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On a class of set-theoretical interpretations of the primitive logic

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§ 0. Introduction.

The main purpose of the present paper is to exhibit an extensive class of set-theoretical interpretations of the *primitive logic* which seems to cover almost all known set-theoretical interpretations, model-theoretical interpretations, and truth-value-theoretic interpretations of logics. The primitive logic has been introduced in my papers [2] and [3].

Throughout this paper, I will denote by upper case letters $A, B, \cdots$ *propositions* as well as *predicates*, and by the corresponding lower case letters $a, b, \cdots$ the *interpretations of the propositions or predicates*. The letters $x, y, \cdots$ are *object variables*. For any set-theoretical interpretation of the primitive logic **LO** of the present paper, a class of subsets of a certain set ω is employed, which can be regarded as the class of closed sets of the space ω by introducing a suitable topology $\mathfrak{T}$ to ω. By introducing another topology $\mathfrak{T}^*$ to the same space ω, I define the set-theoretical interpretation of "*implication*" and "*universal quantification*", which are the *only logical constants* of the primitive logic **LO**. As is shown in my papers [3] and [4], the classical logic **LK** and the intuitionistic logic **LJ** are reducible to the primitive logic **LO**. Accordingly, logical constants of the logics **LK** and **LJ** other than "*implication*" and "*universal quantification*" can be defined in terms of these two logical constants in the primitive logic **LO**. So, the newly defined logical constants are set-theoretically interpreted in accordance with the set-theoretical interpretations of "*implication*" and "*universal quantification*".

The interpretations of "$A \rightarrow B$" and "$(x)A(x)$" are defined as follows: Let $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ be a pair of topologies introduced to the same space ω whose *closure operations* are denoted by "$\{\ \}$" and "$[\ ]$", respectively. Let us further assume that $\{\mathfrak{T}\}$ is a finer topology of ω than $[\mathfrak{T}]$ ($\{a\} \subseteqq [a]$ for every a) and that the topology pair $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ satisfy a certain condition called "*logical*". (See (1.4).) Then, we define "$a \rightarrow b$" and "$(x)a(x)$" as follows:

$$a \rightarrow b \leftrightharpoons [b-a] \cap b ,$$

$$(x)a(x) \leftrightharpoons \{ \bigcup_x a(x)\} ,$$

where "$b-a$" denotes the complementary set of the set a with respect to the set b and "$\mathfrak{F} \rightleftharpoons \mathfrak{G}$" denotes that $\mathfrak{F}$ is defined by $\mathfrak{G}$ (or, $\mathfrak{F}$ stands for $\mathfrak{G}$).

There are a number of interpretations of logics. The following is a list of popular interpretations:

(**A**) The truth-value interpretation of the classical logic as a two-valued logic.

(**B**) Truth-value interpretations of intuitionistic logics as many-valued logics.

(**C**) The set-theoretical interpretation of the classical logic by associating with every proposition a subset of a fixed set.

(**D**) The topological interpretation of intuitionistic logics by associating with every proposition a closed set of a topological space.

There is another interpretation of the proposition part of **LO**. Namely,

(**E**) The set-theoretical interpretation of the proposition part of **LO** by associating every proposition with a subset of a fixed set and by defining $p \rightarrow q$ as denoting 0 if $p \supseteqq q$ and as denoting q otherwise.

These interpretations can be proved to be special cases of interpretations which are introduced by the interpretation just defined of the logic **LO**.

A sufficient condition has been given in my paper [5] and H. Ono [7] for every evaluation to be an interpretation of the logic **LO**[1]. In Section (1), I will describe exactly what we need for a pair of topologies $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ of the same space ω to interpret the primitive logic **LO** set-theoretically in ω by our general method, and I will prove that our interpretation satisfies the condition for being an interpretation of the primitive logic **LO**. In Section (2), our topological interpretation of the primitive logic **LO** is shown to be so general that it covers all the popular interpretations (**A**), (**B**), (**C**), (**D**) and (**E**) with respect to the logical constants "*implication*" and "*universal quantification*".

By applying the reduction of my papers [3] and [4] for our interpretation, we would have set-theoretical interpretations of the classical logic and the intuitionistic logic. I will discuss the matter in Section (3). The reduction of my papers [3] and [4] relies mostly on the *S-closure operation* defined by

$$(A)^S \rightleftharpoons (A \rightarrow S) \rightarrow S$$

for a proposition S, or on the *T-closure operation* defined by

$$(A)_T \rightleftharpoons (x)((A \rightarrow T(x)) \rightarrow T(x))$$

for a predicate T. In the same section, I will also interpret these operations

1) H. Ono pointed out that the condition (**E8**) in [5] should be replaced by the condition (**E8**)* given in his paper [7] (to appear).

set-theoretically, and I will clarify the intrinsic meanings of these operations as well as of logical constants introduced in connection with them.

§ 1. General aspect of set-theoretical interpretations of the primitive logic.

1.1. The primitive logic LO.

I have introduced in my papers [2] and [3] the primitive logic **LO**, which can be regarded as the simplest possible intuitionistic predicate logic but which is powerful enough to faithfully embed in it most of formal theories standing on various logics. The primitive logic **LO** has a pair of logical constants "$\rightarrow$" (implication) and "( )" (universal quantification) and it is characterized by the following inference rules:

(**F**) $\mathfrak{A}$ *is deducible from* $\mathfrak{A}$.

(**I**) $\mathfrak{A}$ *is deducible from* $\mathfrak{B}$ *and* $\mathfrak{B}\rightarrow\mathfrak{A}$.

(**I***) $\mathfrak{A}\rightarrow\mathfrak{B}$ *is deducible from the fact that* $\mathfrak{B}$ *is deducible from* $\mathfrak{A}$.

(**U**) $\mathfrak{A}(t)$ *is deducible from* $(x)\mathfrak{A}(x)$.

(**U***) $(x)\mathfrak{A}(x)$ *is deducible from the fact that* $\mathfrak{A}(t)$ *is deducible for any variable* t *whatever.*

1.2. Interpretation of LO.

Let $\boldsymbol{D}$ be a domain of objects and $\boldsymbol{W}$ be a family of objects containing the designated object 0. The direct product of n $\boldsymbol{D}$'s is denoted by $\boldsymbol{D}^n$. Any mapping from $\boldsymbol{D}^n$ into $\boldsymbol{W}$ is called a $(\boldsymbol{D}^n, \boldsymbol{W})$-matrix and denoted in the form $p(x_1, \cdots, x_n)$. Then,

1) *Any* $(\boldsymbol{D}^n, \boldsymbol{W})$*-matrix of the form* $p(x_1, \cdots, x_n)$ *can be regarded as a special case of* $(\boldsymbol{D}^{n+k}, \boldsymbol{W})$*-matrix of the form* $q(x_1, \cdots, x_n, z_1, \cdots, z_k)$.

2) *Any* $(\boldsymbol{D}^{n+k}, \boldsymbol{W})$*-matrix of the form* $p(x_1, \cdots, x_n, z_1, \cdots, z_k)$ *can be regarded as a* $(\boldsymbol{D}^n, \boldsymbol{W})$*-matrix of the form* $q(x_1, \cdots, x_n)$ *for any fixed sequence* $z_1, \cdots, z_k$.

3) $p(x_1, x_1, \cdots, x_n)$ *can be regarded as a* $(\boldsymbol{D}^n, \boldsymbol{W})$*-matrix of the form* $q(x_1, \cdots, x_n)$.

4) *Any* $(\boldsymbol{D}^n, \boldsymbol{W})$*-matrix* $p(\cdots, x, y, \cdots)$ *can be regarded as a* $(\boldsymbol{D}^n, \boldsymbol{W})$*-matrix of the form* $q(\cdots, y, x, \cdots)$.

Evidently, any $(\boldsymbol{D}^n, \boldsymbol{W})$-matrix represents an n-ary relation. To express the whole class of relations in the primitive logic, we must further assume that a pair of operations "$\rightarrow$" and "( )" is defined for matrices such that

5) $p(x_1, \cdots, x_n)\rightarrow q(z_1, \cdots, z_k)$ *can be regarded as a* $(\boldsymbol{D}^{n+k}, \boldsymbol{W})$*-matrix of the form* $r(x_1, \cdots, x_n, z_1, \cdots, z_k)$.

6) $(x_n)p(x_1, \cdots, x_n)$ *can be regarded as a* $(\boldsymbol{D}^{n-1}, \boldsymbol{W})$*-matrix of the form* $q(x_1, \cdots, x_{n-1})$.

Any system $(\boldsymbol{D}, \boldsymbol{W})$ is called an *interpretation of the primitive logic* **LO** if and only if any n-ary relation which is provable in **LO** is represented by a $(\boldsymbol{D}^n, \boldsymbol{W})$-matrix being equal to 0.

1.3. Sufficient condition for being an interpretation of LO.

According to my paper [5] and H. Ono [7], any system $(\boldsymbol{D}, \boldsymbol{W})$ is an interpretation of **LO** if it satisfies the following conditions:

(E1) $p \to p = 0$,

(E2) $p \to 0 = 0$,

(E3) $0 \to p = p$,

(E4) $p \to (p \to q) = p \to q$,

(E5) $p \to (q \to r) = q \to (p \to r)$,

(E6) $p \to q = 0$ *implies* $(r \to p) \to (r \to q) = 0$,

(E7) $(x)p(x) \to p(t) = 0$,

(E8)* *If* $p \to (q \to r(t)) = 0$ *for every variable* t *whatever, then* $p \to (q \to (x)r(x)) = 0$.

Namely, any matrix representing a provable proposition in **LO** can be proved to be equal to 0 in so far as the conditions **(E1)**—**(E8)*** hold.

1.4. Set-theoretical interpretations.

Any interpretation $(\boldsymbol{D}, \boldsymbol{W})$ of **LO** is called *set-theoretical* if and only if $\boldsymbol{W}$ is formed exclusively by subsets of a certain set ω and the logical combination $p \to q$ as well as the logical operation $(x)p(x)$ for members p, q, and $p(t)$ (for every t which is a parameter running over $\boldsymbol{D}$) of $\boldsymbol{W}$ is defined so as to produce a member of $\boldsymbol{W}$. I will describe now a general method for constructing set-theoretical interpretations of the primitive logic **LO**.

Let ω be a space having a pair of topologies $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$, whose closure operations are denoted by "$\{\ \}$" and "$[\ ]$", respectively. For these topologies, we assume the following properties:

(T1) $p \subseteqq \{p\} \subseteqq [p]$,

(T2) $p \subseteqq q$ *implies* $\{p\} \subseteqq \{q\}$ *as well as* $[p] \subseteqq [q]$,

(T3) $\{\{p\}\} = \{p\}$ *and* $[[p]] = [p]$,

(T4) $[0] = 0$.

The following formulas are easily deducible from **(T1)**—**(T4)**:

(F1) $\{0\} = 0$, $\{\omega\} = [\omega] = \omega$,

(F2) $\{[p]\} = [\{p\}] = [p]$,

(F3) $\{\{p\} \cap \{q\}\} = \{p\} \cap \{q\}$, $[[p] \cap [q]] = [p] \cap [q]$.

For any subset s of ω, we can introduce two operations "$\{\ \}^s$" and "$[\ ]^s$" as follows:

DEFINITION 1.

$$\{p\}^s = \{p\} \cap s, \quad [p]^s = [p] \cap s.$$

THEOREM 1. *For any closed set s with respect to the topology $\{\mathfrak{T}\}$, the set $\{p\}^s$ as well as the set $[p]^s$ is closed with respect to the same topology.*

PROOF. Assume s is closed with respect to $\{\mathfrak{T}\}$. Then, according to **(F2)** and (**F3**),

$$\{\{p\}^s\}=\{\{p\}\cap s\}=\{\{p\}\cap\{s\}\}=\{p\}\cap\{s\}=\{p\}\cap s=\{p\}^s,$$

$$\{[p]^s\}=\{[p]\cap s\}=\{\{[p]\}\cap\{s\}\}=\{[p]\}\cap\{s\}=[p]\cap s=[p]^s.$$

THEOREM 2. *If* $s\supseteqq t$, *then* $\{\{p\}^s\}^t=\{p\}^t$, *and* $[[p]^s]^t=[p]^t$. *If s is closed with respect to the topology* $\{\mathfrak{T}\}$ (*or* $[\mathfrak{T}]$),

$$\{\{p\}^s\}^t=\{p\}^{s\cap t}\quad (\textit{or } [[p]^s]^t=[p]^{s\cap t}).$$

PROOF. Assume $s\supseteqq t$. Then, according to (**T2**),

$$\{\{p\}^s\}^t=\{\{p\}\cap s\}\cap t\supseteqq\{p\}\cap s\cap t=\{p\}\cap t=\{p\}^t$$

$$([[p]^s]^t=[[p]\cap s]\cap t\supseteqq[p]\cap s\cap t=[p]\cap t=[p]^t).$$

On the other hand, according to (**T3**),

$$\{\{p\}^s\}^t=\{\{p\}\cap s\}\cap t\subseteqq\{\{p\}\}\cap t=\{p\}\cap t=\{p\}^t$$

$$([[p]^s]^t=[[p]\cap s]\cap t\subseteqq[[p]]\cap t=[p]\cap t=[p]^t).$$

Next, assume that s is closed with respect to $\{\mathfrak{T}\}$ (or $[\mathfrak{T}]$). Then, according to (**F3**),

$$\{\{p\}^s\}^t=\{\{p\}\cap s\}\cap t=\{\{p\}\cap\{s\}\}\cap t=\{p\}\cap\{s\}\cap t=\{p\}\cap s\cap t=\{p\}^{s\cap t}$$

$$([[p]^s]^t=[[p]\cap s]\cap t=[[p]\cap[s]]\cap t=[p]\cap[s]\cap t=[p]\cap s\cap t=[p]^{s\cap t}).$$

THEOREM 3. *For any set s,*

(**sT1**) $p\cap s\subseteqq\{p\}^s\subseteqq[p]^s\subseteqq s$,

(**sT2**) $p\subseteqq q$ *implies* $\{p\}^s\subseteqq\{q\}^s$ *and* $[p]^s\subseteqq[q]^s$,

(**sT3**) $\{\{p\}^s\}^s=\{p\}^s$ *and* $[[p]^s]^s=[p]^s$,

(**sT4**) $[0]^s=0$.

PROOF. (**sT1**), (**sT2**) and (**sT4**) are easily provable by (**T1**), (**T2**), (**T4**) and (**F3**). (**sT3**) is a special case of Theorem 2.

According to Theorem 3, we can see that the operations "$\{\ \}^s$" and "$[\ ]^s$" induce topologies for the space s, which will be denoted by "$\{\mathfrak{T}\}^s$" and "$[\mathfrak{T}]^s$", respectively.

DEFINITION 2.

$$p\to q\rightleftharpoons[q-p]^q.$$

DEFINITION 3.

$$(x)p(x)\rightleftharpoons\{\bigcup_x p(x)\}.$$

THEOREM 4. $p\to q$ *is closed with respect to the topology* $\{\mathfrak{T}\}$ *if q is so.* $(x)p(x)$ *is closed with respect to the same topology.*

PROOF. By Theorem 1 and (**T3**).

THEOREM 5. $p\to q=0$ *if and only if* $p\supseteqq q$.

Proof. At first, assume $p\rightarrow q=0$. Then, by Theorem 3,

$$0=p\rightarrow q=[q-p]^q \supseteqq (q-p)\cap q=q-p.$$

Hence, $p\supseteqq q$. Next, conversely, assume $p\supseteqq q$. Then, according to Theorem 3,

$$p\rightarrow q=[q-p]^q=[0]^q=0.$$

Theorem 6.

$$p\rightarrow(q\rightarrow r)=[[r-q]^r\cap(r-p)]^r.$$

Proof.

$$\begin{aligned} p\rightarrow(q\rightarrow r)&=[(q\rightarrow r)-p]^{q\rightarrow r}\\ &=[[r-q]\cap r\cap(\omega-p)]\cap[r-q]\cap r\\ &=[[r-q]\cap(r-p)]\cap r, \end{aligned}$$

because $[[r-q]\cap(r-p)]\subseteqq[[r-q]]=[r-q]$ according to (**T2**) and (**T3**).

Theorem 7. *"$\rightarrow$" defined by Definition* 2 *and "( )" defined by Definition* 3 *satisfy* (**E1**)—(**E4**), (**E6**) *and* (**E7**).

Proof. According to Theorem 3,

$$p\rightarrow p=[p-p]^p=[0]^p=0.$$

Hence, (**E1**) holds. Also according to Theorem 3,

$$p\rightarrow 0=[0-p]^0=[0]^0=0.$$

Hence, (**E2**) holds. According to (**T1**),

$$0\rightarrow p=[p-0]^p=[p]\cap p=p.$$

Hence, (**E3**) holds. According to (**T1**) and Theorem 6,

$$p\rightarrow(p\rightarrow q)=[[q-p]^q\cap(q-p)]^q=[[q-p]\cap q\cap(q-p)]^q=[q-p]^q=p\rightarrow q.$$

Hence, (**E4**) holds.

To show (**E6**), let us assume $p\rightarrow q=0$. Then, $p\supseteqq q$ holds according to Theorem 5. Hence, according to (**T2**),

$$r\rightarrow p=[p-r]^p=[p-r]\cap p\supseteqq[q-r]\cap q=[q-r]^q=r\rightarrow q.$$

Consequently, $(r\rightarrow p)\rightarrow(r\rightarrow q)=0$ holds by Theorem 5.

(**E7**) holds by Theorem 5, because

$$(x)p(x)=\{\bigcup_x p(x)\}\supseteqq\bigcup_x p(x)\supseteqq p(t)$$

by (**T1**).

We can not expect that (**E5**) holds in general even for closed sets with respect to the topology $[\mathfrak{T}]$. For example, let $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ be the same topology of a plane ω whose closed sets are the totality of convex sets. Then, $p\rightarrow(q\rightarrow r)$ is not always equal to $q\rightarrow(p\rightarrow r)$ even for closed sets for these topologies satisfying (**E1**)—(**E4**), (**E6**)—(**E8**) as Figure 1 shows:

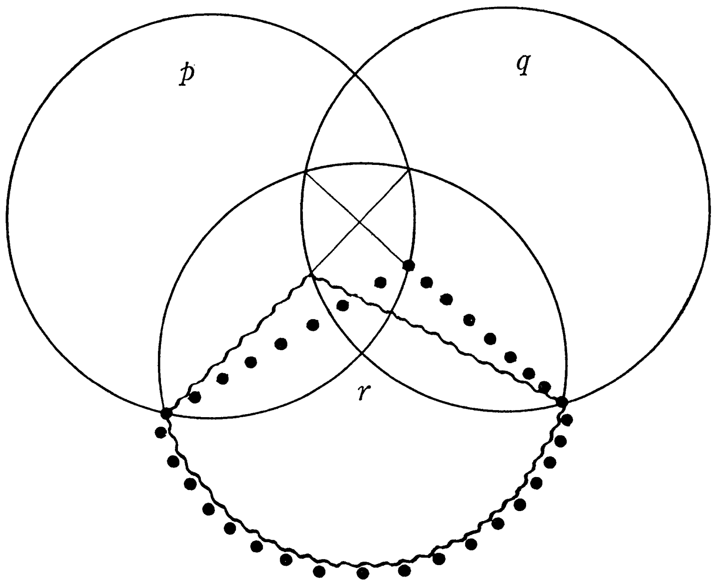

Figure 1.

Dotted line: The boundary of $p \to (q \to r)$,

Wavy line: The bounaary of $q \to (p \to r)$.

Any pair of topologies $(\{\mathfrak{T}\}, [\mathfrak{T}])$ is called *logical* if and only if "$\to$" defined by Definition 2 satisfies (**E5**) for every triple of closed sets p, q and r with respect to the topology $\{\mathfrak{T}\}$ and (**E8**)* for every triple of closed sets p, q and $r(t)$ with respect to the same topology $\{\mathfrak{T}\}$[2].

According to Theorem 6, $(\{\mathfrak{T}\}, [\mathfrak{T}])$ is logical if and only if

$$[[r-p]^r \cap (r-q)]^r = [(r-p) \cap [r-q]^r]^r$$

holds for every triple of closed sets p, q and r with respect to the topology $\{\mathfrak{T}\}$ and

$$[[\{\bigcup_x r(x)\}-q]^{(\bigcup_x r(x))} \cap (\{\bigcup_x r(x)\}-p)]^{(\bigcup_x r(x))} = 0$$

holds for every triple of closed sets p, q and $r(t)$ satisfying

$$[[r(t)-q]^{r(t)} \cap (r(t)-p)]^{r(t)} = 0$$

for any t. Moreover, we can prove easily

THEOREM 8. *Any pair of topologies* $(\{\mathfrak{T}\}, [\mathfrak{T}])$ *is logical if*

(**T5**) $[[r-p] \cap (r-q)] = [(r-p) \cap [r-q]]$

holds for every triple of closed sets p, q and r with respect to the topology $\{\mathfrak{T}\}$

2) This notion "*logical*" is given by H. Ono [7] as an improvement of my original definition. By virtue of his improvement, my original Theorems 8, 12 and 13 remained literally true, and Theorem 9 remained true by some modifications given by him.

and

$$\text{(T6)}\quad [\{\bigcup_x r(x)\}-q]\cap(\{\bigcup_x r(x)\}-p)=0$$

holds for every triple of closed sets p, q and $r(t)$ satisfying

$$[r(t)-q]\cap(r(t)-p)=0$$

for any t.

§ 2. Special cases.

In this section, I will show that the popular interpretations such as (**A**)—(**E**) given in the introduction can be regarded as special cases of set-theoretical interpretations given in the previous section.

At first, I will exhibit two *extremal cases* of logical pairs of topologies. They are indeed extremal, but also pretty extensive in the sense that they cover almost all (**A**)—(**E**). I will also give a theorem concerning *compositions* and *decompositions* of logical pairs of topologies.

2.1. Extremal case I.

THEOREM 9. *Let ω be a space having a topology $\{\mathfrak{T}\}$ and another rough topology $[\mathfrak{T}]$ which has only two closed sets 0 and ω. Then,* (**T5**) *holds. Further, the topology pair $(\{\mathfrak{T}\}, [\mathfrak{T}])$ is logical, if the class of all closed sets with respect to $\{\mathfrak{T}\}$ is totally ordered by set inclusion.* (Modified by H. Ono [7].)

PROOF. It is clear that $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ satisfy the conditions (**T1**)—(**T4**). For the topology $[\mathfrak{T}]$, $[[a]\cap b]$ as well as $[a\cap[b]]$ is ω unless a or b is 0. Hence, $[[a]\cap b]=[a\cap[b]]$. So, the former half holds. The latter half holds too, because $r(t)$ must be always a subset of $p\cap q$.

Now, take $\{\mathfrak{T}\}$ as the topology for which $\{p\}=p$ holds for every subset p of ω. Then, we have the example interpretation (**E**) as a special case of this extremal case. (Former half of Theorem 9.)

Next, take ω as $\{1, \cdots, n\}$, and $\{\mathfrak{T}\}$ as the topology, which has $\emptyset$, $\{1\}, \cdots$, $\{1, \cdots, n\}$ as the totality of its closed sets. Then, we have as a special case of this extremal case, the set-theoretical interpretation of intuitionistic logics as many-valued logics given by Gödel [**1**], which can be regarded as the representative case of the example interpretation (**B**). As the very special case of this interpretation, where we take $\{1\}$ as ω (*i.e.* the case $n=1$), we have the example interpretation (**A**) of the classical logic.

2.2. Extremal case II.

Another extremal case of logical pairs of topologies is the case where $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ coincide and they satisfy the condition

(T7) $[a \cup b] = [a] \cup [b]$.

Before proving that such pairs are all logical, let us state a few theorems for preparation.

THEOREM 10. **(T7)** *implies* $[a \cup b]^s = [a]^s \cup [b]^s$ *for any* s.

Now, I will call any set a *neighborhood* of x with respect to the topology $[\mathfrak{T}]^p$ if and only if it is a set of the form $p - [t]^p$ containing x in it. Then, by making use of Theorem 10, we have

THEOREM 11. *Assume* $a \subseteqq p$. *Then, any* x *is a member of* $[a]^p$ *if and only if every neighborhood of* x *with respect to the topology* $[\mathfrak{T}]^p$ *has at least one member in common with* a.

THEOREM 12. *Any pair* $([\mathfrak{T}], [\mathfrak{T}])$ *of identical topologies* $[\mathfrak{T}]$ *is logical if* $[\mathfrak{T}]$ *satisfies the condition* **(T7)**. *In this case, we can prove further*

$$[[p-q] \cap (p-r)] = [(p-q) \cap [p-r]] = [(p-q) \cap (p-r)]$$

for any closed sets p, q *and* r *with respect to* $[\mathfrak{T}]$.

PROOF. It is enough to show

$$[[p-q] \cap (p-r)] = [(p-q) \cap (p-r)]$$

for any closed sets p, q and r with respect to $[\mathfrak{T}]$. I will prove at first

$$[[p-q^*]^p \cap (p-r^*)]^p = [(p-q^*) \cap (p-r^*)]^p$$

for any closed set p with respect to the topology $[\mathfrak{T}]$ and for any closed sets q^* and r^* with respect to the topology $[\mathfrak{T}]^p$.

Now, let x be any member of $[[p-q^*]^p \cap (p-r^*)]^p$. Then, I will show that any neighborhood $p-[h]^p$ of x has at least one member in common with $(p-q^*) \cap (p-r^*)$.

Namely, according to Theorem 11, any neighborhood $p-[h]^p$ of x has at least one member, say y, in common with $[p-q^*]^p \cap (p-r^*)$. Hence, by virtue of Theorem 10 and by assumption

$$y \in (p-[h]^p) \cap (p-r^*) = p-[h]^p \cup r^* = p-[h]^p \cup [r^*]^p = p-[h \cup r^*]^p .$$

Also, y is a member of $[p-q^*]^p$.

According to Theorem 11, the neighborhood $p-[h \cup r^*]^p$ of the member y of $[p-q^*]^p$ has at least one member, say z, in common with $p-q^*$. By virtue of Theorem 10 and by assumption,

$$\begin{aligned} z \in (p-[h \cup r^*]^p) \cap (p-q^*) &= p-[h \cup r^*]^p \cup q^* \\ &= p-[h \cup r^*]^p \cup [q^*]^p = p-[h \cup r^* \cup q^*]^p . \end{aligned}$$

Hence, by virtue of Theorem 3 and by assumption,

$$z \in p-[h \cup r^* \cup q^*]^p \subseteqq p-[q^*]^p = p-q^* ,$$
$$z \in p-[h \cup r^* \cup q^*]^p \subseteqq p-[r^*]^p = p-r^* .$$

So, $z \in (p-q^*) \cap (p-r^*)$ and $z \in p-[h \cup r^* \cup q^*]^p \subseteqq z-[h]^p$.

Thus, we have proved that any neighborhood $p-[h]^p$ of any member x of $[[p-q^*]^p \cap (p-r^*)]^p$ has at least one member z in common with $(p-q^*) \cap (p-r^*)$. So, according to Theorem 11, x is a member of $[(p-q^*) \cap (p-r^*)]^p$.

Accordingly,

$$[[p-q^*]^p \cap (p-r^*)]^p \subseteqq [(p-q^*) \cap (p-r^*)]^p .$$

On the other hand, according to Theorem 3,

$$[[p-q^*]^p \cap (p-r^*)]^p \supseteqq [(p-q^*) \cap p \cap (p-r^*)]^p = [(p-q^*) \cap (p-r^*)]^p .$$

Hence,

$$[[p-q^*]^p \cap (p-r^*)]^p = [(p-q^*) \cap (p-r^*)]^p$$

holds for any closed set p with respect to $[\mathfrak{T}]$ and for any pair of closed sets q^* and r^* with respect to $[\mathfrak{T}]^p$.

Now, by (**T1**), (**F3**), and assumption,

$$[q \cap p]^p = [q \cap p] \cap p = [[q] \cap [p]] \cap p = [q] \cap [p] \cap p = q \cap p .$$

So, $q \cap p$ is closed with respect to $[\mathfrak{T}]^p$. Similarly, we can prove also that $r \cap p$ is closed with respect to $[\mathfrak{T}]^p$.

Now, by assumption and (**T2**),

$$[[p-q] \cap (p-r)] \subseteqq [p] = p, \qquad [(p-q) \cap (p-r)] \subseteqq [p] = p .$$

Hence, by taking $q \cap p$ and $r \cap p$ in place of q^* and r^*, respectively,

$$\begin{aligned} [[p-q] \cap (p-r)] &= [[p-q] \cap p \cap (p-r)] \cap p = [[p-q]^p \cap (p-r)]^p \\ &= [[p-q \cap p]^p \cap (p-r \cap p)]^p = [(p-q \cap p) \cap (p-r \cap p)]^p \\ &= [(p-q) \cap (p-r)] \cap p = [(p-q) \cap (p-r)] . \end{aligned}$$

Now, take $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ as the same topology, for which $[p]=p$ holds for every subset p of s. Then, we have the example interpretation (**C**) of the classical logic as a special case of this extremal case. As the very special case of this interpretation where $\omega = \{1\}$, we have the example interpretation (**A**) of the classical logic.

Next, take $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ as any topologies equivalent to the same topology of a space satisfying (**T7**). Then, we have the example interpretation (**D**) of the intuitionistic logic as a special case of this extremal case.

2.3. Compositions and decompositions of logical pairs of topologies.

DEFINITION 4. Any pair $(\{\mathfrak{T}\}, [\mathfrak{T}])$ of topologies $\{\mathfrak{T}\}$ and $[\mathfrak{T}]$ of a space ω is called "decomposed" (into the class $\{\cdots, (\{\mathfrak{T}\}_i, [\mathfrak{T}]_i), \cdots\}$ of topologies $\{\mathfrak{T}\}_i$ and $[\mathfrak{T}]_i$ of the topological sub-spaces ω_i of ω having the closure operations "$\{\ \}_i$" and "$[\ ]_i$", respectively), if and only if the following conditions are satisfied:

1) $\omega = \sum_i \omega_i$,

2) $\{p \cap \omega_i\}_i = \{p \cap \omega_i\}$ and $[p \cap \omega_i]_i = [p \cap \omega_i]$,

3) $\{p\} = \sum_i \{p \cap \omega_i\}$ and $[p] = \sum_i [p \cap \omega_i]$,

where each expression of the form $\sum_i s_i$ denotes the set of all members of ω each of which belongs to one and only one s_i. In this case, the pair $(\{\mathfrak{T}\}, [\mathfrak{T}])$ is called the composed pair of pairs $(\{\mathfrak{T}\}_i, [\mathfrak{T}]_i)$ of topologies of the spaces ω_i. This will be denoted shortly by

$$(\{\mathfrak{T}\}, [\mathfrak{T}]) = \sum_i (\{\mathfrak{T}\}_i, [\mathfrak{T}]_i).$$

THEOREM 13. *Assume* $(\{\mathfrak{T}\}, [\mathfrak{T}]) = \sum_i (\{\mathfrak{T}\}_i, [\mathfrak{T}]_i)$. *Then,* $(\{\mathfrak{T}\}, [\mathfrak{T}])$ *is logical if and only if all pairs* $(\{\mathfrak{T}\}_i, [\mathfrak{T}]_i)$ *of topologies are logical.*

PROOF. Throughout this proof, we assume $(\{\mathfrak{T}\}, [\mathfrak{T}]) = \sum_i (\{\mathfrak{T}\}_i, [\mathfrak{T}]_i)$.

At first, let us further assume that $(\{\mathfrak{T}\}, [\mathfrak{T}])$ is logical. To prove in this case that every pair of topologies of the form $(\{\mathfrak{T}\}_i, [\mathfrak{T}]_i)$ is logical, take any subsets p, q and r of ω_i which are all closed with respect to the topology $\{\mathfrak{T}\}_i$. These sets are proved to be closed with respect to the topology $\{\mathfrak{T}\}$ as follows:

$$s = \{s\}_i = \{s \cap \omega_i\}_i = \{s \cap \omega_i\} = \{s\},$$

where s stands for any one of p, q or r. Accordingly, by assumption, **(T2)**, and Theorem 6,

$$\begin{aligned}
[[p-q]_i^p \cap (p-r)]_i^p &= [[(p-q) \cap \omega_i]_i \cap p \cap (p-r) \cap \omega_i]_i \cap p \\
&= [[(p-q) \cap \omega_i] \cap p \cap (p-r) \cap \omega_i] \cap p \\
&= [[p-q]^p \cap (p-r)]^p = [(p-q) \cap [p-r]^p]^p \\
&= [(p-q) \cap [(p-r) \cap \omega_i] \cap p \cap \omega_i] \cap p \\
&= [(p-q) \cap [(p-r) \cap \omega_i]_i \cap p \cap \omega_i]_i \cap p \\
&= [(p-q) \cap [p-r]_i^p]_i^p .
\end{aligned}$$

Hence, $(\{\mathfrak{T}\}_i, [\mathfrak{T}]_i)$ is logical according to Theorem 6.

Next, conversely, let us assume that the pairs $(\{\mathfrak{T}\}_i, [\mathfrak{T}]_i)$ are all logical. To show that the pair $(\{\mathfrak{T}\}, [\mathfrak{T}])$ is logical, take any closed sets p, q and r with respect to the topology $\{\mathfrak{T}\}$. Then, $p \cap \omega_i$, $q \cap \omega_i$, and $r \cap \omega_i$ are all closed with respect to the topology $\{\mathfrak{T}\}_i$. For, taking into account that $\{s \cap \omega_i\} \subseteqq \omega_i$ for $s = p, q, r$ and that all ω_i's are mutually disjoint, we can compute as follows:

$$s \cap \omega_i = \{s\} \cap \omega_i = (\sum_j \{s \cap \omega_j\}) \cap \omega_i = \{s \cap \omega_i\} \cap \omega_i = \{s \cap \omega_i\} = \{s \cap \omega_i\}_i .$$

Accordingly, we can further compute by making use of Theorem 6,

$$
\begin{aligned}
[[p-q]^p \cap (p-r)]^p &= \sum_i [[p-q] \cap p \cap (p-r) \cap \omega_i] \cap p \\
&= \sum_i [(\sum_j [(p-q) \cap \omega_j]) \cap p \cap (p-r) \cap \omega_i] \cap p \\
&= \sum_i [[(p-q) \cap \omega_i]_i \cap p \cap (p-r) \cap \omega_i]_i \cap p \\
&= \sum_i [[p \cap \omega_i - q \cap \omega_i]_i^p \cap (p \cap \omega_i - r \cap \omega_i)]_i^p \\
&= \sum_i [(p \cap \omega_i - q \cap \omega_i) \cap [p \cap \omega_i - r \cap \omega_i]_i^p]_i^p \\
&= \sum_i [(p-q) \cap \omega_i \cap \sum_j [(p-r) \cap \omega_j]_j \cap p]_i \cap p \\
&= \sum_i [(p-q) \cap [p-r] \cap p \cap \omega_i] \cap p \\
&= [(p-q) \cap [p-r]^p]^p .
\end{aligned}
$$

REMARK. According to Theorem 13, we can construct many logical pairs by composing various logical pairs each belonging to the extremal cases I or II.

Pairs of topologies belonging to the extremal case I also satisfy the condition (**T7**). Hence, pairs of topologies thus composed satisfy the same condition. It would be natural to ask the following questions:

(**A**) *Does every logical pair of topologies satisfy the condition* (**T7**)?

(**B**) *Is every logical pair of topologies satisfying the condition* (**T7**) *decomposable into pairs of togologies each belonging to the extremal cases I or II?*

In reality, I am conjecturing that every pair of topologies would be decomposable into pairs of topologies each belonging to the extremal cases I or II, although the above questions are both quite open for me.

§ 3. Intrinsic meanings of two kinds of closure operations and logical constants defined in the primitive logic LO.

According to my papers [3] and [4], the classical logic as well as the intuitionistic logic has been proved to be reducible to the primitive logic **LO** by defining S-closure operation "$(\ )^S$" for a proposition symbol S or T-closure operation "$(\ )_T$" for a predicate symbol T and by defining logical constants other than "*implication*" and "*universal quantification*" suitably in **LO** for a certain class of propositions. In my paper [6], I have proved further that the same device can be extended to any axiomatizable formal theory standing on the classical logic or the intuitionistic logic by taking S as a proposition in general and by taking T as a predicate in general. In the present section, I will try to expose the intrinsic meanings of the S- and T-closure operations and logical constants defined in connection with them in interpreting these operations and logical constants set-theoretically by logical pairs of topologies.

For any topology, there are a certain class Γ of closed sets each of which is equal to the closure of its open kernel and a certain class Ω of open sets each of which is equal to the open kernel of its closure. Closed sets of the class Γ correspond one-to-one to open sets of the class Ω. Let us denote the operation from any set to its open kernel by **K** and the operation from any set to its closure by **C**. Then, the sets of Γ are characterized by the fact that they are invariant with the operation **CK** $[x = \mathbf{CK}(x) = \mathbf{C}(\mathbf{K}(x))]$ and the sets of Ω are characterized by the fact that they are invariant with the operation **KC** $[x = \mathbf{KC}(x) = \mathbf{K}(\mathbf{C}(x))]$.

If we apply the operation **CK** to any set, we have a set belonging to Γ. If we apply the operation **KC** to any set, we have a set belonging to Ω. If we denote by **N** the operation from any set to its complementary set (complementary with respect to ω), the operation **K** is denoted by **NCN**. Accordingly, we obtain sets of Γ if we apply the operation **CNCN** to any sets, and the sets of Γ are characterized by the fact that they are invariant with respect to the operation **CNCN**.

The set-theoretical interpretation of the S-closure operation "$(\ )^S$" with respect to a proposition S is nothing but the operation **CNCN** with respect to the topology $[\mathfrak{T}]^s$. The intrinsic meanings of the logical constants $\overset{s}{\wedge}$, $\overset{s}{\vee}$, $\overset{s}{\rightarrow}$, and $(\overset{s}{\exists})$ can be exhibited easily in connection with the intrinsic meanings of the operation **CNCN** with respect to the topology $[\mathfrak{T}]^s$.

However, the real features of the operation **CNCN** in the extremal cases differ very much from each other as explained later.

The intrinsic meaning of the T-closure operation "$(\ )_T$" and the logical constants $\underset{T}{\wedge}$, $\underset{T}{\vee}$, $\underset{T}{\rightarrow}$, and $(\underset{T}{\exists})$ with respect to a predicate T, can be understood in connection with the intrinsic meanings of the S-closure operation and the logical constants $\overset{S}{\wedge}$, $\overset{S}{\vee}$, $\overset{S}{\rightarrow}$, and $(\overset{S}{\mathrm{E}})$ with respect to a proposition S having a parameter.

3.1. Closure operations and logical constants.

Corresponding to the S-closure $(\mathfrak{A})^S$ and T-closure $(\mathfrak{A})_T$ of a proposition $\mathfrak{A}$ with respect to a proposition S and a predicate T, I define the s-closure $(p)^s$ and the t-closure $(p)_t$ of a subset p of ω with respect to a subset s of ω and a subset $t(u)$ of ω having the parameter u running over $\boldsymbol{D}$, respectively.

DEFINITION 5.

$$(p)^s \rightleftharpoons (p \rightarrow s) \rightarrow s \quad \text{and} \quad (p)_t \rightleftharpoons (u)(p^{t(u)}).$$

Any set p is called s-closed with respect to the set s if and only if p is equal to $(p)^s$. Any set p is called t-closed with respect to the set $t(u)$ having a parameter u running over $\boldsymbol{D}$ if and only if p is equal to $(p)_t$.

I introduce also some combinations and operations which correspond to the logical constants and logical operations defined in **LO**.

DEFINITION 6.

$$p \overset{s}{\wedge} q \rightleftharpoons (p \to (q \to s)) \to s\,,$$

$$p \overset{s}{\vee} q \rightleftharpoons (p \to s) \to ((q \to s) \to s)\,,$$

$$\overset{s}{\neg}\, p \rightleftharpoons p \to s\,,$$

$$(\overset{s}{\exists} x) p(x) \rightleftharpoons (x)(p(x) \to s) \to s\,,$$

$$p \underset{t}{\wedge} q \rightleftharpoons (u)(p \overset{t(u)}{\wedge} q)\,,$$

$$p \underset{t}{\vee} q \rightleftharpoons (u)(p \overset{t(u)}{\vee} q)\,,$$

$$\underset{t}{\neg}\, p \rightleftharpoons (u)(\overset{t(u)}{\neg}\, p)\,,$$

$$(\underset{t}{\exists} x) p(x) \rightleftharpoons (u)((\overset{t(u)}{\exists} x) p(x))\,.$$

The following propositions can be proved easily in the primitive logic **LO** (see my paper [3]).

$$A \to (A)^S\,, \qquad A \to (A)_T\,,$$

$$((A)^S)^S \equiv (A)^S\,, \qquad ((A)_T)_T \equiv (A)_T\,,$$

$$(A \to B) \to ((A)^S \to (B)^S)\,, \qquad (A \to B) \to ((A)_T \to (B)_T)\,,$$

$$(S)^S \equiv S\,, \qquad S \to (A)^S\,, \qquad (x)T(x) \to (A)_T\,,$$

$$(x)((A(x))^S \to A(x)) \to (((x)A(x))^S \to (x)A(x))\,,$$

$$(x)((A(x))_T \to A(x)) \to (((x)A(x))_T \to (x)A(x))\,.$$

According to my paper [5] and H. Ono [7], any set-theoretical expression corresponding to a provable proposition in **LO** is equal to 0 if the conditions (**E1**)—(**E8**)* are satisfied.

Now, let $(\{\mathfrak{T}\}, [\mathfrak{T}])$ be any logical pair of topologies. Then, according to Theorem 7, the conditions (**E1**)—(**E8**)* are satisfied by closed sets with respect to the topology $\{\mathfrak{T}\}$ for " $\to$ " and " ( ) " defined by Definitions 2 and 3. Hence, by virtue of Theorem 5, we can see that the following propositions hold for closed sets with respect to the topology $\{\mathfrak{T}\}$.

1) $p \supseteqq (p)^s$, $p \supseteqq (p)_t$,

2) $((p)^s)^s = (p)^s$, $((p)_t)_t = (p)_t$,

3) $p \supseteqq q$ *implies* $(p)^s \supseteqq (q)^s$ *as well as* $(p)_t \supseteqq (q)_t$,

4) $(s)^s = s$,

5) $s \supseteqq (p)^s$, $(x)t(x) \supseteqq (p)_t$,

6) *If* $p(u)$ *is* s*-closed for every* u *and for a fixed set* s*, then* $(x)p(x)$ *is* s*-closed. If* $p(u)$ *is* t*-closed for every* u *and for a fixed one-dimensional matrix* $t(u)$*, then* $(x)p(x)$ *is* t*-closed.*

The following propositions are provable for any proposition S, for any S-closed proposition A, B, and C, and for any S-closed predicate $A(u)$ in the sense that the proposition $A(u)$ is S-closed for every u. (See my paper [3].)

$$A \overset{S}{\wedge} B \rightarrow A\,, \quad A \overset{S}{\wedge} B \rightarrow B\,, \quad (C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow A \overset{S}{\wedge} B))\,,$$

$$A \rightarrow A \overset{S}{\vee} B\,, \quad B \rightarrow A \overset{S}{\vee} B\,, \quad (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \overset{S}{\vee} B \rightarrow C))\,,$$

$$A(u) \rightarrow (\overset{S}{\exists} x)A(x)\,, \quad (x)(A(x) \rightarrow B) \rightarrow ((\overset{S}{\exists} x)A(x) \rightarrow B)\,,$$

$$(A)^S \equiv \overset{S}{\rightharpoondown}\overset{S}{\rightharpoondown} A\,.$$

Hence, according to my paper [5] and H. Ono [7], we can see by virtue of Theorem 5, that the following propositions hold for any closed set s with respect to the topology $\{\mathfrak{T}\}$, for any s-closed sets p, q, and r, and for any s-closed set $p(u)$ having a parameter u.

s1) $p \overset{s}{\wedge} q \supseteqq p$, $p \overset{s}{\wedge} q \supseteqq q$,

s2) $r \supseteqq p$ *and* $r \supseteqq q$ *imply* $r \supseteqq p \overset{s}{\wedge} q$,

s3) $p \supseteqq p \overset{s}{\vee} q$, $q \supseteqq p \overset{s}{\vee} q$,

s4) $p \supseteqq r$ *and* $q \supseteqq r$ *imply* $p \overset{s}{\vee} q \supseteqq r$,

s5) $p(u) \supseteqq (\overset{s}{\exists} x)p(x)$,

s6) *If* $p(x) \supseteqq q$ *for every* x*, then* $(\overset{s}{\exists} x)p(x) \supseteqq q$,

s7) $\overset{s}{\rightharpoondown}\overset{s}{\rightharpoondown} p = p$.

Also, the following propositions are provable for any predicate T, for any T-closed proposition A, B, and C, and for any T-closed predicate $A(u)$ in the sense that the proposition $A(u)$ is T-closed for every u. (See my paper [3].)

$$A \underset{T}{\wedge} B \rightarrow A\,, \quad A \underset{T}{\wedge} B \rightarrow B\,, \quad (C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow A \underset{T}{\wedge} B))\,,$$

$$A \rightarrow A \underset{T}{\vee} B\,, \quad B \rightarrow A \underset{T}{\vee} B\,, \quad (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \underset{T}{\vee} B \rightarrow C))\,,$$

$$A(u) \rightarrow (\underset{T}{\exists} x)A(x)\,, \quad (x)(A(x) \rightarrow B) \rightarrow ((\underset{T}{\exists} x)A(x) \rightarrow B)\,.$$

Hence, according to my paper [5] and H. Ono [7], we can see by virtue of Theorem 5, that the following propositions hold for any closed set $t(u)$ having a parameter u with respect to the topology $\{\mathfrak{T}\}$, for any t-closed sets p, q, and r, and for any t-closed set $p(u)$ having a parameter u.

t1) $p \underset{t}{\wedge} q \supseteqq p$, $p \underset{t}{\wedge} q \supseteqq q$,

t2) *$r \supseteqq p$ and $r \supseteqq q$ imply $r \supseteqq p \underset{t}{\wedge} q$,*

t3) $p \supseteqq p \underset{t}{\vee} q$, $q \supseteqq p \underset{t}{\vee} q$,

t4) *$p \supseteqq r$ and $q \supseteqq r$ imply $p \underset{t}{\vee} q \supseteqq r$,*

t5) $p(u) \supseteqq (\underset{t}{\exists} x)p(x)$,

t6) *If $p(x) \supseteqq q$ for every x, then $(\underset{t}{\exists} x)p(x) \supseteqq q$.*

REMARK. Let Γ^s be the class of s-closed sets for a set s. Then, every member of Γ^s is closed with respect to the topology $[\mathfrak{T}]^s$. **1)**—**6)** and **s1)**—**s7)** show that Γ^s forms a complemented complete lattice in which

$$p \overset{s}{\wedge} q = p \cup q, \qquad p \overset{s}{\vee} q = p \cap q,$$

$$(x)p(x) = \bigcup_x p(x), \qquad (\overset{s}{\exists} x)p(x) = \bigcap_x p(x)$$

hold for the lattice operations $\cup$, $\cap$, $\bigcup$, and $\bigcap$ in Γ^s.

Let Γ_t be the class of t-closed sets for a set $t(u)$ having a parameter u. Then, **1)**—**6)** and **t1)**—**t6)** show that Γ_t forms a complete lattice having the maximum member $(x)t(x)$ in which

$$p \underset{t}{\wedge} q = p \cup q, \qquad p \underset{t}{\vee} q = p \cap q,$$

$$(x)p(x) = \bigcup_x p(x), \qquad (\underset{t}{\exists} x)p(x) = \bigcap_x p(x)$$

hold for the lattice operations $\cup$, $\cap$, $\bigcup$, and $\bigcap$ in Γ_t.

3.2. Closure operations and logical constants in the extremal case I.

Let us assume that $(\{\mathfrak{T}\}, [\mathfrak{T}])$ is a logical pair of topologies in the extremal case I. Then, $(p)^s$ is s for $p \supseteqq s$, and $(p)^s$ is 0 otherwise. Accordingly, Γ^s turns out to be a class formed by only two members s and 0. The logical constants satisfy

1) *$p \overset{s}{\wedge} q = 0$ if $p = q = 0$, and $p \overset{s}{\wedge} q = s$ otherwise,*

2) *$p \overset{s}{\vee} q = s$ if $p = q = s$, and $p \overset{s}{\vee} q = 0$ otherwise,*

3) *$\overset{s}{\rightarrow} p = 0$ if $p = s$, and $\overset{s}{\rightarrow} p = s$ if $p = 0$,*

4) *$(\overset{s}{\exists} x)p(x) = 0$ if $p(u)$ is 0 for every u, and $(\overset{s}{\exists} x)p(x) = s$ otherwise.*

The intrinsic meaning of $(p)_t$ for a set $t(u)$ having a parameter u depends on how extensive the class of $t(u)$'s for various u's is. For example, let $\{\Gamma\}$ be the class of closed sets with respect to the topology $\{\mathfrak{T}\}$, and every member of $\{\Gamma\}$ be expressible in the form $t(u)$. Then, Γ_t is nothing but $\{\Gamma\}$.

3.3. Closure operations and logical constants in the extremal case II. Let us assume that $([\mathfrak{T}], [\mathfrak{T}])$ is a logical pair of topologies in the extremal case II. Then, the s-closure operation "$(p)^s$" for a closed set p with respect to $[\mathfrak{T}]$ is something like shaving business which restricts p to s taking off all *hairs* of p. Here, I call figuratively *hairs* the set of boundary points of p lying apart from the inner kernel of p.

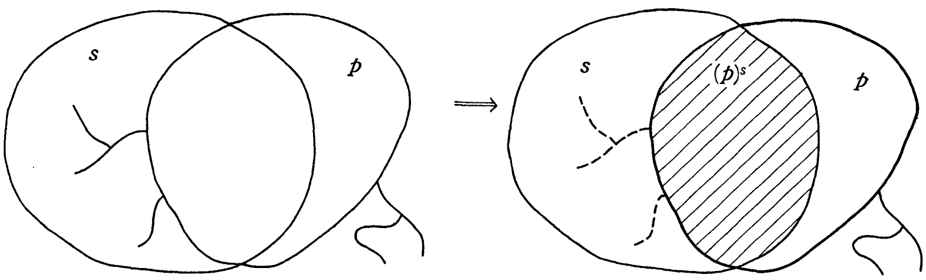

Figure 2.

Γ_s is formed by closed shaved bodies inside s. If p and q are both s-closed (*i. e.* shaved), $p \overset{s}{\wedge} q$ is nothing but the set-theoretical union $p \cup q$, but $p \overset{s}{\vee} q$ is obtained by shaving again the set-theoretical intersection $p \cap q$.

The intrinsic meaning of $(p)_t$ for a set $t(u)$ having a parameter u depends on how extensive the class of $t(u)$'s for various u's is, in this extremal case too. For example, let us assume that every closed set with respect to the topology $[\mathfrak{T}]$ can be expressed in the form $t(u)$. Then, Γ_t is nothing but the whole class of closed sets with respect to the same topology $[\mathfrak{T}]$.

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