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A theorem on the second-order arithmetic with the ω-rule*)

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It is well known that the first order arithmetic with the ω-rule is complete. Moreover, J. R. Shoenfield [3] has shown that the same holds also when the ω-rule is recursively restricted. On the other hand, the second-order arithmetic with the ω-rule is not complete. So Shoenfield has raised a question whether every sentence of the second-order arithmetic provable with the ω-rule is provable with the recursively restricted ω-rule. Concerning this problem, H. Tanaka [4] has shown that every sentence of the second-order arithmetic provable with the ω-rule is provable with the hyper-arithmetically restricted ω-rule.

The purpose of this paper is to give an affirmative answer to Shoenfield's problem stated above. This result can be extended to one corresponding to any higher order arithmetic. We shall use notations and terminologies in [2].

Roughly speaking, the outline of proof is as follows. For a given formula φ, we define a tree T (consisting of formulas) whose only root is φ and which has the following properties: (i) whenever φ is a provable formula, the tree T is well-founded (in a suitable sense), and (ii) in T if $\psi_0, \psi_1, \cdots$ are direct predecessors of ψ then we can effectively construct a "recursive proof" of ψ from information for recursive proofs of ψ_i's. Then by means of Kleene's recursion theorem, there exists a partial recursive function π such that if $\psi, \psi_0, \psi_1, \cdots$ are as above-mentioned and if $\pi(\psi_i)$ is a recursive proof of ψ_i for each i, then $\pi(\psi)$ is a recursive proof of ψ. Thus, if φ is provable, then $\pi(\varphi)$ gives a recursive proof of φ, as is shown by the induction using the well-foundedness of T.

In what follows, we shall carry out a detailed proof based on this idea.

§1. As a formal system of second order arithmetic, we shall use A_ω in [1]. By a familiar way we can assign, to each formula φ of A_ω, a number

*) After the completion of this manuscript the author learned that Lopez-Escobar [5] proved the theorem for the case of the weak second-order logic (not the full system of second-order arithmetic) by a similar way as ours.

$\ulcorner\varphi\urcorner$ called the Gödel number (abbreviated by G. n.) of φ, satisfying the following conditions:

1.1.1. If φ and ψ are distinct, then $\ulcorner\varphi\urcorner \neq \ulcorner\psi\urcorner$;

1.1.2. The number-theoretic predicates, $F(a)$, $AX(a)$, $C(a, b, c)$, $UF_0(a)$, $UF_1(a)$ and $FO(a, b)$ defined below are recursive;

1.1.2.1. $F(a) \equiv \{a$ is the G. n. of a formula$\}$,

1.1.2.2. $AX(a) \equiv \{a$ is the G. n. of an axiom of $A_\omega\}$,

1.1.2.3. $C(a, b, c) \equiv \{a = \ulcorner\varphi\urcorner,\ b = \ulcorner\psi\urcorner,\ c = \ulcorner\chi\urcorner$ and φ is the consequence of ψ and χ by modus ponens$\}$,

1.1.2.4. $UF_0(a) \equiv \{a$ is the G. n. of a formula of the form $(x)\varphi$, where x is a number variable$\}$;

1.1.2.5. $UF_1(a) \equiv \{a$ is the G. n. of a formula of the form $(\alpha^k)\varphi$ where α^k is a function variable$\}$;

1.1.2.6. $FO(a, b) \equiv \{b$ is the G. n. of a function variable and it does not occur in the formula whose G. n. is $a\}$;

1.1.3. There exist recursive functions $g_0(a, b)$, $g_1(a, b)$ and $v(a, b)$ such that

1.1.3.1. whenever $a = \ulcorner(x)\varphi(x)\urcorner$, $g_0(a, n) = \ulcorner\varphi(\bar{n})\urcorner$, where $\bar{n}$ is the numeral for a natural number n;

1.1.3.2. whenever $a = \ulcorner(\alpha^k)\varphi(\alpha^k)\urcorner$ and $b = \ulcorner\beta^k\urcorner$, $g_1(a, b) = \ulcorner\varphi(\beta^k)\urcorner$;

1.1.3.3. whenever $a = \ulcorner\varphi\urcorner$ and $b = \ulcorner\psi\urcorner$, $v(a, b) = \ulcorner\varphi \vee \psi\urcorner$.

Now we define predicates $Pr(a)$, $P^*(p)$ and $Pr^*(a)$ as follows:

1.2.1. If $AX(a)$, then $Pr(a)$;

1.2.2. If $Pr(b)$, $Pr(c)$ and $C(a, b, c)$, then $Pr(a)$;

1.2.3. If $UF_0(a)$ and $Pr(g_0(a, n))$ for all n, then $Pr(a)$;

1.2.4. If $UF_1(a)$, $FO(a, b)$ and $Pr(g_1(a, b))$, then $Pr(a)$;

1.2.5. $Pr(a)$, only as required by 1.2.1–4.

1.3.1. If $AX(a)$, then $P^*(3^a)$;

1.3.2. $P^*(p)$, $P^*(q)$ and $C(a, (p)_1, (q)_1)$, then $P^*(2 \cdot 3^a 5^p 7^q)$;

1.3.3. If $UF_0(a)$ and, for any n, $P^*(\{e\}(n))$ and $(\{e\}(n))_1 = g_0(a, n)$, then $P^*(2^2 \cdot 3^a \cdot 5^e)$;

1.3.4. If $UF_1(a)$, $FO(a, b)$, $P^*(p)$ and $(p)_1 = g_1(a, b)$, then $P^*(2^3 3^a 5^b 7^p)$;

1.3.5. $P^*(p)$, only as required by 1.3.1–4.

1.4. $Pr^*(a) \equiv \exists p\ [P^*(p) \wedge (p)_1 = a]$.

If $P^*(p)$ and $(p)_1 = \ulcorner\varphi\urcorner$, we say that p is a *recursive proof* of φ and φ is *recursively provable*.

It is clear that

1.5. $Pr(a)$ if and only if a is the G. n. of a formula provable in A_ω; and that

1.6. If $Pr^*(a)$, then $Pr(a)$.

The converse of 1.6 will give the affirmative answer to Shoenfield's problem, that is:

1.7. THEOREM. *If $Pr(a)$, then $Pr^*(a)$.*

Before proceeding to the proof of 1.7 we remark the following.

Let X be a set, R be a binary relation over X and A be a subset of X such that aRb and $b \in A$ imply $a \in A$. We say *R well-founds X relative to A* if for each sequence $\{x_n\}_{n=0}^{\infty}$ of elements of X such that $x_{n+1}Rx_n$ for $n \geqq 0$, there exists an n such that $x_n \in A$.

1.8. LEMMA. *Suppose that R well-founds X relative to A. Then the following induction principle holds:*

1.8.1. *If $A \subseteqq Q \subseteqq X$ and $\forall x \in X[\forall y \in X[yRx \Rightarrow y \in Q] \Rightarrow x \in Q]$, then $Q = X$.*

PROOF. Suppose $X-Q$ is not empty, and $x_0 \in X-Q$. By assumption there exists an x_1 such that x_1Rx_0 and $x_1 \in X-Q$. Continuing this procedure, we get a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n+1}Rx_n$ and $x_n \in X-Q$ for all n. But, since R well-founds X relative to A, there must exist an $x_n \in A \subseteqq Q$. This is a contradiction. Hence $Q = X$.

1.8 is a generalization of the so-called bar-induction, and is well known when $A = \phi$.

§ 2. Proof of 1.7.

2.1. The definition of $P(a, \mu)$, where μ ranges over ordinals.

2.1.1. If $AX(a)$, then $P(a, 0)$.

2.1.2. If $P(b, \mu)$, $P(c, \nu)$ and $C(a, b, c)$, then $P(a, \max(\mu, \nu)+1)$.

2.1.3. If $UF_0(a)$ and $P(g_0(a, n), \mu_n)$ for all n, then

$$P(a, \sup\{\mu_n+1 \mid n = 0, 1, \cdots\}).$$

2.1.4. If $UF_1(a)$, $FO(a, b)$ and $P(g_1(a, b), \mu)$, then $P(a, \mu+1)$.

2.1.5. $P(a, \mu)$, only as required by 2.1.1-4.

2.2. LEMMA. $Pr(a) \Leftrightarrow \exists\mu P(a, \mu)$.

PROOF. This is obvious from the definitions of Pr and P.

2.3. LEMMA. *Let $\varphi(\alpha^k)$ be a formula and let β^k be a variable not occurring in $\varphi(\alpha^k)$. Then*

$$P(\ulcorner\varphi(\alpha^k)\urcorner, \mu) \Leftrightarrow P(\ulcorner\varphi(\beta^k)\urcorner, \mu).$$

PROOF. By the induction on μ.

2.4. We define a recursive function $\sigma(a, n, m)$ as follows:

2.4.1. $\sigma(a, 0, m) = 2^{a+1}$;

2.4.2 If $(n)_0 = 0$ and $C((\sigma(a, n, (m)_0))_{(n)_1} \dot{-} 1, (n)_2, (n)_3)$, then

$$\sigma(a, n+1, m) = \begin{cases} \sigma(a, n, (m)_0) \cdot p_{n+1}^{(n)_2+1}, & \text{when } (m)_1 \text{ is even}, \\ \sigma(a, n, (m)_0) \cdot p_{n+1}^{(n)_3+1}, & \text{when } (m)_1 \text{ is odd}; \end{cases}$$

2.4.3. If $(n)_0 = 1$ and $UF_0((\sigma(a, n, (m)_0))_{(n)_1} \dot{-} 1)$, then

$$\sigma(a, n+1, m) = \sigma(a, n, (m)_0) \times p_{n+1} \exp[g_0((\sigma(a, n, (m)_0))_{(n)_1} \dot{-} 1, (m)_1)+1];$$

2.4.4. If $(n)_0 = 2$ and $UF_1((\sigma(a, n, (m)_0))_{(n)_1} \dot{-} 1)$, then

$$\sigma(a, n+1, m) = \sigma(a, n, (m)_0) \times p_{n+1} \exp[g_1((\sigma(a, n, (m)_0))_{(n)_1} \dot{-} 1, s)+1],$$

where $s = \mu x[\forall i \leqq n\, FO((\sigma(a, n, (m)_0))_i \dot{-} 1, x)]$;

2.4.5. Otherwise,

$$\sigma(a, n+1, m) = \sigma(a, n, (m)_0) \cdot p_{n+1} \exp[(\sigma(a, n, (m)_0))_{(n)_1}].$$

Then σ has the following properties:

2.5.1. $\mathrm{Seq}(\sigma(a, n, m)) \wedge lh(\sigma(a, n, m)) = n+1$,

where $\mathrm{Seq}(a)$ means that a is a sequence number i.e., $a \neq 0 \wedge \forall i < lh(a) [(a)_i \neq 0]$.

2.5.2. $F(a) \Rightarrow \forall i \leqq n\ F((\sigma(a, n, m))_i \dot{-} 1)$.

2.5.3. $\forall i \leqq n\ [(\sigma(a, n, (m)_0))_i = (\sigma(a, n+1, m))_i]$.

Let $w(a, n)$ be defined by

2.6.1. $w(a, 0) = (a)_0 \dot{-} 1$, and

2.6.2. $w(a, n+1) = v(w(a, n), (a)_{n+1} \dot{-} 1)$;

and we set

2.6.3. $\tau(a, n, m) = w(\sigma(a, n, m), n)$.

Then we have

2.7.1. $\tau(a, 0, m) = a$,

2.7.2. $F(a) \Rightarrow F(\tau(a, n, m))$,

2.7.3. $F(a) \wedge (n)_0 = 0 \wedge Pr(\tau(a, n+1, 2^m)) \wedge Pr(\tau(a, n+1, 2^m \cdot 3)) \Rightarrow Pr(\tau(a, n, m))$,

2.7.4. $F(a) \wedge (n)_0 = 1 \wedge \forall k\, Pr(\tau(a, n+1, 2^m \cdot 3^k)) \Rightarrow Pr(\tau(a, n, m))$,

2.7.5. $F(a) \wedge (n)_0 = 2 \wedge Pr(\tau(a, n+1, 2^m)) \Rightarrow Pr(\tau(a, n, m))$.

More generally, we have that

2.7.6. $F(a) \wedge \forall l[(l)_0 = m \Rightarrow Pr(\tau(a, n+1, l))] \Rightarrow Pr(\tau(a, n, m))$.

2.8. Let X be the set of all pairs (n, m) of natural numbers and let $(n_1, m_1)R(n_2, m_2)$ hold if and only if $n_1 = n_2+1$ and $(m_1)_0 = m_2$. Moreover let

$$A_a = \{(n, m) \mid \exists i \leqq n\, AX((\sigma(a, n, m))_i \dot{-} 1)\}.$$

2.9. LEMMA. *If $Pr(a)$, then R well-founds X relative to A_a.*

PROOF. Let

$$K(b) \overset{\text{def}}{\Leftrightarrow} \forall n \forall i \forall \alpha [\forall k[(\alpha(k+1))_0 = \alpha(k)] \wedge i \leqq n \wedge (\sigma(a, n, \alpha(n)))_i \dot{-} 1 = b \Rightarrow \exists l[l \geqq n \wedge (l, \alpha(l)) \in A_a]],$$

where α ranges over all number-theoretic functions of one variable. We shall prove that $P(b, \mu) \Rightarrow K(b)$ by the induction on μ. So assume the following:

2.9.1. $\forall c[P(c, \nu) \Rightarrow K(c)]$, for all $\nu < \mu$,

2.9.2. $P(b, \mu)$,

2.9.3. $\forall k[(\alpha(k+1))_0 = \alpha(k)] \wedge i \leqq n \wedge (\sigma(a, n, \alpha(n)))_i \dot{-} 1 = b$.

We must prove that $\exists l[l \geqq n \wedge (l, \alpha(l)) \in A_a]$.

Case 1. $AX(b) \wedge \mu = 0$. Then we may take n as l.

Case 2. $P(c, \nu_1) \wedge P(d, \nu_2) \wedge C(b, c, d) \wedge \mu = \max(\nu_1, \nu_2)+1$.

Then, by 2.9.1, we have $K(c) \wedge k(d)$. Let $t = 3^i 5^c 7^d 11^n$. Then $(t)_0 = 0$ and $t \geqq n$. Successive uses of 2.5.3 with 2.9.3 show that

$$(\sigma(a, t, \alpha(t)))_i \dot{-} 1 = (\sigma(a, n, \alpha(n)))_i \dot{-} 1 = b\,.$$

Hence by definition of σ, $\sigma(a, t+1, \alpha(t+1))$ is $\sigma(a, t, \alpha(t)) \cdot p_{t+1}^{c+1}$ or $\sigma(a, t, \alpha(t)) \cdot p_{t+1}^{d+1}$ according as $(\alpha(t+1))_1$ is even or odd. In either case, using $K(c)$ or $K(d)$ (taking $t+1$ as n and i), we have that

$$\exists l[l \geqq t+1 \wedge (l, \alpha(l)) \in A_a]\,.$$

Hence

$$\exists l[l \geqq n \wedge (l, \alpha(l)) \in A_a]\,.$$

Case 3. $UF_0(b)$ and $P(g_0(b, n), \mu_n)$ for all n and

$$\mu = \sup\{\mu_n+1 \mid n = 0, 1, \cdots\}\,.$$

Then, by the induction hypothesis, we have $\forall j\, K(g_0(b, j))$. Now let $t = 2\cdot 3^i 5^n$. So, $(t)_0 = 1$ and $t \geqq n$. Since

$$(\sigma(a, t, \alpha(t)))_i \dot{-} 1 = (\sigma(a, n, \alpha(n)))_i \dot{-} 1 = b$$

as before, we have that

$$\sigma(a, t+1, \alpha(t+1)) = \sigma(a, t, \alpha(t)) \\ \times p_{t+1} \exp[g_0(b, (\alpha(t+1))_1)+1]\,.$$

Hence, by using $K(g_0(b, (\alpha(t+1))_1))$, we have

$$\exists l[l \geqq t+1 \wedge (l, \alpha(l)) \in A_a]\,.$$

Hence

$$\exists l[l \geqq n \wedge (l, \alpha(l)) \in A_a]\,.$$

Case 4. $UF_1(b) \wedge FO(b, c) \wedge P(g_1(b, c), \nu) \wedge \mu = \nu+1$.

Let $t = 2^2 3^i 5^n$. Then $(t)_0 = 2$ and $t \geqq n$. Since

$$(\sigma(a, t, \alpha(t)))_i \dot{-} 1 = (\sigma(a, n, \alpha(n)))_i \dot{-} 1 = b\,,$$

$$\sigma(a, t+1, \alpha(t+1)) = \sigma(a, t, \alpha(t)) \times p_{t+1} \exp(g_1(b, s)+1)\,,$$

where $s = \mu x[\forall j \leqq t FO((\sigma(a, t, \alpha(t)))_j \dot{-} 1, x)]$.

By 2.3 and the assumption that $P(g_1(b, c), \nu)$, we have $P(g_1(b, s), \nu)$.

Hence $K(g_1(b, s))$, since $\nu < \mu$. So we have that

$$\exists l[l \geqq t+1 \wedge (l, \alpha(l)) \in A_a].$$

Hence

$$\exists l[l \geqq n \wedge (l, \alpha(l)) \in A_a].$$

Thus we have completed the proof of $\forall \mu \forall b[P(b, \mu) \Rightarrow K(b)]$. So by 2.2, if $Pr(a)$, then $K(a)$. By setting $n=i=0$, we have that

$$\forall \alpha[\forall k[(\alpha(k+1))_0 = \alpha(k)] \Rightarrow \exists l[(l, \alpha(l)) \in A_a]].$$

This proves 2.9.

Now it is clear that there exist recursive functions $\xi(a, p, q)$, $\eta(a, l)$, $\gamma(a, p)$, $\zeta(a, p)$ and $\theta(a)$ such that

2.10.1. $F(a) \wedge F(b) \wedge F(c) \wedge F(d) \wedge P^*(p) \wedge P^*(q) \wedge (p)_1 = v(b, c) \wedge (q)_1 = v(b, d) \wedge C(a, c, d) \Rightarrow P^*(\xi(a, p, q)) \wedge (\xi(a, p, q))_1 = v(b, a)$;

2.10.2. $UF_0(a) \wedge F(b) \wedge \forall j[P^*(\{e\}(j)) \wedge (\{e\}(j))_1 = v(b, g_0(a, j))] \Rightarrow P^*(\eta(a, e)) \wedge (\eta(a, e))_1 = v(b, a)$;

2.10.3. $UF_1(a) \wedge F(b) \wedge FO(a, c) \wedge FO(b, c) \wedge P^*(p) \wedge (p)_1 = v(b, g_1(a, c)) \Rightarrow P^*(\gamma(a, p)) \wedge (\gamma(a, p))_1 = v(b, a)$;

2.10.4. $\mathrm{Seq}\,(a) \wedge 2 \leqq lh(a) \wedge \forall i < lh(a) F((a)_i \dot{-} 1) \wedge \exists i < lh(a) \dot{-} 1[(a)_i = (a)_{lh(a) \dot{-} 1}] \wedge P^*(p) \wedge (p)_1 = w(a, lh(a) \dot{-} 1) \Rightarrow P^*(\zeta(a, p)) \wedge (\zeta(a, p))_1 = w(a, lh(a) \dot{-} 2)$;

2.10.5. $\mathrm{Seq}\,(a) \wedge \forall i < lh(a) F((a)_i \dot{-} 1) \wedge \exists i < lh(a) AX((a)_i \dot{-} 1) \Rightarrow P^*(\theta(a)) \wedge (\theta(a))_1 = w(a, lh(a) \dot{-} 1)$.

2.11. LEMMA. *There exists a partial recursive function* $\rho(e, a, n, m)$ *such that*

$$F(a) \wedge \forall l[(l)_0 = m \Rightarrow P^*(\{e\}(l)) \wedge (\{e\}(l))_1 = \tau(a, n+1, l)]$$
$$\Rightarrow \{\rho(e, a, n, m) \textit{ is defined}\} \wedge P^*(\rho(e, a, n, m)) \wedge (\rho(e, a, n, m))_1$$
$$= \tau(a, n, m).$$

PROOF. First, define $\rho_0(e, a, n, m)$ as follows:

2.11.1.1. If $(n)_0 = 0$ and $C((\sigma(a, n, m))_{(n)_1} \dot{-} 1, (n)_2, (n)_3)$, then

$$\rho_0(e, a, n, m) \simeq \xi((\sigma(a, n, m))_{(n)_1} \dot{-} 1, \{e\}(2^m), \{e\}(2^m \cdot 3));$$

2.11.1.2. If $(n)_0 = 1$ and $UF_0((\sigma(a, n, m))_{(n)_1} \dot{-} 1)$, then

$$\rho_0(e, a, n, m) \simeq \eta((\sigma(a, n, m))_{(n)_1} \dot{-} 1, S_1^2(h, e, m)),$$

where h is the Gödel number of $\lambda emj\ \{e\}(2^m 3^j)$;

2.11.1.3. If $(n)_0 = 2$ and $UF_1((\sigma(a, n, m))_{(n)_1} \dot{-} 1)$, then

$$\rho_0(e, a, n, m) \simeq \gamma((\sigma(a, n, m))_{(n)_1} \dot{-} 1, \{e\}(2^m))$$

2.11.1.4. Otherwise,

$$\rho_0(e, a, n, m) \simeq \{e\}(2^m).$$

Then ρ_0 is partial recursive and has the following property:

2.11.2. $F(a) \wedge \forall l[(l)_0 = m \Rightarrow P^*(\{e\}(l)) \wedge (\{e\}(l))_1 = \tau(a, n+1, l)] \Rightarrow P^*(\rho_0(e, a, n, m)) \wedge (\rho_0(e, a, n, m))_1 = w(\sigma(a, n, m) \times p_{n+1} \exp[(\sigma(a, n, m))_{(n)_1}], n+1)$.

To prove 2.11.2, assume that

$$F(a) \wedge \forall l[(l)_0 = m \Rightarrow P^*(\{e\}(l)) \wedge (\{e\}(l))_1 = \tau(a, n+1, l)].$$

Case 1. $(n)_0 = 0$ and $C((\sigma(a, n, m))_{(n)_1} \dot{-} 1, (n)_2, (n)_3)$.

By assumption, $P^*(\{e\}(2^m))$ and

$$\begin{aligned}(\{e\}(2^m))_1 &= \tau(a, n+1, 2^m)\\ &= w(\sigma(a, n+1, 2^m), n+1)\\ &= v(w(\sigma(a, n+1, 2^m), n), (\sigma(a, n+1, 2^m))_{n+1} \dot{-} 1)\\ &= v(w(\sigma(a, n, m), n), (n)_2)\\ &= v(\tau(a, n, m), (n)_2).\end{aligned}$$

Similarly, $P^*(\{e\}(2^m \cdot 3))$ and

$$(\{e\}(2^m \cdot 3))_1 = v(\tau(a, n, m), (n)_3).$$

Hence, by 2.10.1, $P^*(\rho_0(e, a, n, m))$ and

$$\begin{aligned}(\rho_0(e, a, n, m))_1 &= v(\tau(a, n, m), (\sigma(a, n, m))_{(n)_1} \dot{-} 1)\\ &= w(\sigma(a, n, m) \cdot p_{n+1} \exp[(\sigma(a, n, m))_{(n)_1}], n+1).\end{aligned}$$

Case 2. $(n)_0 = 1$ and $UF_0((\sigma(a, n, m))_{(n)_1} \dot{-} 1)$.

By assumption, for any j, $P^*(\{e\}(2^m 3^j))$ and

$$\begin{aligned}(\{e\}(2^m 3^j))_1 &= \tau(a, n+1, 2^m 3^j)\\ &= v(w(\sigma(a, n+1, 2^m 3^j), n), (\sigma(a, n+1, 2^m 3^j))_{n+1} \dot{-} 1)\\ &= v(\tau(a, n, m), g_0((\sigma(a, n, m))_{(n)_1} \dot{-} 1, j)).\end{aligned}$$

Since $\{e\}(2^m 3^j) = \{S_1^2(h, e, m)\}(j)$, by 2.10.2, $P^*(\rho_0(e, a, n, m))$ and

$$\begin{aligned}(\rho_0(e, a, n, m))_1 &= v(\tau(a, n, m), (\sigma(a, n, m))_{(n)_1} \dot{-} 1)\\ &= w(\sigma(a, n, m) \cdot p_{n+1} \exp[(\sigma(a, n, m))_{(n)_1}], n+1).\end{aligned}$$

Case 3. $(n)_0 = 2$ and $UF_1((\sigma(a, n, m))_{(n)_1} \dot{-} 1)$.

By assumption, $P^*(\{e\}(2^m))$ and, as before,

$$(\{e\}(2^m))_1 = v(\tau(a, n, m), g_1((\sigma(a, n, m))_{(n)_1} \dot{-} 1, s)),$$

where $s = \mu x[\forall i \leqq n\, FO((\sigma(a, n, m))_i \dot{-} 1, x)]$.

By the choice of s, we have $FO(\tau(a, n, m), s)$ and

$$FO((\sigma(a, n, m))_{(n)_1} \dot{-} 1, s).$$

Hence $P^*(\rho_0(e, a, n, m))$ and

$$(\rho_0(e, a, n, m))_1 = v(\tau(a, n, m), (\sigma(a, n, m))_{(n)_1} \dot{-} 1)$$
$$= w(\sigma(a, n, m) \cdot p_{n+1} \exp [(\sigma(a, n, m))_{(n)_1}], n+1).$$

Case 4. Otherwise.

By assumption, $P^*(\{e\}(2^m))$ and

$$(\{e\}(2^m))_1 = v(w(\sigma(a, n+1, 2^m), n), (\sigma(a, n+1, 2^m))_{n+1} \dot{-} 1)$$
$$= v(w(\sigma(a, n, m), n), (\sigma(a, n, m))_{(n)_1} \dot{-} 1)$$
$$= w(\sigma(a, n, m) \cdot p_{n+1} \exp [(\sigma(a, n, m))_{(n)_1}], n+1).$$

Hence we have 2.11.2.

Now let $\rho(e, a, n, m) \simeq \zeta(\sigma(a, n, m) \cdot p_{n+1} \exp [(\sigma(a, n, m))_{(n)_1}], \rho_0(e, a, n, m))$.

Then ρ has the desired property in view of 2.10.4 and of the fact that $n \geqq (n)_1$, as well as 2.11.2. This completes the proof of 2.11.

2.12. Consider the following recursion equation:

$$\{f\}(a, n, m) \simeq \begin{cases} \theta(\sigma(a, n, m)), \text{ if } (n, m) \in A_a; \\ \rho(S_1^2(f, a, n+1), a, n, m), \text{ if } (n, m) \notin A_a. \end{cases}$$

By the recursion theorem of Kleene, let f_0 be a solution of this equation for f and let

$$\pi(a, n, m) \simeq \{f_0\}(a, n, m).$$

Then π is partial recursive.

2.13. LEMMA. *If $Pr(a)$, then, for any n and m, $\pi(a, n, m)$ is defined and $P^*(\pi(a, n, m)) \wedge (\pi(a, n, m))_1 = \tau(a, n, m)$.*

PROOF. Suppose $Pr(a)$ and let

$$Q_a = \{(n, m) \mid \pi(a, n, m) \text{ is defined} \wedge P^*(\pi(a, n, m)) \wedge (\pi(a, n, m))_1 = \tau(a, n, m)\}.$$

Then $F(a)$ and $A_a \subseteqq Q_a \subseteqq X$, by the property of θ described above.

Next suppose $(n+1, l) \in Q_a$ for all l such that $(l)_0 = m$, and suppose $(n, m) \notin A_a$. Then $\pi(a, n+1, l)$ is defined and $P^*(\pi(a, n+1, l)) \wedge (\pi(a, n+1, l))_1 = \tau(a, n+1, l)$, for all l with $(l)_0 = m$. Since f_0 is the Gödel number of π, $e = S_1^2(f_0, a, n+1)$ is a Gödel number of $\lambda l\, \pi(a, n+1, l)$.

Hence

$$F(a) \wedge \forall l [(l)_0 = m \Rightarrow P^*(\{e\}(l)) \wedge (\{e\}(l))_1 = \tau(a, n+1, l)].$$

Therefore, by 2.11, $P^*(\rho(e, a, n, m)) \wedge (\rho(e, a, n, m))_1 = \tau(a, n, m)$.

So, by definition, $\pi(a, n, m)$ is defined and $(\pi(a, n, m))_1 = \tau(a, n, m)$ (since $(n, m) \notin A_a$). Therefore $(n, m) \in Q_a$.

Because of well-foundedness of X by R relative to A_a and of 1.8, we now conclude that $Q_a = X$. This proves 2.13.

2.14. Let $\pi(a)=\pi(a, 0, 0)$. By 2.13 and 2.7.1, we have that

2.14.1. $\forall a[Pr(a)\Rightarrow P^*(\pi(a)) \wedge (\pi(a))_1 = a]$.

From this it follows that

$$\forall a[Pr(a)\Rightarrow Pr^*(a)].$$

The proof of our main theorem 1.7 is now complete.

Moreover 2.14.1 shows that we can uniformly construct a recursive proof of each provable formula.

Appendix

Complete list of axioms and rules of inference of the system A_ω.

1. $\varphi \supset (\psi \supset \varphi)$,
2. $(\varphi \supset \psi) \supset ((\varphi \supset (\psi \supset \chi)) \supset (\varphi \supset \chi))$,
3. $\varphi \supset (\psi \supset \varphi \& \psi)$,
4. $\varphi \& \psi \supset \varphi$,
5. $\varphi \& \psi \supset \psi$,
6. $(\varphi \supset \chi) \supset ((\psi \supset \chi) \supset (\varphi \vee \psi \supset \chi))$,
7. $\varphi \supset \varphi \vee \psi$,
8. $\psi \supset \varphi \vee \psi$,
9. $(\varphi \supset \psi) \supset ((\varphi \supset \neg\psi) \supset \neg\psi)$,
10. $\neg\neg\varphi \supset \varphi$,
11. $(\varphi \supset \psi) \supset ((\psi \supset \varphi) \supset (\varphi \equiv \psi))$,
12. $(\varphi \equiv \psi) \supset (\varphi \supset \psi)$,
13. $(\varphi \equiv \psi) \supset (\psi \supset \varphi)$,
14. $(x)\varphi(x) \supset \varphi(\pi)$,
    where π is a term which is free for x in $\varphi(x)$,
15. $((x)(\varphi \supset \psi(x))) \supset (\varphi \supset (y)\psi(y))$,
    where x does not occur free in φ and y does not occur in $\psi(x)$,
16. $(Ex)\varphi(x) \equiv \neg(x)\neg\varphi(x)$,
17. $(\alpha^k)\varphi(\alpha^k) \supset \varphi(\beta^k)$,
18. $(\alpha^k)(\varphi \supset \psi(\alpha^k)) \supset (\varphi \supset (\beta^k)\psi(\beta^k))$,
    where α^k does not occur free in φ and β^k does not occur in $\psi(\alpha^k)$,
19. $(E\alpha^k)\varphi(\alpha^k) \equiv \neg(\alpha^k)\neg\varphi(\alpha^k)$,
20. $\pi=\pi$,
21. $\pi=\rho \supset (\psi(\pi) \equiv \psi(\rho))$,
22. $((E!\,x)\varphi(x) \wedge \varphi((\iota x)\varphi(x))) \vee (\neg(E!\,x)\varphi(x) \wedge (\iota x)\varphi(x)=0)$,
23. $(E\alpha^k)(x_1)\cdots(x_k)(\alpha^k(x_1, \cdots, x_k)=\pi)$,
    where α^k does not occur free in π,
24. $\neg(x+1=0)$,
25. $x+0=x$,

26. $x+(y+1)=(x+y)+1$,
27. $x\times 0=0$,
28. $x\times(y+1)=(x\times y)+x$,
29. $x+1=y+1\supset x=y$,
30. $\alpha^1(0)=0\ \&\ (x)(\alpha^1(x)=\alpha^1(x+1))\supset(x)(\alpha^1(x)=0)$,
31. $\dfrac{\varphi,\ \varphi\supset\psi}{\psi}$,
32. $\dfrac{\varphi(y)}{(x)\varphi(x)}$,
33. $\dfrac{\varphi(\beta^k)}{(\alpha^k)\varphi(\alpha^k)}$,
34. $\dfrac{\varphi(0),\ \varphi(1),\ \varphi(1+1),\ \cdots}{(x)\varphi(x)}$.

Clearly 30 and 32 are redundant on account of 34.

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