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Periods of rational forms on certain elliptic surfaces

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Introduction

In this paper we study some properties of periods of rational forms on a regular elliptic surface $B(\tau, \sigma)$ of basic type and some related problems. (For the theory of elliptic surfaces compare K. Kodaira [8] and A. Kas [**6**]. We mainly follow their notations.) It is known that such surfaces are suitable completions of the affine surfaces defined by

$$(1) \qquad y^2 = 4x^3 - (\tau_{4k}u^{4k} + \cdots + \tau_1 u + \tau_0)x - (u^{6k} + \sigma_{6k-1}u^{6k-1} + \cdots + \sigma_1 u + \sigma_0),$$

where we denote by (u, x, y) the variables of the three dimensional affine space $\boldsymbol{C}^3$ and by $(\tau) = (\tau_{4k}, \cdots, \tau_0)$, $(\sigma) = (\sigma_{6k-1}, \cdots, \sigma_0)$ the parameters. We understand by *a period function* $W(\tau, \sigma)$ of a rational form of the surface $B(\tau, \sigma)$ the holomorphic function of the parameters (τ, σ) of the form: $W(\tau, \sigma) = \int_{\Gamma(\tau,\sigma)} \omega(\tau, \sigma)$, where the rational form $W(\tau, \sigma)$ on $B(\tau, \sigma)$ and the homology class $\Gamma(\tau, \sigma)$ on $B(\tau, \sigma)$ satisfy conditions of continuity with respect to the parameters (τ, σ). Our chief object is to study certain properties of the period functions $W(\tau, \sigma)$.

We sketch our results briefly. After recalling some basic notions and definitions of elliptic surfaces needed below, we study in the first chapter the periods of rational 2-forms on the elliptic surface B_0: $y^2 = 4x^3 - (u^{6k} - 1)$ in details. These periods may be regarded as generalizations of Beta integrals: $\int_0^1 u^{p-1}(u-1)^{q-1}du$ in the theory of hypergeometric functions (F. Klein [**7**]). The study of these periods leads to power series expansions of the period functions $W(\tau, \sigma)$ at the point $(\tilde{\tau}^0, \tilde{\sigma}^0) = ((0, \cdots, 0), (0, \cdots, 0, 1))$. We derive some applications from these power series expansions: the determination of the rank of 'period maps' of individual rational forms, etc.

We also know that these period functions are complete solutions of certain partial linear differential equations $\tilde{D}$ of the second order. In the second chapter we study the structure of $\tilde{D}$ and some related problems. In §4~§5, we find a criterion of 'regularity' for systems of linear partial differential equations. This elementary criterions means merely that the solutions are 'regular' if and only if certain ordinary differential equations, which correspond to the partial equations in question, are regular in the usual sense ([c. f.] Lemma 5.1).

We investigate what an information about the behavior of the equations $\tilde{D}$ at the singular loci is obtained from the regularity condition above. The subjects of these sections are independent of the other sections and are inspired by a work of P. A. Griffiths concerning the behavior of periods at the singular loci. (For the condition of regularity see also Géraud [2]). Next, we investigate the coefficients of the system of linear partial differential equations $\tilde{D}$. We find that these coefficients are expressed in terms of certain matrices $\Omega=(\omega_{ij})$ composed of the periods $\omega_{ij}=\int_{\gamma_j}\omega_i$ of rational 2-forms ω_i along the 2-cycles γ_j on B. Instead of studying the matrices Ω directly, we examine the matrices $X=\Omega^t I^{-1}\,{}^t\Omega$ where I denotes the intersection matrices of γ_j. We obtain a formula which expresses the entries of the matrices X in terms of the data of rational forms ω_i at the singular points of the singular locus of the surfaces B. The process may be regarded as a generalization of well-known Legendre's relation in the theory of algebraic function of one variable. ([c. f.] concluding remarks.)

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§ 0. Preliminaries.

First, we recall briefly some basic notions and definitions of elliptic surfaces (for details, see K. Kodaira [8] and A. Kas [6]). A compact complex analytic surface S is called an elliptic surface when there exists a holomorphic mapping $\Psi: S\to R$ of S onto a compact Riemann surface R such that, for any general point $u\in R$, $C_u=\Psi^{-1}(u)$ is a non-singular elliptic curve. We call S an elliptic surface over R. A fibre C_u is called a singular fibre when C_u is not an elliptic curve. An elliptic surface S is called a basic elliptic surface if it contains a holomorphic section over the base curve R. We use the symbol B to denote a basic elliptic surface. It is known that a basic elliptic surface B is always an algebraic surface (K. Kodaira [8]).

In what follows we restrict our consideration to basic elliptic surfaces over a projective line $\boldsymbol{P}_1(\boldsymbol{C})$. Such a basic elliptic surface B has the following normal form (A. Kas [6]): Let $\boldsymbol{P}_2^{(i)}(\boldsymbol{C})$ $(i=1, 2)$ be projective planes with homogeneous coordinates $(x_0^{(i)}, x_1^{(i)}, x_2^{(i)})$ $(i=1, 2)$ and define an affine coordinate of $\boldsymbol{P}_2^{(1)}(\boldsymbol{C})$ by $x=x_1^{(1)}/x_0^{(1)}$, $y=x_2^{(1)}/x_0^{(1)}$. We define a three dimensional complex manifold $W^{(i)}$ by $W^{(i)}=\boldsymbol{P}_2^{(i)}\times\boldsymbol{C}^{(i)}$ $(i=1, 2)$ and form a manifold $W_k=W^{(1)}\cup W^{(2)}$ $(k=1, 2, \cdots)$ by means of the transformation law

$$(0.1)\qquad uv=1\,,\qquad x_0^{(1)}=x_0^{(2)}\,,\qquad x_1^{(2)}u^{2k}=x_1^{(1)}\,,\qquad x_2^{(2)}u^{3k}=x_1^{(2)}\,,$$

where u, v are affine coordinates of the affine lines $C^{(i)}$ $(i=1, 2)$. Let $g_{4k}(u)=\tau_{4k}u^{4k}+\tau_{4k-1}u^{4k-1}+\cdots+\tau_1u+\tau_0$ and $h_{6k}(u)=u^{6k}+\sigma_{6k-1}u^{6k-1}+\cdots+\sigma_1u+\sigma_0$ be two polynomials of respective degrees $4k$ and $6k$ in the variable u. Then the basic elliptic surface B is defined by

$$(0.2)\qquad \begin{aligned} &(x_2^{(1)})^2\cdot(x_0^{(1)})=4(x_1^{(1)})^3-g_{4k}(u)\cdot(x_1^{(1)})\cdot(x_0^{(1)})^2-h_{6k}(u)(x_0^{(1)})^3 \quad \text{in}\quad W^{(1)},\\ &(x_2^{(2)})^2\cdot(x_0^{(2)})=4(x_1^{(2)})^3-g_{4k}(1/v)\cdot v^{4k}\cdot(x_1^{(2)})\cdot(x_0^{(2)})^2\\ &\qquad -h_{6k}(1/v)\cdot v^{6k}\cdot(x_0^{(2)})^3 \quad \text{in}\quad W^{(2)}. \end{aligned}$$

We denote the elliptic surface B by B^k in order to indicate the degree k of the transformation law (0.1).

Define polynomials $D^k(u)$ and $\tilde{D}^k(v)$, respectively, by $D^k(u)=g_{4k}^3(u)-27\cdot h_{6k}^2(u)$ and $\tilde{D}^k(v)=v^{12k}\cdot D^k(1/v)$. If the polynomials $D^k(u)$ and $\tilde{D}^k(v)$ have no multiple roots, then the surface B^k is non-singular. In what follows we assume that the polynomials $D^k(u)$, $\tilde{D}^k(v)$ have no multiple roots. Define a holomorphic mapping $\Psi: B^k\rightarrow \boldsymbol{P}_1(\boldsymbol{C})$ by

$$(0.3)\qquad \begin{aligned} &\Psi(u,(x^{(1)}))=u, \quad \text{in}\quad W^{(1)},\\ &\Psi(v,(x^{(2)}))=v, \quad \text{in}\quad W^{(2)}. \end{aligned}$$

Now we define an algebraic curve Δ^k in the surface B^k by

$$(0.4)\qquad x_0^{(i)}=0, \quad \text{in}\quad W^{(i)}\ (i=1,2).$$

We can easily verify the following facts:

$(0.5)_1$ $\Psi^{-1}(u)$ is an elliptic curve if and only if $D^k(u)\neq 0$,

$(0.5)_2$ Δ^k is the image of a cross section over the projective line $\boldsymbol{P}_1(\boldsymbol{C})$.

We define an algebraic curve X^k by $x_2^{(i)}=0$ $(i=1,2)$ and define a rational 2-form $\omega_k^{(p,q,r)}$ whose polar locus is the curve X^k by $\omega_k^{(p,q,r)}=(u^px^q/y^r)dx\wedge du$ in terms of the affine coordinate (u,x,y), where p, q, r are non-negative integers, $r\equiv 1 \pmod 2$, and $2(q+1)k+p+2\leqq 3(2r+1)k$. We know that the geometric genus $p_g(B^k)$ of the surface B^k is equal to $k-1$ (Kodaira [8]) and we can easily verify that the holomorphic 2-forms $\omega_k^{(p,0,1)}$ $(p=0,1,\cdots,k-2)$ constitute a base of holomorphic 2-forms on the surface B^k.

In the sequel, to make explicit the dependence of the surface B^k and the curves Δ^k, X^k, $\cdots$ on the parameters $(\tau)=(\tau_{4k},\cdots,\tau_0)$, $(\sigma)=(\sigma_{6k-1},\cdots,\sigma_0)$, we write $B^k(\tau,\sigma)$, $\Delta^k(\tau,\sigma)$, $X^k(\tau,\sigma)$, $\cdots$ for B^k, Δ^k, X^k, $\cdots$.

Chapter I. Power series expansions.

In this chapter, we shall examine the values of the periods of rational 2-forms on certain elliptic surfaces of Fermat type. These values may be regarded as generalizations of Beta integrals: $B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}du$ in the theory of hypergeometric function of one variable (F. Klein [7] p. 7). Moreover we derive some applications from our results.

§1. Periods on an algebraic curve of Fermat type.

Let $C_{(m,n)}$ be the algebraic curve of Fermat type defined by

$$y^n = x^m - 1\,. \tag{1.1}$$

In what follows we assume that $(n, m)=1$ or $m\equiv 0 \pmod n$. Let $G_m=\{\rho, \rho^2, \cdots, \rho^m=1\}$ and $G_n=\{\sigma, \sigma^2, \cdots, \sigma^n=1\}$ be two cyclic groups of order m and n respectively, and define an isomorphism $i(\rho^k, \sigma^l)$ from the product $G_m\times G_n$ into the group of biregular isomorphisms of the algebraic curve $C_{(m,n)}$ by

$$i(\rho^k, \sigma^l)(x, y) = (\zeta_m^k x, \zeta_n^l y)\,, \tag{1.2}$$

$$\left(\text{where } \zeta_m = \exp\frac{2\pi\sqrt{-1}}{m},\ \zeta_n = \exp\frac{2\pi\sqrt{-1}}{n}\right).$$

Concerning the structure of the one dimensional homology group $H_1(C_{(m,n)}, \boldsymbol{Z})$, we know the following (cf. N. Sasakura [**12**])

LEMMA 1.1

(1) $\dim H_1(C_{(m,n)}, \boldsymbol{Z}) = (n-1)(m_0-2)$,

(2) *We can choose a Betti base of* 1*-cycles:* $\Gamma_{i,j}=\Gamma_{i,j}(m, n)$ $(i=1, \cdots, n-1, j=1, \cdots, m_0-2)$ *such that*

$$\{i_{(m,n)}(\rho^k, \sigma^l)\}_*(\Gamma_{1,1}(m, n)) = \Gamma_{l+1,k+1}(m, n)\begin{pmatrix} l=1, \cdots, n-2 \\ k=1, \cdots, m_0-2\end{pmatrix}^{1)}, \tag{1.3}$$

where we mean by m_0 *the number of the ramification points of the curve* $C_{m,n}$ *considered as a ramified covering of the projective line* $\boldsymbol{P}_1(\boldsymbol{C})$.

Thus $m_0=m$ or $m+1$ according as $m\equiv 0 \pmod n$ or $(m, n)=1$. The 1-cycles $\Gamma_{i,j}(m, n)$ meet no ramification points of $C_{m,n}$. The symbol $\{i(\rho^k, \sigma^l)\}_*$ means the operation of the automorphisms $i(\rho^k, \sigma^l)$ on the 1-cycles on the curve $C_{m,n}$. Define rational 1-forms $\omega_{m,n}^{(p,v,l)}$ on the curve $C_{m,n}$ by $\omega_{m,n}^{(p,v,l)}=x^p y^{v-ln}dx$ (where p, v, l are integers $p\geqq 0$, $1\leqq v\leqq n-1$). From (1.3) we obtain

$$\int_{\Gamma_{i,j}(m,n)} \omega_{(m,n)}^{(p,v,l)} = \zeta_n^{v(i-1)} \times \zeta_m^{(p+1)(j-1)} \int_{\Gamma_{1,1}(m,n)} \omega_{m,n}^{(p,v,l)}\,. \tag{1.4}$$

Define the definite integrals $\mathfrak{B}_{(m,n)}(p, v, l)$ by

$$\mathfrak{B}_{(m,n)}(p, v, l) = \int_{\Gamma_{1,1}(m,n)} \omega_{m,n}^{(p,v,l)}\,. \tag{1.5}$$

With respect to definite integrals $\mathfrak{B}_{(m,n)}(p, v, l)$, there are following two types

1) In [12], we treated the case where $n=m$. The other cases can be treated in a similar manner.

of recursion formulas (cf. N. Sasakura [12]).

$(1.6)_1 \qquad n(p+1)\mathfrak{B}_{(m,n)}(p, v, l)+m(v-nl)\mathfrak{B}_{(m,n)}(p+m, v, l+1)=0.$

$(1.6)_2 \qquad n(p+1)\mathfrak{B}_{(m,n)}(p, v, l)-\{n(p+1)+m(v-n(l-1))\}\mathfrak{B}_{(m,n)}(p+m, v, l)=0.$

We know by an elementary calculation that

$$\begin{aligned} &\mathfrak{B}_{(m,n)}(p_0, v, -1)\neq 0 \qquad (\text{if } p_0\neq m-1)\\ &\mathfrak{B}_{(m,n)}(p_0, v, -1)=0 \qquad (\text{if } p_0=m-1). \end{aligned} \tag{1.7}$$

§2. Periods of rational 2-forms on an elliptic surface of Fermat type.

Let $(\tilde{\tau}^0, \tilde{\sigma}^0)$ be a point on the affine space $\boldsymbol{C}^{4k+1}\times\boldsymbol{C}^{6k+1}$ with coordinate $(\tilde{\tau}^0)=(0, \cdots, 0)$, $(\tilde{\sigma}^0)=(0, 0, \cdots, 0, -1)$ and denote the basic elliptic surface corresponding to the point $(\tilde{\tau}^0, \tilde{\sigma}^0)$ by $B^k_{(0)}$. The surface $B^k_{(0)}$ is defined in the affine space $\boldsymbol{C}^3$ by

$$y^2=4x^3-(u^{6k}-1). \tag{2.1}$$

Denote the rational 2-forms $\omega^{(i,j,l)}_{(\tilde{\tau}^0,\tilde{\sigma}^0)}$ on the surface $B^k_{(0)}$ by $\omega^{(i,j,l)}_{(0)}$. For the curves of Fermat type $C_{(3,2)}: t^2=4s^3-1$ [2] and $C_{(6k,6)}: \omega^6=v^{6k}-1$, we define a rational map Φ^k from the product $C_{(3,2)}\times C_{(6k,6)}$ to the surface $B^k_{(0)}$ by

$$\Phi^k((s, t), (v, \omega))=(u, x, y)=(v, s\omega^2, t\omega^3). \tag{2.2}$$

For a cyclic group $G_6=\{\sigma, \sigma^2, \cdots, \sigma^6=1\}$ of order six, we define an operation (σ^l) on the product $C_{(3,2)}\times C_{(6k,6)}$ in the following manner:

$$(\sigma^l)\cdot((s, t)\times(v, \omega))=(\zeta_3^{-l}\cdot s, (-1)^{-l}t)\times(v, \zeta_6^l\cdot\omega).$$

Define a subset $\tilde{C}^k$ of the product $C_{(3,2)}\times C_{(6k,6)}$ by $\tilde{C}^k=C_{(3,2)}\times\left\{C_{(6k,6)}-\bigcup_{j=1}^{6k}(\zeta_{6k}^j, 0)\right\}$ and define a subset $\tilde{B}^k_{(0)}$ of the surface $B^k_{(0)}$ by $\tilde{B}^k_{(0)}=B^k_{(0)}-\bigcup_{j=1}^{6k}\Psi^{-1}(\zeta_{6k}^j)$. In the subset $\tilde{C}^k$, the rational map Φ^k is a regular map with no ramification point and the image $\Phi^k(\tilde{C}^k)$ coincides with $\tilde{B}^k_0$. We know readily that $\Phi^k((s, t)\times(v, \omega))=\Phi^k((s', t')\times(v', \omega'))$ if and only if $(\sigma^l)((s, t)\times(v, \omega))=((s', t')\times(v', \omega'))$ for some integer l. Thus we know that $\tilde{C}^k/G^6$ is biregularly equivalent to the open

2) Strictly speaking the elliptic curve: $t^2=4s^3-1$ is different from the curve $C_{3,2}$: $y^2=s^3-1$. However, the recursion formula (1.6) and the formula (1.7) hold for $t^2=4s^3-1$. Thus we use the same symbol $C_{3,2}$ to denote the curve $t^2=4s^3-1$.

3) In (2.2) the rational map Φ^k is defined in terms of the affine coordinate $(s, t)\times(v, \omega)$. The map Φ^k is extended (as a rational map) to $C_{(3,2)}\times C_{(6k,6)}$ such that the extended mapping is regular and unramified in $C_{(3,2)}\times\{C_{(6k,6)}-\bigcup_{j=1}^{6k}(\zeta_{6k}^j, 0)\}$.

manifold $\tilde{B}^k_{(0)}$.

Let $H_j(B^k_0)$ be the j-dimensional (rational) homology group of B^k_0. Combining the exact sequence of relative cohomology groups concerning the pair $\left(B^k_0, \bigcup_{j=1}^{6k}\Psi^{-1}(\zeta^j_{6k})\right)$ and the Poincaré-Lefschetz duality theorem (cf. Spanier [13] p. 239 and p. 296), we obtain an exact sequence of homology groups (cf. Hodge-Atiyah [5])

$$\text{(2.5)}\qquad \longrightarrow H_1\Big(\bigcup_{j=1}^{6k}\Psi^{-1}(\zeta^j_{6k})\Big)\xrightarrow{\ \tau\ } H_2(\tilde{B}^k_0)\xrightarrow{\ i\ } H_2(B^k_0)\xrightarrow{\ c\ } H_0\Big(\bigcup_{j=1}^{6k}\Psi^{-1}(\zeta^j_{6k})\Big)$$
$$\xrightarrow{\ \tau\ } H_1(\tilde{B}^k_0)\xrightarrow{\ i\ } H_1(B^k_0)\longrightarrow\cdots.$$

On the other hand the surface B^k_0 is regular and the curve $\Psi^{-1}(\zeta^j_{6k})$ is topologically a 2-sphere. Hence we obtain

$$\text{(2.6)}\qquad 0\longrightarrow H_2(\tilde{B}^k_0)\xrightarrow{\ i\ } H_2(B^k_0)\xrightarrow{\ c\ }\bigoplus_{j=1}^{6k}H_0(\Psi^{-1}(\zeta^j_{6k}))\xrightarrow{\ \tau\ } H_1(\tilde{B}^k_0)\longrightarrow 0\,.$$

Let $Z^{(G_6)}_j(\tilde{C}^k)$ and $B^{(G_6)}_j(\tilde{C}^k)$ be vector spaces (over $\boldsymbol{Q}$) composed of G_6-invariant j-dimensional cycles and G_6-invariant j-dimensional boundary cycles, respectively. Put $H^{(G_6)}_j(\tilde{C}^k)=Z^{(G_6)}(\tilde{C}^k)/B^{(G_6)}_j(\tilde{C}^k)$. Then we have $H^{(G_6)}_j(\tilde{C}^k)\cong H_j(\tilde{B}^k_0)$. Now we determine the structure of the homology groups $H^{(G_6)}_1(\tilde{C}^k)$ and $H^{(G_6)}_2(\tilde{C}^k)$. Take C^∞-differentiable loops γ_j on the curve $C_{(6k,6)}$ surrounding the points $(\zeta^j_{6k}, 0)$ once in the positive direction. Then, as a Betti base of the set $C_{(6k,6)}-\bigcup_{j=1}^{6k}(\zeta^j_{6k}, 0)$, we can choose 1-cycles $\Gamma_{l,p}(6k, 6)$ $(l=1, 2, 3, 4, 5,\ p=1, 2, \cdots, 6k-2)$ and 1-cycles γ_j $(j=1, \cdots, 6k-1)$.

Define G_6-invariant 2-cycles $\Gamma_{j,l,m}$ $(j=1, 2, 3,\ l=1, 2, 3, 4, 5,\ m=1, 2, \cdots, 6k-2)$ as follows:

$$\text{(2.7)}\qquad \Gamma_{j,l,m}=\sum_{\sigma^h\in G_6}(\sigma^h)_*(\Gamma_{1,j}(3, 2)\times\Gamma_{l,m}(6k, 6))$$

(The symbol $(\sigma^h)_*$ denotes the operation of σ^h on the cycles on the surface $C_{(2,3)}\times C_{(6k,6)}$).

Then we infer from (1.3) that[4)]

$$\text{(2.8)}\qquad \Gamma_{1,l,m}-\Gamma_{2,l,m}+\Gamma_{3,l,m}=0\,,$$

$$\text{(2.9)}\qquad \Gamma_{j,l,m}=\Gamma_{j-1,l+1,m}=\Gamma_{j-2,l+2,m}=\cdots.$$

On the other hand we have

$$\text{(2.10)}\qquad \sum_{\sigma^l\in G_6}(\sigma^l)_*(\Gamma_{1,j}(3, 2)\times\gamma_i)=0\,.$$

Denote the fundamental class of the algebraic curve $C_{(6k,6)}$ by C and define an (algebraic) 2-cycle θ_0 by $\theta_0=\sum_{\sigma^l\in G_6}(\sigma^l)_*(C_{(3,2)}\times p_2)$ (where p_2 is an arbitrary

4) See [12], formula (2.9).

point on the curve $C_{(6k,6)}$). Now we infer from (1.3) readily the following two facts.

(2.11)$_1$ Any G_6-invariant 2-cycle on the set $\tilde{C}^k$ is homologous (rationally) to a linear combination of the 2-cycles $\Gamma_{j,1,m}$ $(j=1, 2,\ m=1, 2, \cdots, 6k-2)$ and θ_0.

(2.11)$_2$ Among 2-cycles $\Gamma_{j,1,m}$ $(j=1, 2,\ m=1, 2, \cdots, 6k-2)$ and θ_0 there is no homological relation with rational coefficients.

Concerning the one dimensional case, put $\tilde{\gamma}_j=\sum_{\sigma^l\in\sigma_6}(\sigma^l)_*(p_1\times\gamma_j)$ $(j=1, 2, \cdots, 6k-1)$ (where p_1 is a point on $C(3, 2)$). Then we infer readily that

(2.12) the 1-cycles $\tilde{\gamma}_j$ $(j=1, 2, \cdots, 6k-1)$ are representatives of a base of the homology group

$$H_1^{(G_6)}(\tilde{G}^k;\ \boldsymbol{Q}).$$

For any cycle γ on $\tilde{C}^k$, we denote by $\Phi_*^k(\gamma)$ the cycle on $\tilde{B}_0^k$ which is the image of the cycle γ by the mapping Φ^k. Summing up the above results we obtain the following

LEMMA 2.1. (1) *The 2-cycles* $\Phi_*^k(\Gamma_{j,1,m})$ $(j=1, 2,\ m=1, 2, \cdots, 6k-2)$, $\Phi_*^k(\theta_0)$ *form a Betti base of 2-cycles on* $\tilde{B}_0^k$.

(2) *We have the exact sequence*

$$0\longrightarrow H_2(\tilde{B}_0^k)\xrightarrow{\ i\ }H_2(B_0^k)\xrightarrow{\ c\ }\boldsymbol{Q}\ (\cong c(H_2(B_0^k)))\longrightarrow 0. \tag{2.13}$$

(3) *In the sequence* (2.13), *the kernel of the mapping c is represented by the class* (Δ^k), *where* (Δ^k) *denotes the fundamental homology class of the algebraic curve* Δ^k.

For the rational 2-form $\omega_k^{p,q,r}(0)$ on the elliptic surface B_0^k, denote the rational 2-form on $C_{(3,2)}\times C_{(6k,6)}$ induced by the mapping Φ^k by $(\Phi^k)^{(*)}(\omega_k^{p,q,r}(0))$. Then we have

$$(\Phi^k)^*(\omega_k^{p,q,r}(0))=\frac{s^q}{t^r}\,ds\wedge\frac{v^p dv}{\omega^{3r-2(q+1)}}. \tag{2.14}$$

Put $\mathfrak{E}_{k;j,m}^{p,q,r}=\int_{\Phi_*^k(\Gamma_{j,1,m})}\omega_k^{p,q,r}(0)$. We have

$$\mathfrak{E}_{k;j,m}^{p,q,r}=\sum_{\sigma^l\in G_6}\int_{(\sigma^l)_*(\Gamma_{1,j}(3,2))}\frac{s^q}{t^r}ds\times\int_{(\sigma^l)_*(\Gamma_{1,m}(6k,6))}\frac{v^p}{\omega^{3r-2(q+1)}}dv. \tag{2.15}$$

Combining the formulas in the first section and the formula (2.7), we can express the definite integrals $\mathfrak{E}_{k;j,m}^{p,q,r}$ in the following manner.

LEMMA 2.2. *Let* $r=2\tilde{r}+1$.

(a) *If* $q\equiv 2 \pmod 3$, *we have*

$$\mathfrak{E}_{k;j,m}^{p,q,2\tilde{r}+1}=0\qquad (j=1, 2,\ m=1, 2, \cdots, 6k-2). \tag{2.16}$$

(b) *If* $q\equiv 1 \pmod 3$, *we have*

$$(2.17)_1\quad \mathfrak{C}_{k;1,m}^{p,q,2\tilde{r}+1}=6\cdot\zeta_{6k}^{(p+1)(m-1)}\mathfrak{B}_{(3,2)}(q,\,-1,\,\tilde{r})\mathfrak{B}_{(6k-6,6)}\Big(p,\,-5,\,\tilde{r}-1-\frac{q-1}{3}\Big),$$

$$(2.17)_2\quad \mathfrak{C}_{k;2,m}^{p,q,2\tilde{r}+1}=6(-\zeta_3)\zeta_{6k}^{(p+1)(m-1)}\mathfrak{B}_{(3,2)}(q,\,-1,\,\tilde{r})\mathfrak{B}_{(6k-6,6)}\Big(p,\,-5,\,\tilde{r}-1-\frac{q-1}{3}\Big).$$

(c) *If* $q\equiv 0 \pmod 3$, *we have*

$$(2.18)_1\quad \mathfrak{C}_{k;1,m}^{p,q,2r+1}=6\cdot\zeta_{6k}^{(p+1)(m-1)}\mathfrak{B}_{(3,2)}(q,\,-1,\,\tilde{r})\mathfrak{B}_{(6k,-6)}\Big(p,\,5,\,\tilde{r}+1-\frac{q}{3}\Big),$$

$$(2.18)_2\quad \mathfrak{C}_{k;2,m}^{p,q,2r+1}=6\cdot(-\zeta_3^2)\zeta_{6k}^{(p+1)(m-1)}\mathfrak{B}_{(3,2)}(q,\,-1,\,\tilde{r})\mathfrak{B}_{(6k-6,6)}\Big(p,\,5,\,\tilde{r}+1-\frac{q}{3}\Big).$$

§ 3. Some applications. Power series expansions.

For the point $(\tilde{\tau}_0, \tilde{\sigma}_0)$ in $C^{4k+1}\times C^{6k}$ take a sufficiently small spherical neighbourhood $\mathfrak{A}_0$ of $(\tilde{\tau}_0, \tilde{\sigma}_0)$ in $C^{4k+1}\times C^{6k}$. Let $h(0)$ be a homology class on the surface $B^k_{(0)}$. For any point $(\tilde{\tau}_0, \tilde{\sigma}_0)$ in the neighbourhood $\mathfrak{A}_0$, we denote by $h^k(\tau, \sigma)$ the homology class on the surface $B^k(\tilde{\tau}_0, \tilde{\sigma}_0)$ corresponding to the cycle $h^k(0)$. Take any two dimensional homology class $h^k_2(0)$ which belongs to the image $(i)_*(H_2(B^k_0-X^k_0))$. Then we infer readily that the corresponding class belongs to the image $(i)_*(H_2(B^k(\tau, \sigma)-X^k(\tau, \sigma)))$. We can verify that the definite integrals: $\int_{h(\tau,\sigma)}\omega_k^{(p,q,r)}(\tau, \sigma)$ are holomorphic functions in (τ, σ). Moreover we have

$$(3.1)\quad \begin{aligned} \partial/\partial\tau_j\int_{h(\tau,\sigma)}\omega_k^{(p,q,r)}(\tau, \sigma)&=\int_{h(\tau,\sigma)}\frac{\partial\omega_k^{(p,q,r)}(\tau, \sigma)}{\partial\tau_j}, \\ \partial/\partial\sigma_j\int_{h(\tau,\sigma)}\omega_k^{(p,q,r)}(\tau, \sigma)&=\int_{h(\tau,\sigma)}\frac{\partial\omega_k^{(p,q,r)}(\tau, \sigma)}{\partial\sigma_j}. \end{aligned}$$

In the formula (3.1), $\partial\omega^{(p,q,r)}/\partial\tau_j$ and $\partial\omega_k^{(p,q,r)}/\partial\sigma_j$ are rational 2-forms $\frac{r}{2}\frac{u^{p+j}x^{q+1}}{y^{r+2}}du\wedge dx$ and $\frac{r}{2}\frac{u^{p+j}x^{q}}{y^{r+2}}du\wedge dx$ on $B^k_{(\tau,\sigma)}$, respectively. Let $\tilde{\Gamma}_{k,j,m}(0)$ be the homology class to which the 2-cycle $\Gamma_{k,j,m}(0)$ belongs. Put $W_{k;j,m}^{p,q,r}(\tau, \sigma)=\int_{\tilde{\Gamma}_{k;j,m}}\omega_k^{(p,q,r)}(\tau, \sigma)$.

THEOREM 3.1. *We have the following power series expansion of the holomorphic functions* $W_{k;j,m}^{p,q,r}(\tau, \sigma)$ *at the point* (τ_0, σ_0):

$$(3.2)\quad \begin{aligned} &W_{k;j,m}^{(p,q,2r+1)}(\tau, \sigma)\\ &\quad=\sum_{\substack{i_0,\cdots,i_{4k}\geqq 0\\ j_0,\cdots,j_{6k-1}\geqq 0}}C_{k;j,m}^{(p,q,2r+1)}(i_0,\,\cdots,\,i_{4k},\,j_0,\,\cdots,\,j_{6k-1})\tau_0^{i_0}\cdots\,\tau_{4k}^{i_{4k}}(\sigma_0 j_0)\\ &\qquad\times\sigma_1^{j_1}\cdots\,\sigma_{6k-1}^{j_{6k-1}}, \end{aligned}$$

where

$$C_{k;j,m}^{(p,q,2r+1)}(i_0,\cdots,i_k,j_0,\cdots,j_{6k-1}) \tag{3.3}$$
$$=\frac{1}{i_0!\cdots i_k!}\times\frac{1}{j_1!\cdots j_{6k-1}!}(1/2)^{(i_0+\cdots+i_k+j_1+\cdots+j_{6k-1})}$$
$$\times\{(2r+1)(2r+3)\cdots(2r+1+2(i_0+\cdots+j_{6k-1}))\}$$
$$\times\mathfrak{E}_{k;j,m}^{\tilde{p},\tilde{q},\tilde{r}}$$

where

$$\tilde{p}=p+\sum_l li_l+\sum_m mj_m,\qquad \tilde{q}=q+\sum i_l,\qquad \tilde{r}=2r+1+\sum i_l+\sum j_m.$$

For any holomorphic 2-form $\omega_k^{(p,0,1)}(\tau,\sigma)$, we define a holomorphic mapping η_k^p from $\mathfrak{A}_0$ into a $(12k-5)$-dimensional projective space $\boldsymbol{P}_{12k-5}(\boldsymbol{C})$ by

$$\eta_k^p(\tau,\sigma)=(W_{k,1,1}^{(p,0,1)}(\tau,\sigma),\cdots,W_{k,1,6k-2}^{(p,0,1)}(\tau,\sigma),\cdots,W_{k,2,6k-2}^{(p,0,1)}(\tau,\sigma)). \tag{3.4}$$

Let I_k be the intersection matrix with respect to the homology classes $\tilde{\Gamma}_{k,j,m}(\tau,\sigma)$. Obviously the matrix I_k does not depend on the parameters (τ,σ). We have the following bilinear equality (cf. Hodge [4]):

$$\eta_k^{p_1}\cdot(\tau,\sigma)^t I_k^{-1}\cdot{}^t\eta_k^{p_2}(\tau,\sigma)=0,\qquad (p_1,p_2=0,1,\cdots,k-1). \tag{3.5}$$

Let Q_k be a quadric in $\boldsymbol{P}_{12k-5}(\boldsymbol{C})$ defined by

$$(X)\cdot{}^tI_k^{-1}\cdot{}^t(X)=0, \tag{3.6}$$

where $(X)=(X_1,\cdots,X_{12k-4})$ is a homogeneous coordinate of $\boldsymbol{P}_{12k-5}(\boldsymbol{C})$. From the bilinear equality (3.5) we know that the image $\eta_k^p(\mathfrak{A}_0)$ is contained in Q_k. Now we calculate the rank of the mapping η_k^p.

THEOREM 3.2. *The rank of the mapping η_k^p in $\mathfrak{A}_0$ is equal to $10k-2$.*

PROOF. Define a $(10k+2)\times(12k-4)$ matrix $\Omega_{k,1}^p(\tau,\sigma)=[\omega_{l,m}^{(p,k)}(\tau,\sigma)_{\substack{l=1,\cdots,10k+2\\ m=1,\cdots,12k-4}}]$ in the following manner:

$$\omega_{1,m}^{(p,k)}(\tau,\sigma)=W_{k;1,m}^{p,0,1}\qquad(m=1,2,\cdots,6k-2), \tag{3.7$_0$}$$
$$\omega_{1,6k-2+m}^{(p,k)}(\tau,\sigma)=W_{k;2,m}^{p,0,1}\qquad(m=1,2,\cdots,6k-2),$$

$$\omega_{l,m}^{(p,k)}(\tau,\sigma)=\frac{\partial W_{k;1,m}^{p,0,1}}{\partial\tau_{l-2}}\qquad\binom{l=1,2,\cdots,4k+2}{m=1,2,\cdots,6k-2}, \tag{3.7$_{1:\tau}$}$$
$$\omega_{l,6k-2+m}^{(p,k)}(\tau,\sigma)=\frac{\partial W_{k;2,m}^{p,0,1}}{\partial\tau_{l-2}}\qquad\binom{l=2,\cdots,4k+2}{m=1,2,\cdots,6k-2},$$

$$\omega_{l,m}^{(p,k)}(\tau,\sigma)=\frac{\partial W_{k;1,m}^{p,0,1}}{\partial\sigma_{l-(4k+3)}}\qquad\binom{l=4k+3,\cdots,10k+2}{m=1,2,\cdots,6k-2}, \tag{3.7$_{1:\sigma}$}$$
$$\omega_{l,6k-2+m}^{(p,k)}(\tau,\sigma)=\frac{\partial W_{k;2,m}^{p,0,1}}{\partial\tau_{l-(4k+3)}}\qquad\binom{l=4k+3,\cdots,10k+2}{m=1,2,\cdots,6k-2}.$$

Define a $(6k-2)$-vector $\iota_0^{(p)}$ by

$$(3.8)_0 \qquad l_0^{(p)}=(1,\ \zeta_{6k}^{p+1},\ \zeta_{6k}^{2(p+1)},\ \cdots,\ \zeta_{6k}^{(6k-3)(p+1)})\,.$$

Moreover define $6k\times(6k-2)$ matrix $L_{0,1}^{(p)}$ and $(4k+1)\times(6k-2)$ matrix $L_{1,0}^{(p)}$ by

$$(3.8)_{1:\sigma} \qquad L_{0,1}^{(p)}=\frac{1}{2}\begin{bmatrix} 1, & \zeta_{6k}^{p+1}, & \zeta_{6k}^{2(p+1)}, & \cdots, & \zeta_{6k}^{(6k-3)(p+1)} \\ \vdots \\ 1, & \zeta_{6k}^{p+j}, & \zeta_{6k}^{2(p+j)}, & \cdots, & \zeta_{6k}^{(6k-3)(p+j)} \\ \vdots \\ 1, & \zeta_{6k}^{p+6k+1}, & \zeta_{6k}^{2(p+6k+1)}, & \cdots, & \zeta_{6k}^{(6k-3)(p+6k+1)} \end{bmatrix},$$

$$(3.8)_{1:\tau} \qquad L_{1,0}^{(p)}=\frac{1}{2}\begin{bmatrix} 1, & \zeta_{6k}^{p+1}, & \zeta_{6k}^{2(p+1)}, & \cdots, & \zeta_{6k}^{(6k-3)(p+1)} \\ \vdots \\ 1, & \zeta_{6k}^{p+1+j}, & \zeta_{6k}^{2(p+1+j)}, & \cdots, & \zeta_{6k}^{(6k-3)(p+j+1)} \\ \vdots \\ 1, & \zeta_{6k}^{p+1+4k}, & \zeta_{6k}^{2(p+1+4k)}, & \cdots, & \zeta_{6k}^{(6k-3)(p+1+4k)} \end{bmatrix}.$$

Finally define a number $B_0^{(p)}$, $(6k\times 6k)$-diagonal matrix $B_{0,1}^{(p)}$ and $(4k+1)\times(4k+1)$-diagonal matrix $B_{1,0}^{(p)}$ by

$$(3.9)_0 \qquad B_0^{(p)}=2\times 6\times B_{3,2}(0,\ -1,\ 0)\times B_{(6k-6,6)}(p,\ -1,\ 0)\ ,$$

$$(3.9)_{0,\sigma} \qquad B_{0,1}^{(p)}=2\times 6\times B_{3,2}(0,\ -1,\ 1)\times\begin{bmatrix} B_{(6k,6)}(p,\ -5,\ 0) & & & \\ & B_{(6k,6)}(p+1,\ -5,\ 0) & & \\ & & \ddots & \\ & & & B_{(6k,6)}(p+6k,\ -5,\ 0) \end{bmatrix},$$

$$(3.9)_{0,\tau} \qquad B_{1,0}^{(p)}=2\times 6\times B_{3,2}(1,\ -1,\ 1)\times\begin{bmatrix} B_{(6k,6)}(p,\ -5,\ 0) & & & \\ & B_{(6k,6)}(p+1,\ -5,\ 0) & & \\ & & \ddots & \\ & & & B_{(6k,6)}(p+4k,\ -5,\ 0) \end{bmatrix}.$$

Then, by Theorem 1, we have

$$(3.10) \qquad \Omega_{k,1}^{p}(0)=\begin{bmatrix} B_0^{(p)}\times l_0 \\ B_{1,0}^{(p)}\times L_{1,0}^{(p)} \\ B_{0,1}^{(p)}\times L_{0,1}^{(p)} \end{bmatrix}\times\begin{bmatrix} E_{6k-2} & 0 \\ & \\ 0 & E_{6k-2} \end{bmatrix}$$

where we denote by E_{6k-2} the $(6k-2)\times(6k-2)$ unit matrix. Now, from the formula (1.7) and the equation (3.10), we infer that rank $\Omega_k^p(0)=10k-1$.

REMARK. By a result of A. Kas and a simple calculation, we know that the systems of basic elliptic surfaces $B_0^k(\tau,\sigma)$ essentially depend on $10k-2$ parameters. This fact and Theorem 2 show that *the periods of any holomorphic 2-form* $\omega_k^{(p,0,1)}(\tau,\sigma)$ describe the structure of the basic elliptic surfaces $B^k(\tau,\sigma)$ in the neighbourhood $\mathfrak{A}_0$.

We continue the arguments of the previous page. Define

$$(6k)^2\times(12k-4) \text{ matrix } \Omega_{k;(2,0)}^{p}(\tau,\sigma)\ (=\omega_{l,m}^{(p,k),(2,0)}(\tau,\sigma))$$

$(6k\times(4k+1))\times(12k-4)$ matrix $\Omega^{p}_{k;(1,1)}(\tau, \sigma)$ $(=\omega^{(p,q),(1,1)}_{l,m}(\tau, \sigma))$

and $(4k+1)^2\times(12k-4)$ matrix $\Omega^{p}_{k;(0,2)}(\tau, \sigma)$ $(=\omega^{(p,k),(0,2)}_{l,m}(\tau, \sigma))$

in the following manner:

$$(3.11)_{(2,0)} \qquad \omega^{(p,k),(2,0)}_{i+j,m}(\tau, \sigma)=\frac{\partial^2 W^{(p,0,1)}_{1,m}}{\partial\tau_i\partial\tau_j},$$

$$\omega^{(p,k),(2,0)}_{i+j,6k-2+m}(\tau, \sigma)=\frac{\partial^2 W^{(p,0,1)}_{2,m}}{\partial\tau_i\partial\sigma_j} \qquad (m=1, 2, \cdots, 6k-2),$$

$$(3.11)_{(1,1)} \qquad \omega^{(p,k),(1,1)}_{i+j,m}(\tau, \sigma)=\frac{\partial^2 W^{(p,0,1)}_{1,m}}{\partial\tau_i\partial\sigma_j},$$

$$\omega^{(p,k),(0,2)}_{i+j,6k-2+m}(\tau, \sigma)=\frac{\partial^2 W^{(p,0,1)}_{2,m}}{\partial\tau_i\partial\tau_j} \qquad (m=1, 2, \cdots, 6k-2),$$

$$(3.12)_{(0,2)} \qquad \omega^{(p,k),(0,2)}_{i+j,m}(\tau, \sigma)=\frac{\partial^2 W^{(p,0,1)}_{1,m}}{\partial\sigma_i\partial\sigma_j},$$

$$\omega^{(p,k),(0,2)}_{i+j,m}(\tau, \sigma)=\frac{\partial^2 W^{(p,0,1)}_{2,m}}{\partial\sigma_i\partial\sigma_j} \qquad (m=1, 2, \cdots, 6k-2).$$

Define a matrix $\Omega^{p}_{k;2}(\tau, \sigma)$ by

$$(3.13) \qquad \Omega^{p}_{k;2}(\tau, \sigma)=\begin{bmatrix} \Omega^{p}_{k;1}(\tau, \sigma) \\ \Omega^{p}_{k;(2,0)}(\tau, \sigma) \\ \Omega^{p}_{k;(1,1)}(\tau, \sigma) \\ \Omega^{p}_{k;(0,2)}(\tau, \sigma) \end{bmatrix}.$$

Then by a similar calculation as in the proof of Theorem 3.2 we have

THEOREM 3.3.

(3.14) *The rank of the matrix* $\Omega^{p}_{k;2}(\tau, \sigma)$ *is equal to* $12k-4$.[5]

It is well known that a 2-cycle γ_2 on an algebraic surface S is algebraic if and only if $\int_{\gamma_2}\omega^2=0$ for all holomorphic two forms ω^2 on the surface S (Lefschetz [**10**], Kodaira-Spencer [**9**]), so the Picard number $\rho(B_k(\tau, \sigma))$ of the surface $B(\tau, \sigma)$ is equal to 2 for general values of the parameters (τ, σ).

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5) As is seen from the process of the proof of Theorem 2.2 and the fact (3.14), the essential part of Theorem 2.2 and Theorem 2.3 lies in whether or not the systems of rational forms with poles on $X(\tau, \sigma)$ in question are cohomologically independent when considered as C^∞-differentiable 2-forms in $S(\tau, \sigma)-X(\tau, \sigma)$. It will be probably done by some modifications of the method of Hodge-Atiyah [**5**] or that of Griffiths [**3**], though the polar divisor $X(\tau, \sigma)$ is not an ample divisor. However we hope that the power series expansion (3.3) gives some other information than the one obtained cohomological independence mentioned above.

Chapter II. A system of linear partial differential equation.

In this chapter, we discuss a system of certain partial differential equations of the second order whose solutions are the period functions $W^{p;0;1}_{k;0;m}(\tau, \sigma)$. This may be regarded as a generalization of Hypergeometric differential equations of one and several variables (cf. F. Klein [7] and Appell et Kampe de Feriet [1]).

§ 4. Some algebraic preparations.

In this section we discuss somewhat general situation than in the previous chapter. Let $\mathfrak{A}$ be a connected domain in the n-dimensional affine space $\boldsymbol{C}^n$, and let $\mathfrak{D}$ be an analytic set of codimension one in the domain $\mathfrak{A}$. Letting $\mathfrak{B}=\mathfrak{A}-\mathfrak{D}$, we denote by $\pi_1(\mathfrak{B})$ the fundamental group of the domain $\mathfrak{B}$. Let f be a multivalued holomorphic function defined on $\mathfrak{B}$. Let $\mathfrak{F}$ be an N-dimensional linear space (over the complex number field $\boldsymbol{C}$) composed of multivalued holomorphic functions defined on $\mathfrak{B}$. For any element $\pi \in \pi_1(\mathfrak{B})$, we denote by (π) the operation of the element π on the linear space $\mathfrak{F}$. We assume that $\mathfrak{F}$ is invariant under the operation (π) for any $\pi \in \pi_1(\mathfrak{B})$. Let $f_1, \cdots, f_N \in \mathfrak{F}$ be a base of the vector space $\mathfrak{F}$, and define an N-dimensional vector $\mathfrak{f}$ by $\mathfrak{f}=(f_1, \cdots, f_N)$. For any partial differential operator $D_{i_1,\cdots,i_n}=\partial^{i_1,\cdots,i_n}/\partial z_1^{i_1}\cdots\partial z_n^{i_n}$, put $D_{(I)}\mathfrak{f}=D_{i_1,\cdots,i_n}\mathfrak{f}=(D_{i_1,\cdots,i_n}f_1, \cdots, D_{i_1,\cdots,i_n}f_N)$. Let $\mathfrak{S}$ be a set of pairs of indices (l_i, m_i) $(i=1, 2, \cdots, N-(n+1))$. For the sake of simplicity, we denote by D_j the differential operator $\partial/\partial z_j$ $(j=1, \cdots, n)$ and by $D_{(l_i,m_i)}$ the differential operator $\partial^2/\partial z_{l_i}\partial z_{m_i}$ with $(l_i, m_i)\in\mathfrak{S}$.

Define an $N\times N$ matrix $F(z)$ by

$$F(z)=\begin{bmatrix} \mathfrak{f} \\ D_1\mathfrak{f} \\ \vdots \\ D_n\mathfrak{f} \\ D_{(l_1,m_1)}\mathfrak{f} \\ \vdots \\ D_{(l_{N-n-1},m_{N-n-1})}\mathfrak{f} \end{bmatrix}. \tag{4.1}$$

Now we assume that

$$\det F(z)\neq 0 \quad \text{in } \mathfrak{B}. \tag{4.2}$$

For an arbitrary differential operator $D_{i_1,\cdots,i_n}$, we define *single valued holomorphic functions* $A_0^{(I)}$, $A_j^{(I)}$ $(j=1, \cdots, n)$, $A_{(l_i,m_i)}^{(I)}$ $(i=1, \cdots, N-(n+1))$ in the following manner:

6) We assume that $N\geqq n+1$ in this section.

$$(4.3)\qquad {}^tD_{(I)}\mathfrak{f} = {}^tF(z)\begin{bmatrix} A_0^{(I)} \\ A_j^{(I)} \\ A_{(l_i,m_i)}^{(I)} \end{bmatrix}.$$

Then, obviously, we obtain the following easy

PROPOSITION 4.1. *For an arbitrary function* $f\in\mathfrak{F}$ *and a differential operator* $D_{i_1,\cdots,i_n}$, *we obtain the following (uniquely determined) differential equation*

$$(4.4)\qquad D_{i_1,\cdots,i_n}f = \sum_{(l_i,m_i)\in\mathfrak{S}} A^{(i_1,\cdots,i_N)}D_{(l_i,m_i)}f + \sum_{j=1}^{N} A_j^{(i_1,\cdots,i_N)}(z)D_jf + A_0^{(i_1,\cdots,i_N)}(z)f.$$

§ 5. A criterion on the behavior of the solutions at the singular divisors.

In this section, let the situation be the same as in the beginning of the section 4, except the following two differences. First we restrict the domain $\mathfrak{A}$ and the divisor $\mathfrak{D}$ to those defined by $\mathfrak{A}=\{(z_1,\cdots,z_n)\in \boldsymbol{C}^n;\ |z_i|<\varepsilon_i$, with sufficiently samll $\varepsilon_i\}$, $\mathfrak{D}=\{(z_1,\cdots,z_n)\in\mathfrak{A};\ z_1=0\}$. Secondly we omit the condition: $N\geqq n+1$. For the generator π of the fundamental group $\pi_1(\mathfrak{A}-\mathfrak{D})$, we denote by $\rho_1,\cdots,\rho_k$ the eigenvalues of the transformation (π) (of the linear space $\mathfrak{F}$). For a base $(F_1,\cdots,F_N)$ of the linear space $\mathfrak{F}$, we define a matrix $W(z_1,\cdots,z_n)$ by

$$(5.1)\qquad W(z_1,\cdots,z_n)=\begin{bmatrix} F_1 & F_2 & & F_N \\ \partial F_1/\partial z_1, & \partial F_2/\partial z_1, & \cdots, & \partial F_N/\partial z_1 \\ \partial^2 F_1/\partial z_1^2, & \partial^2 F_2/\partial z_1^2, & \cdots, & \partial^2 F_N/\partial z_1^2 \\ \\ \partial^{N-1}F_1/\partial z_1^{N-1}, & \partial^{N-1}F_2/\partial z_1^{N-1}, & \cdots, & \partial^{N-1}F_N/\partial z_1^{N-1} \end{bmatrix}.$$

Hereafter we assume that

$$(5.2)\qquad \det W(z_1,\cdots,z_n)\neq 0 \text{ in the open domain } \mathfrak{A}-\mathfrak{D}.$$

We write functions $F_1(z_1,z_2,\cdots,z_n),\cdots,F_N(z_1,z_2,\cdots,z_n)$ as $F_1(z_1;z_2,\cdots,z_n)$, $\cdots,F_N(z_1;z_2,\cdots,z_n)$ when we consider $(z_2,\cdots,z_n)$ as parameters and z_1 as the variable. Then obviously the assumption (5.2) implies that $F_1(z_1;z_2,\cdots,z_n)$, $\cdots,F_N(z_1;z_2,\cdots,z_n)$ are linearly independent as functions of the variable z_1. We define *single-valued holomorphic functions* $P_{N-1},\cdots,P_0$ in the open domain $\mathfrak{A}-\mathfrak{D}$ by

$$(5.3)\qquad \begin{bmatrix} P_{N-1} \\ P_{N-2} \\ \vdots \\ P_0 \end{bmatrix} = \begin{bmatrix} F_1^{(N-1)}, F_1^{(N-2)}, \cdots, F_1 \\ \vdots \\ F_N^{(N-1)}, F_N^{(N-2)}, \cdots, F_N \end{bmatrix}^{-1} \begin{bmatrix} F_1^{(N)} \\ \vdots \\ F_N^{(N)} \end{bmatrix}$$

where $F_i^{(N)}=\dfrac{\partial^N}{\partial z_1^N}$.

Let $D_{(z_2,\cdots,z_n)}$ be the ordinary differential equation in the variable z_1 defined by

(5.4) $$d^N F/dz_1^N = P_{N-1}\cdot d^{N-1}F/dz_1^{N-1}+P_{N-2}\cdot d^{N-2}F/dz_1^{N-2}+\cdots+P_1\cdot dF/dz_1+P_0F.$$

We denote the linear space $\mathfrak{F}$ by $\mathfrak{F}_{(z_2,\cdots,z_n)}$, when we consider $(z_2,\cdots,z_n)$ as parameters. Then it follows from (5.3) that the linear space $\mathfrak{F}_{(z_2,\cdots,z_n)}$ coincides with the linear space of the solutions of the differential equation (5.4). Consider the following condition (R) imposed on the linear space $\mathfrak{F}$:

(R) There are (sufficiently small) positive numbers $(\varepsilon_2',\cdots,\varepsilon_n')$ and a sufficiently large, positive number L such that the condition

(5.5) $$\lim_{\substack{|z_1|\to 0\\ \arg z_1:\ \text{bounded}}} |z_1|^L\cdot|f|=0,$$

holds for any functions $f\in\mathfrak{F}$ uniformly in the parameter $(z_2,\cdots,z_n)$ with $|z_i|<\varepsilon_i'$.

Let the Laurent expansions of the holomorphic functions $P_{N-1},\cdots,P_0$ be

(5.6) $$P_j=\sum_{q=-\infty}^{\infty} p_{j,q}\cdot z_1^q;\ \text{where } p_{j,q}\text{'s are holomorphic in } (z_2,\cdots,z_n).$$

If the condition (R) is true, we infer (from a well known theorem on linear ordinary differential equations [E. Picard [11]]) that

$(\tilde{R})$ $$P_{N-1}\cdot z_1,\ P_{N-2}\cdot z_1^2,\ \cdots,\ P_0\cdot z_1^N \text{ are holomorphic in } (z_1,\cdots,z_n).$$

Conversely, assume that the condition $(\tilde{R})$ on the functions P_j $(j=0,\cdots,N-1)$ holds. First we prove the following

PROPOSITION 5.1. *There is a positive integer $\tilde{M}$ which is independent of the parameters $(z_2,\cdots,z_n)$ such that*

(5.7) $$\lim_{\substack{|z_1|\to 0\\ \arg z_1:\ \text{bounded}}} |\tilde{f}(z_1)_{(z_2,\cdots,z_n)}|\cdot|z_1|^{\tilde{M}}=0$$

for all the solutions $\tilde{f}(z_1)_{(z_2,\cdots,z_n)}$ of the ordinary differential equation (5.4).

PROOF. The determining equation for the equation (5.4) has the following form

(5.8) $$P_{(z_2,\cdots,z_n)}(\rho):\ \rho(\rho-1)(\rho-2)\cdots(\rho-(N-1)) = \sum_{j=1}^{N} P_{N-j,-j}(z_2,\cdots,z_n)\rho(\rho-1)\cdots(\rho-(N-j)+1).$$

Since the coefficients $P_{N-j,-j}(z_2,\cdots,z_n)$ are holomorphic in the parameters $(z_2,\cdots,z_n)$ there exist a positive number $\tilde{L}$ and (sufficiently small) positive numbers $(\varepsilon_2'',\cdots,\varepsilon_N'')$ such that all the roots $\rho(z_2,\cdots,z_n)$ of the equation (5.8) satisfy $|\rho(z_2,\cdots,z_n)|<\tilde{L}$ for $|z_i|<\varepsilon_i''$ $(i=1,\cdots,n)$. Let $\tilde{\rho}(z_2,\cdots,z_n)$ be a root of the equation (5.8) whose real part $\operatorname{Re}\tilde{\rho}(z_2, \cdots, z_n)$ is maximal. The equation (5.4) has the solutions of the following form:

$$(5.9)\qquad \tilde{F}(z_1)_{(z_2,\cdots,z_n)} = \tilde{z}_1^{\tilde{\rho}(z_2,\cdots,z_n)} \cdot \tilde{G}(z_1)_{(z_2,\cdots,z_n)},$$

where $\tilde{G}(z_1)_{(z_2,\cdots,z_n)}$ is a holomorphic function of the variable z_1 and $\tilde{G}(0)_{(z_2,\cdots,z_n)} \neq 0$.

For any solution $f(z_1)_{(z_2,\cdots,z_n)}$ of the equation (5.4) we define a function $g(z_1)_{(z_2,\cdots,z_n)}$ by $g = d(f/\tilde{F})dz_j$. Then the function $g_{(z_2,\cdots,z_n)}$ is a solution of the ordinary differential equation (of order $N-1$):

$$(5.10)\qquad \sum_{q=0}^{N-1} R_q \cdot \frac{d^q g(z_1)_{(z_2,\cdots,z_n)}}{dz_1^q} = 0,$$

where

$$(5.11)\qquad R_q = \binom{N}{N-(q+1)} \cdot \frac{d^{N-(q+1)}\tilde{F}}{dz_1^{N-(q+1)}} - \sum_{i=q+1}^{N-1} \binom{i}{i-(q+1)} \cdot P_i \cdot \frac{d^{i-(q+1)}\tilde{F}}{dz_1^{i-(q+1)}}.$$

Write the determining equation (P_{N-1}) of the equation (5.10) in the following form

$$(5.12)\qquad (P_{N-1}):\ \rho_1(\rho_1-1)\cdots(\rho_1-(N-2)) = \sum_{j=0}^{N-2} \gamma_{N-1-j,j}(z_2, \cdots, z_n) \cdot \rho_1(\rho_1-1)\cdots(\rho_1-(N-j)-2).$$

Then we infer from the equations (5.10) and (5.11) that the coefficients $\gamma_{N-1-j,j}$ are expressed as polynomials of the roots $\tilde{\rho}_{(z_2,\cdots,z_n)}$ and the holomorphic functions $\gamma_{N-1-j,j}(z_2, \cdots, z_n)$. There exist a positive number L_1 and positive numbers $(\varepsilon_2''', \cdots, \varepsilon_n''')$ such that the roots $\rho_1(z_2, \cdots, z_n)$ of the determining equation (5.12) satisfy the inequality: $|\rho_1(z_2, \cdots, z_n)| < L$ for $|z_i| < \varepsilon_i''$. Letting $\tilde{\rho}_1(z_2, \cdots, z_n)$ be a root of the equation (5.12) whose real part is maximal, we have two linearly independent solutions $\tilde{F}_1, \tilde{F}_2$ of the equation (5.4) of the following forms:

$$(5.13)\qquad \begin{aligned} \tilde{F}_1(z_1)_{(z_2,\cdots,z_n)} &= \tilde{z}_1^{\tilde{\rho}} \cdot \tilde{G}_1(z_1)_{(z_2,\cdots,z_n)}, \\ \tilde{F}_2(z_1)_{(z_2,\cdots,z_n)} &= \tilde{F}_1(z_1) \cdot \int \tilde{z}_1^{\tilde{\rho}_1} \cdot \tilde{G}_2(z_1)_{(z_2,\cdots,z_n)}, \end{aligned}$$

where $\tilde{G}_1(z_1)_{(z_2,\cdots,z_n)}$ and $\tilde{G}_2(z_1)_{(z_2,\cdots,z_n)}$ are holomorphic in the variable z_1 and $\tilde{G}_i(0) \neq 0$ $(i=1, 2)$. Repeating these procedures, we obtain N linearly independent solutions $\tilde{F}_1(z_1)_{(z_2,\cdots,z_n)}, \cdots, \tilde{F}_N(z_1)_{(z_2,\cdots,z_n)}$ of the equation (5.4) of the following forms:

$$(5.14)\qquad \begin{aligned} \tilde{F}_1(z_1)_{(z_2,\cdots,z_n)} &= \tilde{z}_1^{\tilde{\rho}(z_2,\cdots,z_n)} \cdot \tilde{G}_1(z_1)_{(z_2,\cdots,z_n)}, \\ \tilde{F}_2(z_1)_{(z_2,\cdots,z_n)} &= F_1 \cdot \int \tilde{z}_1^{\tilde{\rho}_1(z_2,\cdots,z_n)} \cdot \tilde{G}_2(z_1)_{(z_2,\cdots,z_n)} dz_1, \\ \tilde{F}_3(z_1)_{(z_2,\cdots,z_n)} &= \tilde{F}_1 \cdot \int \tilde{F}_2 \times \tilde{G}_3(z_1)_{(z_2,\cdots,z_n)} dz_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where $\tilde{G}_i(z_1)_{(z_2,\cdots,z_n)}$ are holomorphic in the variable z_1 and the roots $\tilde{\rho}(z_2, \cdots, z_n)$, $\tilde{\rho}_1(z_2, \cdots, z_n)$, $\tilde{\rho}_{N-1}(z_2, \cdots, z_n)$ are bounded in the domains $|z_i| < \tilde{\varepsilon}_i$ for some positive numbers $\tilde{\varepsilon}_i$. q. e. d.

Now we prove the following

PROPOSITION 5.2. *The determining equation* (5.8) *is independent of the parameters* $(z_2, \cdots, z_n)$.

PROOF. In order to prove this proposition, it is enough to prove that

(5.15) the coefficients $P_{N-1,-1}(z_2, \cdots, z_n)$, $P_{N-2,-2}(z_2, \cdots, z_n)$, $\cdots$, $P_{0,-N}(z_2, \cdots, z_n)$ are independent of the parameters $(z_2, \cdots, z_n)$.

We prove the assertion (5.15) by induction on the dimension of the linear space $\mathfrak{F}$.

(I) Put $N=1$ and let f_0 be a base of the space $\mathfrak{F}$. Then f_0 satisfies the following differential equation: $\frac{\partial f}{\partial z_1} = P_0 \cdot f_0$. By the previous Proposition 5.1, the function f_0 can be expanded in the following form:

$$f_0 = z_1^{\tilde{\lambda}_0}(f_{0,0} + f_{1,0}z_1 + \cdots + f_{m,0}z_1^m + \cdots), \tag{5.16}$$

where the coefficients $f_{0,0}, f_{1,0}, \cdots$ are holomorphic in the variables $(z_2, \cdots, z_n)$ and the function $f_{0,0}$ is not identically equal to 0. The exponent $\tilde{\lambda}_0$ satisfies $\log \tilde{\lambda}_0 = \rho_0$, where ρ_0 is an eigenvalue of the operation (π) on the space $\mathfrak{F}$. Now the function P_0 is expressed in the form:

$$P_0 = \frac{\frac{\partial f}{\partial z_1}}{f} = \frac{\rho_0 \cdot f_{0,0} + (f_{0,0} + \rho f_{1,0})z_1 + \cdots}{z_1(f_{0,0} + \cdots)}.$$

So we have: $P_{0,-1} = \rho_0$ for all values $(z_2, \cdots, z_n)$.

(II) Now suppose that the assertion is verified in the case where $\dim \mathfrak{F} \leqq N-1$ and consider the case in which $\dim \mathfrak{F} = N$. Let $\tilde{\rho}_1$ be an eigenvalue of the operation (π) on the space $\mathfrak{F}$. Moreover let $\tilde{f}$ be a function in the space $\mathfrak{F}$ such that

$$(\pi)(\tilde{f}_1) = \tilde{\rho}_1 \cdot \tilde{f}_1. \tag{5.17}$$

By the Proposition 5.2, we can write the function $\tilde{f}_1$ in the form:

$$\tilde{f}_1 = z_1^{\tilde{\lambda}_1} \cdot (\tilde{f}_{0,1} + \tilde{f}_{1,1}z_1 + \cdots + \tilde{f}_{m,1}z_1^m + \cdots), \tag{5.18}$$

where the functions $\tilde{f}_{0,1}, \tilde{f}_{1,1}, \cdots$, are holomorphic in the variables $(z_2, \cdots, z_n)$ and the function $\tilde{f}_{0,1}$ is not identically equal to zero. The exponent $\tilde{\lambda}_1$ satisfies the relation: $\log \tilde{\lambda}_1 = \rho_1$. Let $\mathfrak{D}_1$ be the divisor defined by

$$\mathfrak{D}_1: \ \tilde{f}_{0,1} + \tilde{f}_{1,1}z_1 + \cdots + \tilde{f}_{m,1}z_1^m + \cdots = 0. \tag{5.19}$$

Let $\tilde{\mathfrak{D}}_1$ be the subdivisor of the divisor $\mathfrak{D}$ defined by $\tilde{\mathfrak{D}}_1 = \mathfrak{D} \cap \mathfrak{D}_1$. Take a point P in the divisor $\mathfrak{D}$ which does not belong to the subdivisor $\tilde{\mathfrak{D}}_1$. Take a sufficiently small neighbourhood U_P of the point P in the space of the variables $(z_1, \cdots, z_n)$ such that the neighbourhood U_P has no common point with the divisor $\tilde{\mathfrak{D}}_1$. For such a neighbourhood U_P, we define a linear space $\mathfrak{G}_P$ by putting $\mathfrak{G}_P=\left\{\tilde{g}=\frac{\partial}{\partial z_1}(f/\tilde{f})\,;\, f\in\mathfrak{F}\right\}$. Define an ordinary differential equation (of order $n-1$ depending on parameters $z_2, \cdots, z_n$)

$$\sum_{q=0}^{N-1}\tilde{R}_q\cdot\frac{d^q\cdot\tilde{g}}{dz_1^q}=0\,, \tag{5.20}$$

where the coefficients $\tilde{R}_q$ are defined by

$$\tilde{R}_q=\binom{N}{N-(q+1)}\cdot\frac{d^{N-(q+1)}\tilde{f}}{dz_1^{N-(q+1)}}-\sum_{i=q+1}^{N-1}\binom{i}{i-(q+1)}\cdot P_i\cdot\frac{d^{i-(q+1)}\tilde{f}}{dz_1^{i-q-1}}\,. \tag{5.21}$$

Then the linear space of the solutions of the equation (5.20) coincides with $\mathfrak{G}_P$. By the assumption $(\tilde{R})$, the meromorphic function $\tilde{R}_q/\tilde{R}_{N-1}$ is expanded at the point $P=(0, z_2(p), \cdots, z_n(p))$ in the following form

$$\tilde{R}_q/\tilde{R}_{N-1}=\frac{1}{z_1^{N-1-q}}\times\{\tilde{r}_0^q+\tilde{r}_1^q z_1+\tilde{r}_2^q z_1^2+\cdots\}\,, \qquad (q=0, 1, \cdots, N-2)\,, \tag{5.22}$$

where the coefficients $\tilde{r}_0^q, \tilde{r}_1^q, \tilde{r}_2^q, \cdots$ are holomorphic in the variables $(z_2-z_2(p)), \cdots, (z_n-z_n(p))$. Especially the first term $\tilde{r}_0^q$ has the following form:

$$\begin{aligned}\tilde{r}_0^q=&\binom{N}{N-(q+1)}\cdot\rho\cdot(\rho-1)\cdots(\rho-(N-q-2))\\ &-\sum_{i=q+1}^{N-1}\binom{i}{i-(q+1)}\cdot P_{i,-(N-i)}\rho(\rho-1)\cdots(\rho-(i-q-2))\,,\\ &\qquad (q=0, \cdots, N-2)\,.\end{aligned} \tag{5.23}$$

On the other hand, by the assumption of the induction we know that the terms $\tilde{r}_0^q$ are independent of the variables $(z_2, \cdots, z_n)$. Hence we conclude that the functions $P_{N-1,-1}, \cdots, P_{1,-(N-1)}$ are independent of the variables $(z_2, \cdots, z_n)$. Finally by the equations (5.4), we see that $P_{0,-N}$ is also independent of the variables $(z_2, \cdots, z_n)$. Thus we conclude that the functions $P_{N-1,-1}, \cdots, P_{1,-(N-1)}, P_{0,-N}$ are constant on the divisor $\mathfrak{D}$, while the functions $P_{N-1,-1}, \cdots, P_{1,-(N-1)}, P_{0,-N}$ are holomorphic on the divisor $\mathfrak{D}$. q. e. d.

By means of this proposition we can speak of 'the characteristic roots' of the determining equations of the family of the differential equations $D_{(z_2,\cdots,z_n)}$ depending on the parameters $(z_2, \cdots, z_n)$. Denote by μ_1 the root of the determining equation (5.8) whose real part $\mathrm{Re}\,\mu_1$ is maximal among the roots of the equation (5.8). Take a non-vanishing holomorphic function $h_{0,0}$ on the divisor $\mathfrak{D}$ and define in a usual manner holomorphic functions $h_{1,0}(z_2,$

$\cdots, z_n), \cdots, h_{n,0}(z_2, \cdots, z_n)$ by

$$(5.24)\quad \{\mu_1(\mu_1-1)\cdots(\mu_1-(n-1))-\mu_1(\mu_1-1)\cdots(\mu_1-(n-2))P_{N-1,-1}\cdots P_{0,-N}\}h_{1,0}$$
$$=\{\mu_1(\mu_1-1)\cdots(\mu_1-(n-2))P_{N-1,0}+\mu_1(\mu_1-1)\cdots(\mu_1-(n-3))P_{N-2,-1}$$
$$+\cdots+h_0\cdot P_{0,-(n-1)}\}h_{0,0},\cdots.$$

Define a holomorphic function H_0 (of the variables $(z_1, \cdots, z_n)$) by $H_0=(h_{0,0}+h_{1,0}z_1+\cdots+h_{n,0}z_1^n+\cdots)$ and put $y_1(z_1, \cdots, z_n)=z_1^{\mu_1}\cdot H_0(z_1; z_2, \cdots, z_n)$. Then the function $y_1(z_1; z_2, \cdots, z_n)$ is a solution of the differential equation $D_{(z_2,\cdots,z_n)}$. Define a differential equation $D^{(1)}_{(z_2,\cdots,z_n)}$ by

$$(5.25)\qquad D^{(1)}_{(z_2,\cdots,z_n)}:\ \sum_{q=0}^{N-1}P_q^{(1)}\cdot\frac{d^qf_1}{dz_1^q}=0,$$

where

$$P_q^{(1)}=\binom{N}{N-(q+1)}\cdot\frac{d^{n-(q+1)}y_1}{dz_1^{n-(q+1)}}-\sum_{i=q+1}^{N-1}\binom{i}{i-(q+1)}\cdot P_i\cdot\frac{d^{i-(q+1)}y_1}{dz_1^{i-(q+1)}}.$$

Then we infer readily that the determining equation of the differential equation (2.25) does not depend on the parameters $(z_2, \cdots, z_n)$. Repeating these procedures, we obtain a series consisting of N holomorphic functions $H_0, H_1, H_2, \cdots, H_{N-1}$ such that

$$(5.26)\qquad H_i=(h_{0,i}+h_{1,i}z_1+h_{2,i}z_1^2+\cdots+h_{m,i}z_1^m+\cdots)$$

where the functions $h_{0,i}, h_{1,i}, \cdots, h_{m,i}$ are holomorphic in the variables $(z_2, \cdots, z_n)$ and $h_{0,i}(0)\neq 0$. Moreover we obtain a series of exponents $\mu_1, \cdots, \mu_N$ such that the functions

$$(5.27)\qquad \begin{aligned} y_1(z_1; z_2, \cdots, z_n)&=z_1^{\mu_1}\cdot H_1,\\ y_2(z_1; z_2, \cdots, z_n)&=y_1\cdot\int z_1^{\mu_2}\cdot H_2dz_1,\\ &\cdots\cdots\cdots\cdots,\\ y_N(z_1; z_2, \cdots, z_n)&=y_{N-1}\cdot\int z_1^{\mu_N}\cdot H_Ndz_1 \end{aligned}$$

form a base of the space of the solutions of the ordinary differential equation $D_{(z_2,\cdots,z_n)}$. Now, for the base $F_1, \cdots, F_N$ of the linear space $\mathfrak{F}$, the functions $F_1(z_1; z_2, \cdots, z_n), \cdots, F_N(z_1; z_2, \cdots, z_n)$ form a base of the space of the solutions of the equations $D_{(z_2,\cdots,z_n)}$. Thus there is a $N\times N$ matrix $C_{(z_2,\cdots,z_n)}$ depending on the parameters $(z_2, \cdots, z_n)$ such that

$$(5.28)\qquad C_{(z_2,\cdots,z_n)}\begin{bmatrix} y_1\\ \vdots\\ y_N\end{bmatrix}=\begin{bmatrix} F_1\\ \vdots\\ F_N\end{bmatrix}$$

holds. The matrix $C_{(z_2,\cdots,z_n)}$ are determined by

$$(5.29)\qquad C_{(z_2,\cdots,z_n)}\cdot\begin{bmatrix} y_1,\ \partial y_1/\partial z_1,\ \cdots,\ \partial^{N-1}y_1/\partial z_1^{N-1} \\ \vdots \\ y_N,\ \partial y_N/\partial z_1,\ \cdots,\ \partial^{N-1}y_N/\partial z_1^{N-1}\end{bmatrix} = \begin{bmatrix} F_1,\ \partial F_1/\partial z_1,\ \cdots,\ \partial^{N-1}F_1/\partial z_1^{N-1} \\ \vdots \\ F_N,\ \partial F_N/\partial z_1,\ \cdots,\ \partial^{N-1}F_N/\partial z_1^{N-1}\end{bmatrix}.^{7)}$$

By the equation (5.29), the entries $c_{j,k}(z_2, \cdots, z_n)$ of the matrix $C_{(z_2,\cdots,z_n)}$ are holomorphic in $(z_2, \cdots, z_n)$. Thus for any function $f(z_1, z_2, \cdots, z_n)$ in the space $\mathfrak{F}$, we can find a set of holomorphic functions $d_1(z_2, \cdots, z_n), \cdots, d_N(z_2, \cdots, z_n)$ in the parameters $(z_2, \cdots, z_n)$ such that

$$(5.30)\qquad f(z_1;\ z_2, \cdots, z_n)=\sum_{j=1}^{N} d_j(z_2, \cdots, z_n)\cdot y_j\,.$$

On the other hand, there exists a positive number $\widetilde{L}$ such that

$$(5.31)\qquad \lim_{\substack{|z_1|\to 0\\ \arg z_1:\ \text{bounded}}} |z_1|^{\widetilde{L}}\,|y_j|=0 \qquad (j=1, \cdots, N)$$

hold uniformly in $|z_2|<\tilde{\varepsilon}_2, \cdots, |z_n|<\tilde{\varepsilon}_n$, where $\tilde{\varepsilon}_2, \cdots, \tilde{\varepsilon}_N$ are sufficiently small positive numbers.

We reformulate these results briefly in the following form:

LEMMA 5.1. *If the coefficients P_j defined by* (5.3) *of the differential equation $D_{(z_2,\cdots,z_n)}$ satisfy the condition* ($\tilde{R}$), *then the linear space $\mathfrak{F}$ is 'regular' in the sense that it satisfies the condition* (R).

REMARK. In a recent result of P. A. Griffiths, it has been discussed that the period functions of the rational forms are 'regular' in the sense that they satisfy the condition (R). The above criterion of 'regularity' is inspired by his result.

Now we examine what conditions are imposed on the coefficients $A_j^{(i_1,\cdots,i_N)}$, $A_{l_i m_i}^{(i_1,\cdots,i_N)}$ of the systems of partial differential equations (4.4) by the assumption (5.2). We assume that the singular locus is defined by the equation $z_1=0$ and for a certain value i_0, the pair of index $(1, i_0)$ is contained in the set $\mathfrak{p}=\{(l_1, m_1), \cdots\}$ (cf. § 4). Then we can express the derivatives $\partial^k F/\partial z_1^k$ $(k=2, \cdots)$ in the following form:

$$\frac{\partial^k F}{\partial z_1^k}=\sum_i B_{l_i,m_i}^{(k)}\cdot\frac{\partial^2 F}{\partial z_{l_i}\partial z_{m_i}}+\sum_j B_j^k\cdot\frac{\partial F}{\partial z_j}+B_0^k F\,.$$

The coefficients $B_{l_i,m_i}^{(k)}$, B_j^k, B_0^k satisfy the following recursion formulas:

7) Note that the matrix $C_{(z_2,\cdots,z_n)}$ does not depend on the variable z_1.

$$(5.32)_{(l_i,m_i)} \qquad B^{(k+1)}_{l_i,m_i} = \frac{\partial B^{(k)}_{l_i,m_i}}{\partial z_1} + \sum_j B^k_{l_j,m_j} \cdot A^{(l_j,m_j,1)}_{l_i,m_i} + \sum_p B^k_p \cdot A^{(1,p)}_{l_j,m_j},$$

$$(5.32)_{(i)} \qquad B^{(k+1)}_i = \sum_j B^k_{l_j,m_j} \cdot A^{(l_j,m_j,1)}_i + \frac{\partial B^{(k)}_i}{\partial z_1} + \sum_p B^k_p \cdot A^{(1,p)}_i \qquad (i \neq 0, 1),$$

$$(5.32)_{(1)} \qquad B^{(k+1)}_1 = \sum_j B^k_{l_j,m_j} \cdot A^{(l_j,m_j,1)}_1 + \frac{\partial B^{(k)}_1}{\partial z_1} + \sum_p B^k_p \cdot A^{(1,p)}_1 + B^k_0,$$

$$(5.32)_{(0)} \qquad B^{(k+1)}_0 = \sum_j B^k_{l_j,m_j} \cdot A^{(l_j,m_j,1)}_0 + \frac{\partial B^k_0}{\partial z_1} + \sum_p B^k_p \cdot A^{(1,p)}_0.$$

The expressions $A^{(l_j,m_j,1)}_{l_i,m_i}, \cdots$ are determined by

$$(5.33) \qquad \partial^3 F/\partial z_{l_j} \partial z_{m_j} \partial z_1 = \sum_i A^{(l_j,m_j,1)}_{l_i,m_i} \cdot \frac{\partial^2 F}{\partial z_{l_i} \partial z_{m_i}} + \sum_p A^{(l_j,m_j,1)}_p \cdot \frac{\partial F}{\partial z_p} + A^{(l_j,m_j,1)}_0 F.$$

Moreover the coefficients $A^{(1,p)}_{l_i,m_i}, \cdots$ are determined by

$$(5.34) \qquad \frac{\partial^2 F}{\partial z_1 \partial z_p} = \sum_i A^{(1,p)}_{l_i,m_i} \frac{\partial^2 F}{\partial z_{l_i} \partial z_{m_i}} + \sum_q A^{(1,p)}_q \cdot \frac{\partial F}{\partial z_q} + A^{(1,0)}_0 F.$$

Besides the matrix $W_{(z_2,\cdots,z_n)}$, we define matrices $W_0, \cdots, W_{N-1}$ by

$$(5.35) \qquad W_0 = \begin{bmatrix} \partial^N F_1/\partial z_1^N, & \cdots, & \partial^N F_N/\partial z_1^N \\ \partial F_1/\partial z_1, & \cdots, & \partial F_N/\partial z_1 \\ \cdots & \cdots & \cdots \\ \partial^{N-1} F_1/\partial z_1^{N-1}, & \cdots, & \partial^{N-1} F_N/\partial z_1^{N-1} \end{bmatrix},$$

$$\cdots, W_{N-1} = \begin{bmatrix} F_1, & \cdots, & F_N \\ \partial F_1/\partial z_1, & \cdots, & \partial F_N/\partial z_1 \\ \cdots & \cdots & \cdots \\ \partial^{N-2} F_1/\partial z_1^{N-2}, & \cdots, & \partial^{N-2} F_N/\partial z_1^{N-2} \\ \partial^N F_1/\partial z_1^N, & \cdots, & \partial^N F_N/\partial z_1^N \end{bmatrix}.$$

Then we can express the matrices $W, W_0, \cdots, W_{N-1}$ in the following form:

$$(5.36) \qquad W = B \cdot F(z),\ W_0 = B_0 \cdot F(z),\ \cdots,\ W_{N-1} = B_{N-1} \cdot F(z),$$

where the matrix $F(z)$ is defined by the equation (4.1) and $B, B_0, \cdots, B_{N-1}$ are defined as follows:

$$(5.37) \qquad B = \begin{bmatrix} B^{(0)}_{l_1,m_1}, & \cdots, & B^{(0)}_1, & \cdots, & B^{(0)}_0 \\ & \cdots & & \cdots & \\ B^{(N-1)}_{l_1,m_1}, & \cdots, & B^{(N-1)}_1, & \cdots, & B^{(N-1)}_0 \end{bmatrix},$$

$$B_0=\begin{bmatrix} B_{l_1,m_1}^{(N)}, \cdots, B_1^{(N)}, \cdots, B_n^{(N)}, B_0^{(N)} \\ B_{l_1,m_1}^{(1)}, \cdots, B_1^{(1)}, \cdots, B_n^{(1)}, B_0^{(1)} \\ \cdots \quad \cdots \\ B_{l_1,m_1}^{(N-1)}, \cdots, B_1^{(1)}, \cdots, B_1^{(1)}, B_0^{(1)} \end{bmatrix},$$

$$\vdots$$

$$B_{N-1}=\begin{bmatrix} B_{l_1,m_1}^{(0)}, \cdots, B_1^{(0)}, \cdots, B_0^{(0)} \\ \cdots \quad \cdots \\ B_{l_1,m_1}^{(N-2)}, \cdots, B_1^{(N-2)}, \cdots, B_0^{(N-2)} \\ B_{l_1,m_1}^{(N)}, \cdots, B_1^{(N)}, \cdots, B_0^{(N-2)} \end{bmatrix}.$$

Thus the coefficients $P_{N-1}, \cdots, P_0$ are expressed as follows:

(5.38) $$P_{N-1}=\det W_{N-1}/\det W=\det B_{N-1}/\det B,$$
$$\cdots\cdots\cdots\cdots,$$
$$P_0=\det W_0/\det W=\det B_0/\det B.$$

Now we give an example in a simple case, namely, in the case in which $\dim \mathfrak{F}=4$ and the number of the variables are two. In this case, the system of differential equations (4.4) are expressed in the form:

(5.39) $$\frac{\partial^2 F}{\partial z_i^2}=A_{1,2}^{(i,i)}\cdot\frac{\partial^2 F}{\partial z_1\partial z_2}+\sum_{j=1}^{2}A_j^{(i,i)}\cdot\frac{\partial F}{\partial z_j}+A_0^{(i,i)}\cdot F.$$

LEMMA 5.2. *For any positive integer m_0, there is a system of differential equations $\mathfrak{D}_{m_0}$ of type* (5.39) *satisfying the following two conditions:*

(5.40) *The space $\mathfrak{F}_{m_0}$ of the solutions of the equation D_{m_0} is 'regular' in the sense that the condition* (R) *is satisfied.*

(5.41) *The coefficient $A_{1,2}^{(1,1)}$ of the equation D_{m_0} is expressed in the form $A_{1,2}^{(1,1)}=\dfrac{a_0+a_1z+a_2z^2+\cdots}{z_1^{m_0}}$, where $a_i(z_2)$ are holomorphic in z_2, and $a_0(0)\neq 0$.*

PROOF. Take four holomorphic functions f_1, f_2, f_3, f_4 in the following manner:

$(5.42)_1$ $$f_1=1, \quad f_2=z_2,$$

$(5.42)_2$ $$f_3=z_1^{\rho_3}(f_{03}+f_{13}z_1+\cdots+f_{m_03}z_1^{m_0}+\cdots)$$
$$f_4=z_1^{\rho_4}(f_{04}+f_{14}z_1+\cdots+f_{m_04}z_1^{m_0}+\cdots).$$

Now we impose the following conditions on the functions f_3 and f_4:

$(5.43)_1$ $$\rho_3\rho_4(\rho_3-\rho_4)\neq 0, \quad f_{03}\cdot f_{04}\neq 0,$$

$(5.43)_2$ $f_{03}, \cdots, f_{m_0-1,3}$ and $f_{04}, \cdots, f_{m_0-1,4}$ are constants,

$$(5.43)_3 \qquad \frac{\partial f_{m03}}{\partial z_2} \cdot f_{m04} - \frac{\partial f_{m04}}{\partial z_2} \cdot f_{m03} \neq 0 \quad \text{at} \quad z_2 = 0 .$$

Then the coefficients $A_{1,2}^{(1,1)}$, $A_j^{(1,1)}$ $(j=1, 2)$, $A_0^{(1,1)}$ are determined by

$$(5.44) \qquad \begin{bmatrix} \partial^2 F_1/\partial z_1^2 \\ \partial^2 F_2/\partial z_1^2 \\ \partial^2 F_3/\partial z_1^2 \\ \partial^2 F_4/\partial z_1^2 \end{bmatrix} = \begin{bmatrix} \partial^2 F_1/\partial z_1 \partial z_2, \partial F_1/\partial z_1, \partial F_1/\partial z_2, F_1 \\ \vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ \vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ \partial^2 F_4/\partial z_1 \partial z_2, \partial F_4/\partial z_1, \partial F_4/\partial z_2, F_4 \end{bmatrix} \begin{bmatrix} A_{1,2}^{(1,1)} \\ A_1^{(1,1)} \\ A_2^{(1,1)} \\ A_0^{(1,1)} \end{bmatrix} .$$

The assertion of the theorem is easily verified by the relation (5.44).

REMARK. The example shows that in order to determine the coefficients of a differential equation such as (4.4), some other informations about the behavior of the functions are needed. We shall examine the behavior in the following sections.

§ 6. A bilinear equality between rational 2-forms (without residues) on the surface $B_{(\tau,\sigma)}$.

We begin with a general situation. Let V^n be a compact complex manifold of complex dimension n and let W^{n-1} be a non-singular divisor of V^n. Fix a positive definite Hermitian metric ds^2 on the manifold V^n. For any sufficiently small positive number ε, denote the tubular neighbourhood of radius ε by $N_\varepsilon(W^{n-1})$. Denote $N_\varepsilon(W^{n-1}) - W^{n-1}$ by N_ε. Let ω^n be a meromorphic n-form on the manifold V^n whose polar locus is the divisor W^{n-1}. We assume that ω^n has no residue around W^{n-1}, i.e., $\int_{\gamma_n} \omega^n = 0$ for any n-cycle contained in the open manifold N_ε. By de Rham's theorem n-form ω^n is a derived form considered as a C^∞-differentiable form in N_ε. Take a C^∞-differentiable $(n-1)$-form Ψ^{n-1} in N_ε such that

$$(6.1) \qquad d\Psi^{n-1} = \omega^n , \qquad \text{in} \quad N_\varepsilon .$$

Take a series of positive numbers $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < \varepsilon$. Define a C^∞-differentiable function χ in N_ε such that

$$(6.2) \qquad \begin{cases} \chi = 1 & \text{in} \quad N_\varepsilon - N_{\varepsilon 2} \\ \chi = 0 & \text{in} \quad N_{\varepsilon 1} . \end{cases}$$

Define a C^∞-differentiable $(n-1)$-form $\tilde{\Psi}$ by $\tilde{\Psi} = \chi \cdot \Psi^{n-1}$. Then we have $\omega^n = d\tilde{\Psi}$ in $N_\varepsilon - N_{\varepsilon_2}$. Now we define a closed C^∞-differentiable n-form $\tilde{\omega}$ by

$$(6.3) \qquad \begin{cases} \tilde{\omega} = \omega & \text{in} \quad V^n - N_{\varepsilon_3} \\ \tilde{\omega} = d\tilde{\Psi} & \text{in} \quad N_{\varepsilon_4} . \end{cases}$$

Recall the following exact sequence of homology groups (Hodge-Atiyah [5], loc. cit.)

$$(6.4)\quad \cdots \longrightarrow H_{n-1}(W^{n-1}) \xrightarrow{\tau} H_n(V^n - W^{n-1}) \xrightarrow{i_*} H_n(V^n) \xrightarrow{c} H_{n-2}(W^{n-1}) \longrightarrow \cdots.$$

Choose a base $\{\gamma_1, \cdots, \gamma_\beta, \gamma_{\beta+1}, \cdots, \gamma_{b_n}\}$ of the homology group $H_n(V^n)$ such that $\gamma_1, \cdots, \gamma_\beta$ form a base of the subspace $i_*(H_n(V^n - W^{n-1}))$. Denote by I the intersection matrix with respect to the base $\{\gamma_1, \cdots, \gamma_{b_n}\}$. Let ω_1^n, ω_2^n be two meromorphic n-forms having the divisor W^{n-1} as polar loci. We assume ω_1^n and ω_2^n to have no residues along W^{n-1}. Denote by $\tilde{\omega}_i^n$ $(i=1, 2)$ the corresponding C^∞-differentiable n-forms in the construction (6.1)—(6.3). Denote by $\omega_{i,j}$ $(i=1, 2, j=1, \cdots, b_n)$ the periods of the n-form ω_i^n on the homology classes γ_j $(j=1, \cdots, \beta)$ and by $\tilde{\omega}_{i,j}$ $(i=1, 2, j=1, \cdots, b_n)$ the periods of the n-form $\tilde{\omega}_i^n$ on the homology classes γ_j $(j=1, \cdots, b_n)$. Define the vector $\tilde{\eta}_i$ by $\tilde{\eta}_i = (\tilde{\omega}_{i,1}, \cdots, \tilde{\omega}_{i,b_n})$. We note that

$$(6.5)\qquad \omega_{i,j} = \tilde{\omega}_{i,j} \qquad (j=1, \cdots, \beta).$$

The well-known bilinear equality asserts that

$$(6.6)\qquad \tilde{\eta}_1 \cdot {}^tI^{-1} \cdot \tilde{\eta}_2 = \int_{V^n} \tilde{\omega}_1^n \wedge \tilde{\omega}_2^n.$$

From the equation (6.6), we obtain

$$(6.7)\qquad \tilde{\eta}_1 \cdot {}^tI^{-1} \cdot \tilde{\eta}_2 = \int_{N_\varepsilon} \tilde{\omega}_1^n \wedge \tilde{\omega}_2^n = \int_{N_\varepsilon} (d\Psi_1^n) \wedge (d\Psi_2^n).$$

REMARK. In the case of an algebraic curve, we obtain the following well-known relation from the formulas (6.1)—(6.7) in the following manner. Let $P_1, \cdots, P_N$ be points of the algebraic curve V^1 and let ω_1', ω_2' be two rational 1-forms on the curve V^1 which are regular in $V^1 - \bigcup_{j=1}^{N} P_j$. For each point P_j, let $\omega_i = \left(\sum_{q=q_0}^{\infty} a_{j,q}^{(i)} \cdot t_j^q\right) dt_j$ $(i=1, 2)$ be the expansion of the 1-forms ω_i in terms of the local coordinate t_j at the point P_j. From the assumption we have $a_{j,-1}^{(i)} = 0$. The integral $\Psi_{i,j} = \int \omega_i$ of the 1-form on $N_\varepsilon(P_j)$ is $\Psi_{i,j} = \sum_{q=q_0} a_{j,q}^{(i)}/q+1 \cdot t_j^{q+1}$. Obviously, we have $d\Psi_{i,j} = \omega_j$ in $N_\varepsilon(P_j)$. Choosing a suitable C^∞-function χ_j in $N_\varepsilon(P_j)$ satisfying the equations corresponding to (6.2), we define 1-forms $\tilde{\omega}_j$ $(j=1, 2)$ as above. Then the expression:

$$\begin{aligned}\int_{N_\varepsilon} (d\tilde{\Psi}_1^m) \wedge (d\tilde{\Psi}_2^n) &= \left(\int_{\tilde{N}_\varepsilon(P_j)} d(\tilde{\Psi}_1^m \wedge d\tilde{\Psi}_2^n) = \int_{\partial \tilde{N}_\varepsilon} \Psi_1^n \wedge \omega_2\right) \\ &= \int_{|t_j|=\varepsilon} \left(\sum_{q=q_0}^{\infty} \frac{a_{j,q}^{(1)}}{q+1} t_j^{q+1}\right) \cdot \left(\sum_{q=q_0}^{\infty} a_{j,q}^{(2)} t_j^{q+1}\right) dt \\ &= 2\pi\sqrt{-1} \sum_{q=q_0} \frac{a_{j,q}^{(1)}}{q+1} \times a_{j,-(q+2)}^{(2)}.\end{aligned}$$

Thus, for a Betti base $\{\gamma_1, \cdots, \gamma_{b_1}\}$ of V^1 and the corresponding intersection matrix I, we obtain

$$(\gamma_1)^t I^{-1} \cdot {}^t(\gamma_2) = \sum_{j=1}^{N} 2\pi\sqrt{-1} \cdot \sum_q \frac{a_{j,q}^{(1)} a_{j,-(q+2)}^{(2)}}{q+1}$$

which is an well-known formula in the theory of algebraic functions of one variable.

Now we consider the surface $B = B(\tau, \sigma)$ and the curve $X = X(\tau, \sigma)$ introduced in § 0. We have the exact sequence corresponding to (6.3):

$$\text{(6.3)}' \qquad \cdots \longrightarrow H_1(X) \xrightarrow{\tau} H_2(B-X) \xrightarrow{i_*} H_2(B) \xrightarrow{c} H_0(X) \longrightarrow 0 .$$

We know that the dimension of the image: $i_* H_2(B-X)$ is $b_2(B)-1$.

For 2-cycles $\tilde{\gamma}_1, \cdots, \tilde{\gamma}_{b_1-2}$ which represent, together with the algebraic cycle $\varDelta$, a base of the subgroup $i_*(H_2(B-X))$, we define the intersection matrix I_k'. We note that the matrix I_k' is non-singular. Let ω_1 and ω_2 be two 2-forms whose polar loci are the divisor X. Then we have the following bilinear equality:

$$\text{(6.8)} \qquad \tilde{\eta}_1 {}^t I_{k'}^{-1} {}^t \tilde{\eta}_2 = \int_{N_\varepsilon(X)} d\tilde{\Psi}_1 \wedge d\tilde{\Psi}_2 ,$$

where the vectors $\tilde{\eta}_i$ $(i=1, 2)$ and the 1-forms $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ have the same meaning as before. Our task is to express the integral $\int_{N_\varepsilon(X)} d\tilde{\Psi}_1 \wedge d\tilde{\Psi}_2$ in terms of the data of rational 2-forms ω_j $(j=1, 2)$ at the singular locus $X = X_{(\tau,\sigma)}$.

Define a divisor T on the surface B by

$$\text{(6.9)} \qquad \begin{aligned} &12x^2 - g(u) = 0, &&\text{in } \mathfrak{A}_u , \\ &12(x')^2 - g_1(v) = 0, &&\text{in } \mathfrak{A}_v , \end{aligned}$$

where we put $\mathfrak{A}_u = W_1 \cap B$, $\mathfrak{A}_v = W_2 \cap B$ and $x' = x/u^{2k}$, $g_1(v) = g\left(\frac{1}{v}\right) \cdot v^{4k}$. Define a regular function t in $\mathfrak{A}_u$ by $t = 12x^2 - g(u)$. Hereafter we assume that (6.10) the curves X and T intersect transversely at the points $p_j \in X \cap T$ $(j=1, 2, \cdots, 12k)$[8]. Then, we can choose local coordinates of the surface B in the tubular neighbourhood $N_\varepsilon(X)$ in the following manner:

For points $p \notin T$, we choose (u, y) as local coordinates.

For points $p \in T$, we choose (t, y) as local coordinates.

By a simple calculation, we have the following transformation law:

8) This condition is equivalent to the following two equivalent conditions.
(i) Every singular fibre has only one ordinary double point.
(ii) The discriminant D(u) has no multiple roots.

(6.11) $$\frac{\partial u}{\partial y}=-\frac{48xy}{S}, \qquad \frac{\partial u}{\partial t}=\frac{t}{S},$$

where $S=24x\left(\frac{\partial g}{\partial u}x+\frac{\partial h}{\partial u}\right)-\frac{\partial g}{\partial u}t$. S is a regular function in the space $\mathfrak{A}_u$. We note that the function S does not vanish at the points p_j $(\in X\cap T)$ under the assumption (6.10). Define three systems of algebraic curves $L_{\tilde{u}}$, $L_{\tilde{t}}$ and $L_{\tilde{y}}$ on B by

(6.12)$_{\tilde{u}}$ $$L_{\tilde{u}}: u=\tilde{u},$$

(6.12)$_{\tilde{t}}$ $$L_{\tilde{t}}: 12x^2-g(u)=\tilde{t},$$

(6.12)$_{\tilde{y}}$ $$L_{\tilde{y}}: y=\tilde{y}.$$

For any point $p\in X$, let $\tilde{t}(p)$ and $\tilde{u}(p)$ denote, respectively, the values at p of the rational functions t and u. For any point p, let $t(p)$ and $u(p)$ denote, respectively, the values at p of the rational functions t and u.

Let δ be a sufficiently small positive number and define open subsets $L_t(p)$ of $L_{t(p)}$ and $L_u(p)$ of $L_{u(p)}$ as follows:

(6.13)$_t$ $$L_t(p)=\{Q\in L_{t(p)}\,;\ \mathrm{dis}\,(Q,p)<\delta\},$$

(6.13)$_u$ $$L_u(p)=\{Q\in L_{u(p)}\,;\ \mathrm{dis}\,(Q,p)<\delta\}.$$

For the points $p_j\in X\cap T$ $(j=1,\cdots,12k)$, we take a small 'spherical' neighbourhood $\mathfrak{B}_l^{(j)}=\{p_j'\in X:\ \mathrm{dis}\,(p_j,p_j')<\lambda_l\}$ $(l=1,2,3,4;\ \lambda_1<\lambda_2<\lambda_3<\lambda_4)$ in X such that the following conditions are satisfied:

(6.14)$_{\tilde{u}}$ $L_{\tilde{u}(q_1)}\cap L_{\tilde{u}(q_2)}=\emptyset$ for all points $q_1, q_2\in X-\bigcup_{j=1}^{12k}\mathfrak{B}_1^{(j)}$, $q_1\neq q_2$,

(6.14)$_{\tilde{t}}$ $L_{\tilde{t}(q_1)}\cap L_{\tilde{t}(q_2)}=\emptyset$ for all points $q_1, q_2\in \bigcup_{j=1}^{12k}\mathfrak{B}_4^{(j)}$, $q_1\neq q_2$,

(6.14)$'$ The set $L_{\tilde{u}}(p,\delta)$, $p\in X-\bigcup_{j=1}^{12k}\mathfrak{B}_1^{(j)}$, and the set $L_{\tilde{t}}(p,\delta)$, $p\in\bigcup_{j=1}^{12k}\mathfrak{B}_4^{(j)}$, are simply connected.

Let $X_1=X-\bigcup_{j=1}\mathfrak{B}_X^i$ and define open connected sets N_0 and N_j $(j=1,\cdots,12k)$ in the surface B by

(6.15) $$N_0=\bigcup_{p\in X_1}L_{\tilde{u}(p)}, \qquad N_j=\bigcup_{p\in\mathfrak{B}_4^j}L_{\tilde{t}(p)}.$$

By the conditions (6.14)$_{\tilde{u}}$ and (6.14)$_{\tilde{t}}$, we can define a complex mapping $\mathrm{Proj}_u=Pr_u$ from the open set N_0 onto the set X_1 and $\mathrm{Proj}_t^{(j)}=Pr_t^{(j)}$ from the open set N_j onto $\mathfrak{B}_4^{(j)}$ by

(6.16)$_u$ $Pr_u(Q)=p$, where p is the point on X such that $L_u(p)$ contains the point Q $(\in N_0)$, (6.16)$_t$ $Pr_t^{(j)}(Q)=p$, where p is the point on X such that $L_t(p)$ contains the point Q $(\in N_j)$.

Take a sufficiently small positive number ε such that the following conditions are satisfied:

(6.17) The tubular neighbourhood $N_\varepsilon(X)$ of the curve X (with radius ε) is contained in the union $\bigcup_{j=1}^{12k} N_j \cup N_0$.

(6.18) The tubular neighbourhood $N_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$ of $\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)}$ is contained in the intersection:

$$\{\bigcup_{p\in\mathfrak{B}_4^{(j)}-\mathfrak{B}_1^{(j)}} L_u(p)\}\cap\{\bigcup_{p\in\mathfrak{B}_4^{(j)}-\mathfrak{B}_1^{(j)}} L_t(p)\}.$$

For the rational 2-form $\omega^{(p,q,2r+1)}=(u^p x^q/y^{2r+1})dx\wedge du$, we have the following transformation law:

(6.19) $$\omega^{(p,q,2r+1)}=(u^p x^q y^{2r} t)dy\wedge du=(u^p x^q/y^{2r}S)dy\wedge dt.$$

Define holomorphic 1-forms $\Psi_0^{(p,q,2r+1)}$ in the neighbourhood $\hat{N}_0$[9] and $\Psi_j^{(p,q,2r+1)}$ in the neighbourhood $\hat{N}_j$ by

(6.20)$_0$ $$\Psi_0^{(p,q,2r+1)}=\Big(\int^{(u,y)} u^p x^q/y^{2r}t\, dy\Big)du,$$

(6.20)$_j$ $$\Psi_j^{(p,q,2r+1)}=\Big(\int^{(t,y)} u^p x^q/y^{2r}S\, dy\Big)dt,$$

where the first and the second integrations are extended over differentiable paths in $L_u(p)$ and $L_t(p)$, respectively. Then, obviously, we have

(6.21) $$d\Psi_0^{(p,q,2r+1)}=\omega^{(p,q,2r+1)} \text{ in the neighbourhood } N_0,$$

(6.22) $$d\Psi_j^{(p,q,2r+1)}=\omega^{(p,q,2r+1)} \text{ in the neighbourhood } N_j.$$

We note that the 1-forms $\Psi_0^{(p,q,2r+1)}$ and $\Psi_j^{(p,q,2r+1)}$ are *single-valued* in view of (6.20)$_0$ and (6.20)$_j$. We know by (6.21) and (6.22) that the 1-form $\Psi_0-\Psi_j$ is a closed holomorphic 1-form in the neighborhood $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$.[10]

Now we prove the following

PROPOSITION 6.1. *The* 1-*form* $\Psi_0-\Psi_j$ *is a derived* 1-*form.*

PROOF. The one dimensional homology group $H_1(\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)});\boldsymbol{Z})$ of the neighbourhood $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$ is isomorphic to the direct sum $\boldsymbol{Z}\oplus\boldsymbol{Z}$. Define two closed arcs $\gamma_l^{(j)}$ $(l=1, 2)$ in the neighbourhood $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$ in the following manner:

---

9) $\hat{N}_0=N_0-X$, $\quad \hat{N}_j=N_j-X$,

10) $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})=N_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})-X$.

(6.23)$_1$ $\quad \gamma_1^{(j)}:\ y=y_1^{(j)},\quad u=u(t_0e^{2\pi\sqrt{-1}\theta},0)\quad (1\geqq\theta\geqq 0)$,

(6.23)$_2$ $\quad \gamma_2^{(j)}:\ u=u_0^{(j)},\quad y=|y_2|\cdot e^{2\pi\sqrt{-1}\theta}\quad (1\geqq\theta\geqq 0)$,

where the numbers u_0, y_1, y_2, t_0 are chosen such that

(6.24)$_1$ the C^∞-differentiable curve $\gamma_1^{(j)}$ is contained in the neighbourhood $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$,

(6.24)$_2$ the C^∞-differentiable curve $\gamma_2^{(j)}$ is contained in the neighbourhood $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$,

and where $u(t_0e^{2\pi\sqrt{-1}\theta},0)$ denotes the value at the point $(t_0e^{2\pi\sqrt{-1}\theta},0)$ of the rational function $u(t,y)$ of (t,y).

Now the 1-form $u^px^q/y^{2r}t\,du$ on the algebraic curve $L_{\tilde{y}}$ is regular in the neighbourhood of the point $p_y:(y,t)=(\tilde{y},0)$. So, we have $\int_{\gamma_{1,y}^{(j)}}u^px^q/y^{2r}t\,du=0$ for all the values $\tilde{y}$ where $\gamma_{1,\tilde{y}}^{(j)}$ denotes the 1-cycle: $y=\tilde{y}$, $u=u(t_0e^{2\pi\sqrt{-1}\theta},0)$ $(1\geqq\theta\geqq 0)$. Consequently, we obtain $\int_{\gamma_1^{(j)}}\Psi_0^{(p,q,2r+1)}=0$.

We note that the projection Pr_t induces a biregular isomorphism $Pr_y^{(j)}$ from the neighbourhood $\mathfrak{B}(p_y,\lambda)$ of the algebraic curve $L_{\tilde{y}}$ onto X $(=L_0)$. The closed C^∞-differentiable curve $\gamma_1^{(j)}$ on the curve L_{y_1} is mapped (diffeomorphically) onto the C^∞-differentiable curve $\tilde{\gamma}_2^{(j)}$ on the algebraic curve X $(=L_0)$.

The integral $\int_{\gamma_2^{(j)}}\Psi_j^{(p,q,2r+1)}$ is expressed in the following manner:

$$\int_{\gamma_2^{(j)}}\Psi_j^{(p,q,2r+1)}=\int_{\gamma_2^{(j)}}\Big(\int^{(y_1,t)}u^px^q/y^{2r}S\,dy\Big)\,dt \tag{6.25}$$
$$=\int^{y_1}\Big(\int_{(\tilde{y},t)\in\gamma_{2,y}^{(j)}}u^px^q/y^{2r}S\,dt\Big)\,dy\,.$$

Because of the transformation laws (6.11) we conclude in a similar manner as above that $\int_{\gamma_2^{(j)}}\Psi_j=0$. q. e. d.

Put

$$f_j^{(p,q,2r+1)}=\int^{(t,y)}(\Psi_0^{(p,q,2r+1)}-\Psi_j^{(p,q,2r+1)})\,.$$

From Proposition 6.1, we know that f_j is a single valued holomorphic function in the neighbourhood $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$. Choose a series of positive numbers $\lambda_2<\sigma_1^{(j)}<\sigma_2^{(j)}<\sigma_3^{(j)}<\sigma_4^{(j)}<\sigma_5^{(j)}<\sigma_6^{(j)}<\lambda_3$. Define a C^∞-differentiable function χ_j in the neighbourhood $\hat{N}_\varepsilon(\mathfrak{B}_3^{(j)}-\mathfrak{B}_2^{(j)})$ such that

$$\begin{cases}\chi_j=1, \text{ in the neighbourhood } N_\varepsilon(\mathfrak{B}_{\sigma_5}^{(j)}-\mathfrak{B}_{\sigma_2}^{(j)}),\\ \chi_j=0, \text{ in the neighbourhood } N_\varepsilon(\mathfrak{B}_{\sigma_1}^{(j)}) \text{ and in the set } X-N_\varepsilon(\mathfrak{B}_{\sigma_6}^{(j)})\,.\end{cases} \tag{6.26}$$

Define a C^∞-differentiable 1-form $\tilde{\Psi}_j^{(p,q,2r+1)}$ in the neighbourhood $\hat{N}_\varepsilon(\mathfrak{V}_3^{(j)}-\mathfrak{V}_2^{(j)})$ by

(6.27) $$\tilde{\Psi}_j^{(p,q,2r+1)}=d(\chi\cdot f_j^{(p,q,2r+1)})$$

and define a C^∞-differentiable 1-form $\tilde{\Psi}^{(p,q,2r+1)}$ in the neighbourhood $\hat{N}_\varepsilon(X)$ by

(6.28) $$\begin{cases}\tilde{\Psi}^{(p,q,2r+1)}=\Psi_0^{(p,q,2r+1)}, \text{ in the set: } \hat{N}_\varepsilon(X)-\bigcup_{j=1}^{12k}\hat{N}_\varepsilon(\mathfrak{V}_{\sigma 3}^{(j)}),\\ \tilde{\Psi}^{(p,q,2r+1)}=\Psi_j^{(p,q,2r+1)}+\tilde{\Psi}_j^{(p,q,2r+1)}, \text{ in the set: } \hat{N}_\varepsilon(\mathfrak{V}_{\sigma_4}^{(j)}).\end{cases}$$

Then we have the following identity in the neighbourhood $\hat{N}_\varepsilon(X)$:

(6.29) $$d\tilde{\Psi}^{(p,q,2r+1)}=\omega^{(p,q,2r+1)}.$$

Take a series of positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$ such that $0<\varepsilon_1<\varepsilon_2<\varepsilon_3<\varepsilon_4<\varepsilon_5<\varepsilon_6$ and define a C^∞-function χ in the whole surface B by

(6.30) $$\begin{cases}\chi=1, \text{ in the open set } N_{\varepsilon_5}(X)-N_{\varepsilon_2}(X),\\ \chi=0, \text{ in the set } N_{\varepsilon_1}(X), \text{ and in the set } B-N_{\varepsilon_6}(X).\end{cases}$$

Finally, we define a closed C^∞-differentiable 2-form $\tilde{\omega}^{(p,q,2r+1)}$ by

(6.31) $$\begin{cases}\tilde{\omega}^{(p,q,2r+1)}=\omega^{(p,q,2r+1)} \text{ in the set } B-N_{\varepsilon_3}(X),\\ \tilde{\omega}^{(p,q,2r+1)}=d(\chi\cdot\tilde{\Psi}^{(p,q,2r+1)}) \text{ in the set } N_{\varepsilon_4}(X).\end{cases}$$

Let $\omega^{(p_i,q_i,2r_i+1)}$ $(i=1,2)$ be rational 2-forms on the surface B with polar locus X and let $\tilde{\omega}^{(p_i,q_i,2r_i+1)}$ $(i=1,2)$ be closed C^∞-differentiable 2-forms corresponding to the rational 2-forms $\omega^{(p_i,q_i,2r_i+1)}$ by means of the above construction. We mean by C^∞-differentiable 1-forms $\tilde{\Psi}^{(p_i,q_i,2r_i+1)}$ $(i=1,2)$ the 1-forms corresponding to the rational 2-forms $\omega^{(p_i,q_i,2r_i+1)}$ $(i=1,2)$ by the formula (6.28). For any tubular neighbourhood $N_\varepsilon(X)$ of X, we choose a C^∞-differentiable retract $\rho\colon N_\varepsilon(X)\to X$ and, for any subset $\mathfrak{V}$ of X, we define

$$\tau_\varepsilon(\mathfrak{V})=\rho^{-1}(\mathfrak{V})\cap\partial N_\varepsilon(X),$$

where $\partial N_\varepsilon(X)$ denotes the boundary of $N_\varepsilon(X)$. Then we have

(6.32) $$\begin{aligned}\int_B\tilde{\omega}^{(p_1,q_1,2r_1+1)}\wedge\tilde{\omega}^{(p_2,q_2,2r_2+1)}&=\int_{N_{\varepsilon_3}}d\tilde{\Psi}^{(p_1,q_1,2r_1+1)}\wedge d\tilde{\Psi}^{(p_2,q_2,2r_2+1)}\\&=\int_{\tau_{\varepsilon_3}(X)}\tilde{\Psi}^{(p_1,q_1,2r_1+1)}\wedge d\tilde{\Psi}^{(p_2,q_2,2r_2+1)}\\&=(-1)\int_{\tau_{\varepsilon_4}(X)}d\tilde{\Psi}^{(p_1,q_1,2r_1+1)}\wedge\tilde{\Psi}^{(p_2,q_2,2r_2+1)}.\end{aligned}$$

From the equations $(6.20)_0$, $(6.22)_j$, (6.28) and (6.30), we infer that $d\tilde{\Psi}^{(p_i,q_i,2r_i+1)}=\omega^{(p_i,q_i,2r_i+1)}$ $(i=1,2)$ at any point $Q\in\tau_{\varepsilon_3}(X)$. From the equations $(6.20)_{0,j}$

and (6.28) we see that $\tilde{\Psi}^{(p_i,q_i,2r_i+1)}=\Psi_0^{(p_i,q_i,2r_i+1)}$ $(i=1, 2)$ at any point $Q\in\tau_{\varepsilon_3}(X)\cap(N_\varepsilon(X)-N_\varepsilon(\mathfrak{B}(p_j;\sigma_3))$. So we have

$$\int_B \tilde{\omega}^{(p_1,q_1,2r_1+1)}\wedge\tilde{\omega}^{(p_2,q_2,2r_2+1)} = \sum_{j=1}^{12k}\int_{\tau_{\varepsilon_3}(\mathfrak{B}(p_j;\lambda_4))}\tilde{\Psi}^{(p_1,q_1,2r_1+1)}\wedge\omega^{(p_2,q_2,2r_2+1)}. \tag{6.33}$$

Finally, from the equations (6.28) and (6.30), we infer that $\tilde{\Psi}^{(p_i,q_i,2r_i+1)}=df_j+\Psi_j^{(p_i,q_i,2r_i+1)}$ $(i=1, 2)$ at any point $Q\in\tau_{\varepsilon_3}(\mathfrak{B}_4^{(j)})$. So we have

$$\begin{aligned}\sum_{j=1}^{12k}\int_{\tau_{\varepsilon_3}(\mathfrak{B}_4^{(j)})}&\tilde{\Psi}^{(p_1,q_1,2r_1+1)}\wedge\omega^{(p_2,q_2,2r_2+1)}\\ &=\sum_{j=1}^{12k}\int_{\tau_{\varepsilon_3}(\mathfrak{B}_4^{(j)})}df_j^{(p_1,q_1,2r_1+1)}\wedge\omega^{(p_2,q_2,2r_2+1)}\\ &=\sum_{j=1}^{12k}\int_{\tau_{\varepsilon_3}(\partial\mathfrak{B}_4^{(j)})}f_j^{(p_1,q_1,2r_1+1)}\omega^{(p_2,q_2,2r_2+1)}.\end{aligned}\tag{6.34}$$

Now we know that 2-form $f_j^{(p_1,q_1,2r_1+1)}\omega^{(p_2,q_2,2r_2+1)}$ is holomorphic in the neighbourhood $N_\varepsilon(\mathfrak{B}(p_j;\lambda_3)-\mathfrak{B}(p_j;\lambda_2))$ and $\tau_{\varepsilon_3}(\partial\mathfrak{B}(p_j;\lambda_4))$ forms a Betti base of two cycles on the neighbourhood $N_\varepsilon(\mathfrak{B}(p_j;\lambda_3)-\mathfrak{B}(p_j;\lambda_2))$. We summarize the above results in the following

LEMMA 6.1. *Let $\tilde{\gamma}_j$ $(\subset\mathfrak{B}(p_j:\tilde{\sigma}_j))$ be a C^∞-closed curve which surrounds the point p_j $(\in X\cap T)$ once, where $\tilde{\sigma}_j$ is a sufficiently small positive number. Let $\tilde{\varepsilon}$ be a sufficiently small positive number. Then we have*

$$\begin{aligned}\int_B \tilde{\omega}^{(p_1,q_1,2r_1+1)}\wedge\tilde{\omega}^{(p_2,q_2,2r_2+1)} &=\sum_{j=1}^{12k}\int_{\tau_{\tilde{\varepsilon}}(\tilde{\gamma}_j)}f_j^{(p_1,q_1,2r_1+1)}\omega^{(p_2,q_2,2r_2+1)}\\ &\left(=\sum_{j=1}^{12k}\int_{\tau_{\tilde{\varepsilon}}(\tilde{\gamma}_j)}\omega^{(p_1,q_1,2r_1+1)}f_j^{(p_2,q_2,2r_2+1)}\right).\end{aligned}\tag{6.35}$$

Now we shall obtain the explicit values of the integrals (6.35). We know that the holomorphic function $u(y, t)$ and $x(y, t)$ are even functions in the variable y, i.e., $u(y, t)=u(-y, t)$, $x(y, t)=x(-y, t)$ in view of the equation (6.9). By a simple calculation we find that the partial derivatives $\partial^q u/\partial y^q$, $\partial^q x/\partial y^q$ $(q=0, 1, 2, \cdots)$ are rational functions in the variables (y, u, x) whose denominators are powers of S. Hence we can express the function $u^p x^q/S$ by means of the coordinates (y, t) in the neighbourhood $N_\varepsilon(\mathfrak{B}(p_j;\lambda_2))$ in the form

$$u^p x^q/S=\sum_{f=0}^{\infty}\tilde{l}_{j,2f}^{(p,q)}(t)y^{2f}, \tag{6.36}$$

where the coefficients $\tilde{l}_{j,2f}^{(p,q)}(t)=[\partial^{2f}/\partial y^{2f}(u^p x^q/S)]_{\mathrm{Proj}\,\tilde{t}}$ [11] are rational functions on the curve X which are regular in the sets: $\mathfrak{V}(p_j;\lambda_2)$. Thus the 1-form $\Psi_j^{(p,q,2r+1)}$ is expressed in the form

$$(6.37)\qquad \Psi_j^{(p,q,2r+1)}=\Big\{\sum_{f=0}^{\infty}\frac{\tilde{l}_{j,2f}^{(p,q)}(t)}{2(f-r)+1}(y^{2(f-r)+1})\Big\}dt\,.$$

We express the function $u^p x^q/t$ in the neighbourhood $N_\varepsilon(\mathfrak{V}_4^{(j)}-\mathfrak{V}_3^{(j)})$ in terms of the coordinates (y,u) in the form

$$(6.38)\qquad u^p x^q/t=\sum_{f=0}^{\infty}\tilde{m}_{2f}^{(p,q)}(u)\,y^{2f}\,,$$

where the coefficients $\tilde{m}_{2f}^{(p,q)}(u)=[\partial^{2f}/\partial y^{2f}(u^p x^q/t)]_{\mathrm{Proj}\,\tilde{u}}$ are rational functions on the curve X which are regular in the set $\mathfrak{V}(p_j;\lambda_2)-p_j$. Moreover we obtain

$$(6.39)\qquad \Psi_0^{(p,q,2r+1)}=\Big\{\sum_{f=0}^{\infty}\frac{m_{2f}^{(p,q)}(u)}{2(f-r)+1}(y^{2(f-r)+1})\Big\}du\,.$$

We expand the derivatives $\partial u/\partial y$ and $\partial u/\partial t$ in the following forms (cf. (6.11))

$$(6.40)\qquad \partial u/\partial y=y\Big(\sum_{f=0}^{\infty}\tilde{n}_{2f}(t)y^{2f}\Big)\,,\qquad \partial u/\partial t=t\Big(\sum_{f=0}^{\infty}\tilde{s}_{2f}(t)y^{2f}\Big)\,.$$

By the equations (6.39) and (6.40), we have

$$(6.41)\qquad \phi_0^{(p,q,2r+1)}=\Big[\sum_{f=0}^{\infty}\frac{\tilde{m}_{2f}^{(p,q)}(u)}{2(f-r)+1}(y^{2(f-r)+1})\Big]\times\Big[y\Big(\sum_{f=0}^{\infty}\tilde{n}_{2p}(t)y^{2f}\Big)dy+t\Big(\sum_{f=0}^{\infty}\tilde{s}_{2f}(t)y^{2f}\Big)dt\Big]\,.$$

Now we write the 1-form $\phi_0^{(p,q,2r+1)}-\phi_j^{(p,q,2r+1)}$ in the following form

$$(6.42)\qquad \phi_0^{(p,q,2r+1)}-\phi_j^{(p,q,2r+1)}=\Big\{\sum_i a_{j,2i+1}^{(p,q,2r+1)}y^i\Big\}dy+\Big\{\sum_i b_{j,2i+1}^{(p,q,2r+1)}y^i\Big\}dt\,,$$

where the coefficients $a_{j,i}^{(p,q,2r+1)}$, $b_{j,i}^{(p,q,2r+1)}$ are rational functions on the curve X. Recall that the 1-form $\phi_0^{(p,q,2r+1)}-\phi_j^{(p,q,2r+1)}$ is closed and has no period on any 1-cycles in the neighbourhood $N(\mathfrak{V}_4^{(j)}-\mathfrak{V}_3^{(j)})$. Hence we have

$$(6.43)_1\qquad \partial a_{j,i}^{(p,q,2r+1)}/\partial t=(i+1)b_{j,i+1}^{(p,q,2r+1)}\,,$$

$$(6.43)_2\qquad a_{j,-1}^{(p,q,2r+1)}=0\,,$$

$$(6.43)_3\qquad \int_{\tilde{\gamma}_j}b_{j,i}^{(p,q,2r+1)}(t)dt=0 \text{ for any 1-cycles } \tilde{\gamma}_j \text{ in the set } \mathfrak{V}^{(j)}(p_j;\lambda_2)-p_j\,.$$

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11) The symbol $[\partial^{2f}/\partial y^{2f}(u^p x^q/S)]_{\mathrm{Proj}\,\tilde{t}}$ denotes the value of $[\partial^{2f}/\partial y^{2f}(u^p x^q/S)]$ at the point $\mathrm{Proj}_{\tilde{t}}(x,y,u)$.

Thus we obtain

(6.44) $$f_j^{(p,q,2r+1)}=\int^{(t,y)}(\phi_0^{(p,q,2r+1)}-\phi_j^{(p,q,2r+1)})$$
$$=-\sum_s a_{2s+1}^{(p,q,2r+1)}(t_0)/(2s+1)\cdot y_0^{s+1}+\sum_s a_{2s+1}^{(p,q,2r+1)}(t)/(2s+1)\cdot y^{2s+1}.$$

Now we expand the rational 2-form $\omega^{(p,q,2r+1)}$ at the point p_j in terms of the coordinate (t, y) in the form

(6.45) $$\omega^{(p,q,2r+1)}=u^p x^q/y^{2r}S\, dy\wedge dt$$
$$=\Big(\sum_{f=0}^{\infty}\tilde{l}_{j,2f}^{(p,q)}(t)y^{2(f-r)}\Big)dy\wedge dt.$$

Let $\omega^{(p_i,q_i,2r_i+1)}$ $(i=1, 2)$ be 2-forms and let $a_{2s+1}^{(p_i,q_i,2r_i+1)}(t)$, $\tilde{l}_{j,2f}^{(p_i,q_i)}(t)$ be the rational functions on X corresponding to the forms $\omega^{(p_i,q_i,2r_i+1)}$ by means of the equations (6.44) and (6.45). Also we denote by $\tilde{\omega}^{(p_i,q_i,2r_i+1)}$ closed C^∞-differentiable 2-forms corresponding to the forms $\omega^{(p_i,q_i,2r_i+1)}$ in the process (6.9)-(6.31). Then we have the following

THEOREM 6.1.

(6.46) $$\int_B\tilde{\omega}^{(p_1,q_1,2r_1+1)}\wedge\tilde{\omega}^{(p_2,q_2,2r_2+1)}$$
$$=2\pi\sqrt{-1}\cdot\sum_{j=1}^{12k}\sum_{2s+2f=2r_2}\int_{\tilde{\gamma}(p_j)}\tilde{l}_{j,2f}^{(p_1,q_1)}a_{2s+1}^{(p_2,q_2,2r_2+1)}$$
$$=2\pi\sqrt{-1}\cdot\sum_{j=1}^{12k}\sum_{2s+2f=2r_1}\int_{\tilde{\gamma}(p_j)}\tilde{l}_{j,2f}^{(p_2,q_2)}a_{2s+1}^{(p_1,q_1,2r_1+1)},$$

where $\tilde{\gamma}(p_j)$ denotes a 1-cycle in the set $\mathfrak{V}(p_j\,;\lambda_2)-p_j$ which surrounds the point p_j once in the positive direction.

Now we shall obtain the values of the integral (6.46) for the cases: $r=0, 1, 2$. These values are necessary in order to know the coefficients of the systems of the differential equations. First we have

$(6.47)_1$ $$\tilde{l}_0^{(p,q)}=u^p\cdot x^q/S,$$
$$\tilde{l}_2^{(p,q)}=\frac{1}{2}\cdot\Big\{\frac{u^p x^q}{S^3}-\frac{1}{S^2}\cdot(p\cdot u^{p-1}\cdot(24x)\cdot x^q+q\cdot g'\cdot x^{q-1}\cdot u^p)\Big\},$$

$(6.47)_2$ $$\tilde{n}_0=(-48)x/S,\qquad \tilde{n}_2=-\frac{48}{2}\times\Big\{\frac{x(g'S_x+24xS_u)}{S^3}-\frac{g'}{S^2}\Big\},$$

$(6.47)_3$ $$\tilde{m}_0^{(p,q)}=u^p\cdot x^q/t,\qquad \tilde{m}_2=u^p/t^3\cdot\{q\cdot t\,x^{q-1}-24\cdot x^{q-1}\}.$$

From these tables we obtain[12)]

12) As the expansion of rational 2-form $\omega^{(p,q,2r+1)}$ $(=1/y^{2r}(\sum_{f=0}^{\infty}\tilde{l}_{2f}^{(p,q)}\cdot y^{2f})dy\wedge dt)$ does not contain terms of odd degrees in the variable y, we need only the terms $a_{2f}^{(p,q,2r+1)}$ of even degrees in y in order to determine the values (6.46).

$$(6.48)_0 \qquad a_{j,2i}^{(p,q,1)}=0 \quad (i\leqq 0), \qquad a_{j,2}^{(p,q,1)}=-48\cdot\frac{u^p\cdot x^{q+1}}{t\cdot S},$$

$$(6.48)_1 \qquad a_{j,0}^{(p,q,1)}=0 \quad (i\leqq -1), \qquad a_{j,0}^{(p,q,3)}=48\cdot\frac{u^p\cdot x^{q+1}}{t\cdot S},$$

$$(6.48)_2 \qquad a_{j,2i}^{(p,q,5)}=0 \quad (i\leqq -2), \qquad a_{j,-2}^{(p,q,5)}=16\cdot\frac{u^p\cdot x^{q+1}}{t\cdot S}.$$

From the tables $(6.48)_0$-$(6.48)_2$, we obtain the following

COROLLARY TO THEOREM 6.1.

$$(6.49)_{(0,0)} \qquad \int_B \tilde{\omega}_0^{(p_1,q_1,1)}\wedge\tilde{\omega}_0^{(p_2,q_2,1)}=0,$$

$$(6.49)_{(0,1)} \qquad \int_B \tilde{\omega}^{(p_1,q_1,1)}\wedge\tilde{\omega}^{(p_2,q_2,3)}=0,$$

$$(6.49)_{(0,2)} \qquad \int_B \tilde{\omega}^{(p_1,q_1,1)}\wedge\tilde{\omega}^{(p_2,q_2,5)}=-16\times(2\pi\sqrt{-1})^2\sum_{j=1}^{12k} u_j^{p_1+p_2}\cdot x_j^{q_1+q_2+1}/S_j^2,$$

$$(6.49)_{(1,1)} \qquad \int_B \tilde{\omega}^{(p_1,q_1,3)}\wedge\tilde{\omega}^{(p_2,q_2,3)}=48(2\pi\sqrt{-1})^2\cdot\sum_{j=1}^{12k}\frac{u_j^{p_1+p_2}x_j^{q_1+q_2+1}}{S_2^j}.$$ [13]

Finally, we shall explain the connection between the bilinear equality and the determination of coefficients of (partial) linear differential equations of periods functions $W^{(p,0,1)}$'s. Let $W_{k;j}^p(\tau,\sigma)$ $(=W_{k,j}^{(p,0,1)}(\tau,\sigma))$ be the period functions of the rational forms $\omega_k^{(p,0,1)}$ $(j=1,\cdots,12k-4)$ and let $\eta_k^p(\tau,\sigma)$ be the vector defined by $\eta_k^p(\tau,\sigma)=(W_{k,j}^{(p,0,1)}(\tau,\sigma))$ $(j=1,\cdots,12k-4)$. For the differential operator $D=\partial/\partial\tau_i,\cdots,\partial^2/\partial\tau_i,\partial\sigma_j,\cdots$ let $D\eta_k^p=[DW_{k,j}^{(p)}]_{j=1,\cdots,12k-4}$. We infer from Theorem 3.2 that the matrix

$$(6.50) \qquad \Omega_k^p(\tau,\sigma)=\begin{bmatrix}\eta_k^p(\tau,\sigma)\\ \partial/\partial\tau_i(\eta_k^p(\tau,\sigma))\\ \vdots\\ \partial^2/\partial\tau_i\partial\sigma_j(\eta_k^p(\tau,\sigma))\end{bmatrix}$$

is of rank $(12k-4)$. Write the differential operator D_l $(l=1,2,\cdots,12k-5)$ for $\partial/\partial\tau_i,\cdots,\partial^2/\partial\tau_i\partial\sigma_j,\cdots$ such that

$$(6.51) \qquad \det\begin{bmatrix}\eta_k^p\\ D_1\eta_k^p\\ \vdots\\ D_{12k-5}\eta_k^p\end{bmatrix}\neq 0.$$

13) The bilinear equality between rational 2-forms are available in more general situations now. We shall discuss in later.

We write the matrix in (6.51) by $\tilde{\Omega}_k^p$. For any differential operator D define functions C_0, C_j $(j=1, 2, \cdots, 6k-5)$ by

$$ {}^t[D\eta_k^p] = {}^t\tilde{\Omega}_k^p \begin{bmatrix} C_0^{(p,k)} \\ C_j^{(p,k)} \end{bmatrix}. \tag{6.52}$$

Now consider the systems $\mathfrak{D}_k^{(p)}$ of the partial linear differential equations:

$$ D \cdot F = C_0^{(p,k)} \cdot F + \sum_{j=1}^{12k-5} C_j^{(p,k)} D_j F \tag{6.53}$$

and let $\mathfrak{F}$ be the linear space composed of the solutions of $\mathfrak{D}_k^{(p)}$. Then we infer from § 4 and (6.51) that period functions $W_k^{(p)}$ constitutes the complete solutions of $\mathfrak{D}_k^{(p)}$. Moreover we know by Theorem 3.3 that the differential operators D_j in (6.51) are chosen such that the order of $D_j \leqq 2$.

On the other hand, by means of Cramer's rule, functions C_0, C_j are expressed as follows:

$$ C_0 = \det \varDelta_0 / \det \tilde{\Omega}_k^p, \qquad C_j = \det \varDelta_j / \det \tilde{\Omega}_k^p \cdots, \tag{6.54}$$

where $\varDelta_0, \varDelta_j, \cdots$, which are determined by Cramer's rule, are composed of periods functions. We know that $\varDelta_0 {}^tI^{-1} \cdot {}^t\varDelta_0$, $\varDelta_0 {}^tI^{-1} \cdot {}^t\varDelta_0 \cdots$ are expressed as rational functions of (τ, σ) explicitly by the procedure of § 6. Thus the arguments of § 6 enable us to investigate the coefficients $C_0, C_j, \cdots$, in detail.

These are what we call the connection between the differential equations of period function and bilinear relation.[14)]

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14) Beyond this point, more detailed study of the differential equations $\tilde{D}$, for example, the order of poles of coefficients of $\tilde{D}, \cdots$, are not obtained to us.

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