, and let $\mathcal{N}$ be a non-closed analytic subgroup of $\mathcal{G}$. Suppose that $\mathcal{N}$ contains a normal analytic subgroup $\mathcal{L}$ of codimension one, which is closed in $\mathcal{G}$. Then we can find a non-closed one-parameter subgroup $\mathcal{U}$ of $\mathcal{N}$ with the closure $\mathcal{T}$ such that $\bar{\mathcal{N}}=\mathcal{T}\mathcal{L}$ and $\mathcal{T}\cap\mathcal{L}$ is a finite group.*

PROOF. We can find a one-parameter subgroup $\mathcal{H}$ of $\mathcal{N}$ such that $\bar{\mathcal{N}}=\bar{\mathcal{H}}\mathcal{N}$, see e.g. Goto [**2**]. The fact that $\mathcal{H}\mathcal{L}=\mathcal{N}$ implies that $\bar{\mathcal{N}}=\bar{\mathcal{H}}\mathcal{L}$. $\bar{\mathcal{H}}$ is a toral group and $\bar{\mathcal{H}}\cap\mathcal{L}$ is a closed subgroup of $\bar{\mathcal{H}}$. Hence we can find a toral subgroup $\mathcal{T}$ of $\bar{\mathcal{H}}$ such that $\bar{\mathcal{H}}=\mathcal{T}(\bar{\mathcal{H}}\cap\mathcal{L})^0$ and $\mathcal{T}\cap(\bar{\mathcal{H}}\cap\mathcal{L})^0=\{I\}$, from which it follows that $\mathcal{T}\cap\mathcal{L}$ is finite. Let $\mathcal{U}$ be the one-parameter subgroup contained in $\mathcal{T}\cap\mathcal{N}$. Because $\mathcal{N}=\mathcal{U}\mathcal{L}$ and $\bar{\mathcal{U}}$ is compact, we see that $\bar{\mathcal{U}}=\mathcal{T}$.

Q. E. D.

We apply the Lemma for $\mathcal{N}=\mathcal{Z}\mathcal{L}$, and obtain the one-parameter subgroup $\mathcal{U}=\exp \boldsymbol{R}U$ in $\mathcal{Z}\mathcal{L}$ and the toral subgroup $\bar{\mathcal{U}}=\mathcal{T}$. By taking a suitable $\mathcal{U}$ we may suppose that there exists an analytic curve $l(\lambda)$ in $\mathcal{L}$ such that

$$\exp \lambda Z=\exp \lambda U\cdot l(\lambda) \qquad \text{for} \quad \lambda\in\boldsymbol{R}\,.$$

We set

$$v(\lambda)=\exp \lambda Y\cdot \exp \lambda U \qquad \text{for} \quad \lambda\in\boldsymbol{R}\,.$$

$v(\lambda)$ is an analytic curve with $\exp \lambda X\cdot\mathcal{L}=v(\lambda)\mathcal{L}$. We note here that $v(\lambda)$ is not necessarily a one-parameter subgroup.

As we have seen $\bar{\mathcal{Y}}\mathcal{T}\mathcal{L}=\bar{\mathcal{Y}}\overline{\mathcal{Z}\mathcal{L}}$ is a locally compact set. $\bar{\mathcal{Y}}\mathcal{T}$ is closed because $\mathcal{T}$ is compact. Let us define an analytic mapping ϕ, from the direct product group $\bar{\mathcal{Y}}\times\mathcal{T}$ into $\mathcal{GL}(n,\boldsymbol{R})/\mathcal{L}$, by

$$\bar{\mathcal{Y}}\times\mathcal{T}\ni(y,t)\mapsto\phi(y,t)=yt\mathcal{L}\,.$$

Let $\mathcal{F}$ denote the set of all elements (y,t) in $\bar{\mathcal{Y}}\times\mathcal{T}$ with $\phi(y,t)=\mathcal{L}$. Then $\mathcal{F}$ is a subgroup of $\bar{\mathcal{Y}}\times\mathcal{T}$; because if $y_1t_1=l_1$ and $y_2t_2=l_2$ for $y_i\in\bar{\mathcal{Y}}$, $t_i\in\mathcal{T}$, $l_i\in\mathcal{L}$, $(i=1,2)$, then

$$(y_2y_1^{-1})(t_2t_1^{-1})=t_1t_1^{-1}y_1^{-1}y_2t_2t_1^{-1}=t_1(l_1^{-1}l_2)t_1^{-1}\in\mathcal{L}\,.$$

$\mathcal{F}$ is a closed subgroup, and $\phi(y_1,t_1)=\phi(y_2,t_2)$ holds if and only if $(y_2,t_2)\in\mathcal{F}(y_1,t_1)$. Thus we have an analytic homeomorphism between the abelian Lie group $(\bar{\mathcal{Y}}\times\mathcal{T})/\mathcal{F}$ and the submanifold $\bar{\mathcal{Y}}\mathcal{T}\mathcal{L}/\mathcal{L}$ of $\mathcal{GL}(n,\boldsymbol{R})/\mathcal{L}$.

Next we shall prove that $\mathcal{F}$ is a finite group. We set $\mathcal{F}_1=\bar{\mathcal{Y}}\cap\mathcal{T}\mathcal{L}$ and $\mathcal{F}_2=\mathcal{T}\cap\mathcal{L}$. Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite groups. If (y,t) is in $\mathcal{F}$, then since $yt\in\mathcal{L}$ implies that $y\in\mathcal{T}\mathcal{L}$, we have $y\in\mathcal{F}_1$. On the other hand, if both (y,t_1) and (y,t_2) belong to $\mathcal{F}$, then $t_2^{-1}t_1=(yt_2)^{-1}(yt_1)\in\mathcal{L}\cap\mathcal{T}=\mathcal{F}_2$. Hence $\mathcal{F}$ is a finite group.

Next we set $\tilde{v}(\lambda)=(\exp\lambda Y,\exp\lambda U)$ for $\lambda\in\boldsymbol{R}$. Then $\tilde{v}$ is a one-parameter subgroup of $\bar{\mathcal{Y}}\times\mathcal{T}$.

When $\mathcal{Y}$ is a closed straight line, $\tilde{v}(\boldsymbol{R})$ is clearly closed, and so is $\mathcal{F}\tilde{v}(\boldsymbol{R})$ by the finiteness of $\mathcal{F}$. Hence $\phi(\tilde{v}(\boldsymbol{R}))=v(\boldsymbol{R})\mathcal{L}/\mathcal{L}$ is closed in $\mathcal{Y}\mathcal{T}\mathcal{L}/\mathcal{L}$, i.e. the orbit $\mathcal{X}\mathcal{L}/\mathcal{L}$ is locally compact.

Now we may suppose that $\bar{\mathcal{Y}}$ is a toral group. Let us denote by $\mathcal{W}$ the closure of the one-parameter subgroup $\tilde{v}(\boldsymbol{R})$ in the toral group $\bar{\mathcal{Y}}\times\mathcal{T}$. Then $\phi(\mathcal{W})$ is the closure of $\phi(\tilde{v}(\boldsymbol{R}))=\mathcal{X}\mathcal{L}/\mathcal{L}$ in the coset space $\mathcal{GL}(n, \boldsymbol{R})/\mathcal{L}$. Since $\mathcal{W}\mathcal{T}/\mathcal{T}$ is a toral group with $\mathcal{W}$, $\phi(\mathcal{W})$ is a torus.

§ 5. Proof of Theorem 2.

Let $\mathcal{L}$ be an algebraic subgroup, and let $\mathcal{X}=\exp \boldsymbol{R}X$ be a one-parameter subgroup of $\mathcal{GL}(n, \boldsymbol{R})$. For g in $\mathcal{GL}(n, \boldsymbol{R})$, the group $g\mathcal{L}g^{-1}$ is also algebraic. Hence the play of $\mathcal{X}$ in $\mathcal{L}$

$$\mathcal{P}=\mathcal{P}(\mathcal{X}, \mathcal{L})=\bigcap_{\lambda\in\boldsymbol{R}} \exp \lambda X\cdot\mathcal{L}\cdot\exp(-\lambda X)$$

is algebraic, as an intersection of algebraic groups. In particular, $\mathcal{P}^0$ is of finite index in $\mathcal{P}$. Let Q be the extended play: $Q=\mathcal{X}\mathcal{P}$. Obviously, $Q^0=\mathcal{X}\mathcal{P}^0$ and Q is a closed subgroup if and only if Q^0 is.

When Q is not closed, by the Lemma in § 4, we can find a non-closed one-parameter subgroup $\mathcal{Y}$ of Q^0 such that $\bar{Q}^0=\bar{\mathcal{Y}}\mathcal{P}^0$ and $\bar{\mathcal{Y}}\cap\mathcal{P}^0$ is finite. We set $\bar{\mathcal{Y}}=\mathcal{T}$. Using $Q^0=\mathcal{X}\mathcal{P}^0=\mathcal{Y}\mathcal{P}^0$ we have that $\mathcal{T}\mathcal{L}=\overline{\mathcal{X}\mathcal{L}}$. Hence $\overline{\mathcal{X}\mathcal{L}}/\mathcal{L}\simeq\mathcal{T}/\mathcal{T}\cap\mathcal{L}$ as analytic manifolds. Since $\mathcal{T}\cap\mathcal{L}$ is a finite group as a subgroup of $\mathcal{T}\cap\mathcal{P}$, the toral group $\mathcal{T}/\mathcal{T}\cap\mathcal{L}$ is of dimension at least two, and $\mathcal{X}\mathcal{L}/\mathcal{L}=\mathcal{Y}\mathcal{L}/\mathcal{L}$ corresponds to an everywhere dense one-parameter subgroup in the toral group.

Next, let us suppose that Q is closed. If X normalizes L, then $Q^0=\mathcal{X}\mathcal{L}^0$ is closed, and so is $\mathcal{X}\mathcal{L}$. Hence we may assume, after this, that

$$X=Y+Z, \qquad [Y, Z]=0, \qquad Z\in L,$$
$$[Y]\cap L=\{0\} \qquad \text{and} \qquad Y\neq 0.$$

Since $\exp \boldsymbol{R}Z\subset\mathcal{P}$, the one-parameter subgroup $\mathcal{Y}=\exp \boldsymbol{R}Y$ is contained in Q. Because the closure $\bar{\mathcal{Y}}$ is contained in $[\mathcal{Y}]$, the intersection $\bar{\mathcal{Y}}\cap\mathcal{P}$ is a finite group. Hence for $\bar{\mathcal{Y}}\mathcal{P}$ to be in Q, it is necessary that $\dim\bar{\mathcal{Y}}=1$, i. e. $\mathcal{Y}$ is closed. Applying (2.3) for $[\mathcal{Y}]$ and $\mathcal{L}$, we see that $\mathcal{Y}\mathcal{L}$ is closed in the locally compact set $[\mathcal{Y}]\mathcal{L}$.

§ 6. Examples.

EXAMPLE 1. We choose real numbers α, β and γ such that the system $\{1, \alpha, \beta, \gamma\}$ is linearly independent over the rationals. We set

$$e=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad i=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad 0=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

and define matrices L and X in $M(8, \boldsymbol{R})$ by

$$L=\begin{pmatrix} e-i & 0 & 0 & 0 \\ 0 & e-\alpha i & 0 & 0 \\ 0 & 0 & e-i & 0 \\ 0 & 0 & 0 & e-\beta i \end{pmatrix}$$

and

$$X=\begin{pmatrix} e & -\gamma e & 0 & 0 \\ \gamma e & e & 0 & 0 \\ 0 & 0 & e & -\gamma e \\ 0 & 0 & \gamma e & e \end{pmatrix}.$$

For a real number λ,

$$\exp \lambda L=\exp \lambda \cdot \begin{pmatrix} r(-\lambda) & 0 & 0 & 0 \\ 0 & r(-\alpha\lambda) & 0 & 0 \\ 0 & 0 & r(-\lambda) & 0 \\ 0 & 0 & 0 & r(-\beta\lambda) \end{pmatrix}$$

where

$$r(\mu)=\exp \mu i=\begin{pmatrix} \cos \mu & -\sin \mu \\ \sin \mu & \cos \mu \end{pmatrix},$$

and

$$\exp \lambda X=\exp \lambda \cdot \begin{pmatrix} \cos \gamma\lambda \cdot e & -\sin \gamma\lambda \cdot e & 0 & 0 \\ \sin \gamma\lambda \cdot e & \cos \gamma\lambda \cdot e & 0 & 0 \\ 0 & 0 & \cos \gamma\lambda \cdot e & -\sin \gamma\lambda \cdot e \\ 0 & 0 & \sin \gamma\lambda \cdot e & \cos \gamma\lambda \cdot e \end{pmatrix}.$$

Both $\mathcal{L}=\exp \boldsymbol{R}L$ and $\mathcal{X}=\exp \boldsymbol{R}X$ are closed straight lines.

The Lie algebra $[L]$ is of dimension four and is given by $[L]=\boldsymbol{R}I\oplus T$, where T is the Lie algebra of a toral group $\mathcal{T}$ and T has a basis:

$$H_1=\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Next, setting $Y=X-I$ we have that $[X]=\boldsymbol{R}I+\boldsymbol{R}Y$ and $[X]\cap[L]=\boldsymbol{R}I$. The one-parameter subgroup $\mathcal{Y}=\exp \boldsymbol{R}Y$ is a circle. We set $Y=\gamma H_0$ and

$$v(\lambda)=\exp (\lambda\gamma H_0)\cdot \exp (\lambda H_1+\lambda\alpha H_2+\lambda\beta H_3), \qquad \lambda \in \boldsymbol{R}.$$

$v(\lambda)$ is a curve in $\mathcal{Y}\mathcal{T}$ and $v(\lambda)\mathcal{L}=\exp \lambda X\cdot \mathcal{L}$.

Since $\mathcal{Y}\cap\mathcal{T}=\{\pm I\}$, the set $\mathcal{M}=\mathcal{Y}\mathcal{T}$ can be identified with the toral group $(\mathcal{Y}\times\mathcal{T})/\{\pm(I, I)\}$ and $\mathcal{M}$ is a torus. The curve $v(\lambda)$ is obviously everywhere dense in $\mathcal{M}$. Because $\det(m)=1$ for $m \in \mathcal{M}$ and $\det(\exp \lambda L)=\exp(8\lambda)$, we see that the mapping $\mathcal{M}\times\mathcal{L} \ni (m, l)\mapsto ml \in \mathcal{M}\mathcal{L}$ is one-one. Thus, we have proved that the closure of the orbit $\mathcal{X}\mathcal{L}/\mathcal{L}$ can be identified with the four-dimensional torus $\mathcal{M}$.

Next, let us prove the following two propositions concerning our example:

(i) $\mathcal{P}(\mathcal{X}, \mathcal{L})=\{I\}$, *and so the extended play* $\mathcal{X}$ *is closed.*

(ii) *For any toral subgroup* $\mathcal{H}$ *of* $\mathcal{GL}(8, \boldsymbol{R})$, $\mathcal{HL}$ *cannot contain* $\mathcal{XL}$.

PROOF OF (i). Computing $[X, L]$ we see that $\{X, L\}$ cannot be a basis of two-dimensional Lie algebra. Hence $\mathcal{P}=\mathcal{P}(\mathcal{X}, \mathcal{L})$ is discrete. Since $\mathcal{X}$ is connected and $\mathcal{X}$ normalizes $\mathcal{P}$, $\mathcal{X}$ must centralize $\mathcal{P}$. On the other hand, the equality

$$\left[\begin{pmatrix} r(-\lambda) & 0 \\ 0 & r(-\alpha\lambda)\end{pmatrix}, \begin{pmatrix} 0 & -\gamma e \\ \gamma e & 0\end{pmatrix}\right]=0$$

implies that $r(\lambda)=r(\alpha\lambda)$, i.e. $\alpha\lambda-\lambda=2m\pi$ for some integer m. If we have moreover that $\beta\lambda-\lambda=2n\pi$ for some integer n, then λ must vanish by the linear independence of $\{1, \alpha, \beta\}$. This proves that $\mathcal{P}=\{I\}$.

PROOF OF (ii). Suppose that $\mathcal{HL}\supset\mathcal{XL}$. Since $\mathcal{HL}$ is closed, we have that $\mathcal{HL}\supset\overline{\mathcal{XL}}=\mathcal{ML}$. Because a toral subgroup of $\mathcal{GL}(8, \boldsymbol{R})$ is of dimension at most four, and $\dim(\mathcal{ML})=5$, we have that $\dim\mathcal{H}=4$. From the equality $\dim(\mathcal{HL})+\dim(\mathcal{H}\cap\mathcal{TL})=\dim\mathcal{H}+\dim(\mathcal{TL})$, we have $\dim(\mathcal{H}\cap\mathcal{TL})=3$. On the other hand, $\mathcal{T}$ is the largest compact subgroup of $\mathcal{TL}$. Hence we have $\mathcal{H}\cap\mathcal{TL}=\mathcal{T}$, whence $\mathcal{H}\supset\mathcal{T}$. Since $\mathcal{TL}=\exp\boldsymbol{R}I\cdot\mathcal{T}$, $\mathcal{H}\supset\mathcal{T}$ implies that $\mathcal{HL}$ is an abelian group, which contradicts the fact that $[X, L]\neq 0$.

EXAMPLE 2. Let $\mathcal{L}$ be a closed connected subgroup or an algebraic subgroup, and let $\mathcal{X}$ be a one-parameter subgroup, of $\mathcal{GL}(n, \boldsymbol{R})$. Suppose the orbit $\mathcal{XL}/\mathcal{L}$ is a locally compact straight line. Let $\mathcal{B}$ be the boundary of the orbit, i.e. $\mathcal{B}$ is the complement of $\mathcal{XL}/\mathcal{L}$ in $\overline{\mathcal{XL}}/\mathcal{L}$. We can find examples of $\mathcal{X}$ and $\mathcal{L}$ such that $\mathcal{B}$ is empty, one end-point, or two end-points. Also $\mathcal{B}$ can be a single point in the closure of the orbit which is a circle. In these cases, *the boundary points are all fixed points of the one-parameter group* $\mathcal{X}$. When $\mathcal{L}$ is algebraic and $\mathcal{X}=\exp\boldsymbol{R}X$, with X algebraic, by the orbit theorem we can prove this is true.

However the following example shows that it is not true in general.

For an irrational number α we consider the orbit of the one-parameter group $\exp\boldsymbol{R}X$:

$$X=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

passing through the point $(1, 1, 1, 1, 1)$ in $\boldsymbol{R}^5$. Then the boundary of the orbit is a torus of two dimension, although the isotropy group is algebraic.

University of Pennsylvania

Bibliography

[1] C. Chevalley, Théorie des groupes de Lie, t. 2, Hermann et Cie., Paris, 1951.
[2] M. Goto, Faithful representations of Lie groups I, Math. Japon., 1 (1948), 107–119.
[3] M. Goto, Orbits of one-parameter groups I, J. Math. Soc. Japan, 22 (1970), 113–122.
[4] D. Montgomery and L. Zippin, Topological transformation groups, Interscience, New York, 1955.