J. Math. Soc. Japan
Vol. 21, No. 2, 1969

A note on the large inductive dimension of totally normal spaces

Dedicated to Professor Atuo Komatu for his 60th birthday

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(Received Aug. 8, 1968)

This paper gives a sum theorem, a characterization theorem, a product theorem and a coincidence theorem for the large (or equivalently strong) inductive dimension of totally normal spaces. The definitions and notations for the large and small inductive dimension can be seen in Nagata's dimension theory [11]. The concept of the total normality was introduced by C.H. Dowker in [1], offering a well designed class of spaces for considering the large inductive dimension, as follows: A space X is totally normal if i) X is normal and ii) for every open set G of X there exists a locally finite (in G) collection of cozero-sets $\{G_\alpha\}$ with $\cup G_\alpha = G$. We say that G_α is a cozero-set if there is a real-valued continuous function f_α defined on the whole space X with $G_\alpha = \{x : f_\alpha(x) \neq 0\}$. In [1] the following are proved:

(a) A hereditarily paracompact space is totally normal.

(b) A perfectly normal space is totally normal.

(c) There exists a totally normal space which is neither paracompact nor perfectly normal.

(d) Every subset of a totally normal space is totally normal.

(e) If X is a totally normal space, then the subset theorem and the sum theorem for the large inductive dimension are true as follows: i) If $Y \subset X$, then $\operatorname{Ind} Y \leqq \operatorname{Ind} X$. ii) If Y_i, $i = 1, 2, \cdots$, are closed in X, then $\operatorname{Ind}(\cup Y_i) = \sup \operatorname{Ind} Y_i$.

All spaces considered in this paper are Hausdorff.

THEOREM 1. *Let X be a totally normal space having the weak topology*[1] *with respect to a closed covering $\{F_\alpha : \alpha \in A\}$. If $\operatorname{Ind} F_\alpha \leqq n$ for each $\alpha \in A$, then $\operatorname{Ind} X \leqq n$.*

PROOF (by double induction). The theorem is evidently true for the case:

1) According to K. Morita [7] a space X has the weak topology with respect to its closed covering $\{F_\alpha : \alpha \in A\}$ if the following condition is satisfied: A subset S of X is closed if and only if for an arbitrary subset B of A with $S \subset \bigcup\{F_\alpha : \alpha \in B\}$, $S \cap F_\alpha$ is closed for each $\alpha \in B$.

$n=-1$. Assume that the theorem is true for dimension $n-1$. Let us prove the theorem is true for dimension n. Well order the index set A such that A consists of all ordinals α less than or equal to some fixed ordinal η:

$$A=\{\alpha : 0 \leqq \alpha \leqq \eta\}.$$

Let S and T be a disjoint pair of closed sets of X. Set

$$K_\alpha = \bigcup_{\beta \leqq \alpha} F_\beta .$$

Let (P_α) be the proposition such that there exists, for every $\beta \leqq \alpha$, a triple of sets G_β, C_β, H_β satisfying the following six conditions:

(1) $\{G_\beta, C_\beta, H_\beta\}$ is disjoint.

(2) $K_\beta = G_\beta \cup C_\beta \cup H_\beta$.

(3) Both G_β and H_β are open in K_β.

(4) $S \cap K_\beta \subset G_\beta$ and $T \cap K_\beta \subset H_\beta$.

(5) For every $\gamma \leqq \beta$, $G_\gamma = G_\beta \cap K_\gamma$, $C_\gamma = C_\beta \cap K_\gamma$, $H_\gamma = H_\beta \cap K_\gamma$.

(6) $\operatorname{Ind} C_\beta \leqq n-1$.

Since $\operatorname{Ind} K_0 = \operatorname{Ind} F_0 \leqq n$, (P_0) is evidently true. Let α be an ordinal with $0<\alpha \leqq \eta$. Assuming (P_β) for every $\beta<\alpha$, let us prove (P_α). Set

$$C=\cup\{C_\beta : \beta<\alpha\}.$$

Then C is closed as follows. If $\beta<\alpha$, then $C \cap K_\beta = C_\beta$ by (5). Since C_β is closed in K_β by (1), (2) and (3), $C_\beta \cap F_\beta$ is closed in F_β. Since $C_\beta \cap F_\beta = C \cap F_\beta$, $C \cap F_\beta$ is closed in F_β. Thus C is closed in $\bigcup_{\beta<\alpha} F_\beta$ and hence so in X.

Since $C \cap F_\beta \subset C \cap K_\beta = C_\beta$, $\operatorname{Ind} C \cap F_\beta \leqq n-1$ by (6). Since C has the weak topology with respect to

$$\{C \cap F_\beta : \beta<\alpha\},$$

we obtain

$$\operatorname{Ind} C \leqq n-1$$

by the induction hypothesis. Set

$$G=\cup\{G_\beta : \beta<\alpha\},$$

$$H=\cup\{H_\beta : \beta<\alpha\},$$

$$K=\cup\{K_\beta : \beta<\alpha\}.$$

Then G and H are open in K by (3) and

$$K=G \cup C \cup H$$

by (2). Moreover $\{G, C, H\}$ is disjoint by (1) and (4). It is evident that

$$S \cap K \subset G \quad \text{and} \quad T \cap K \subset H .$$

Since a totally normal space is hereditarily normal by (d) and $S \cup G$ and $T \cup H$

are relatively closed sets in $X-C$, there exist relatively open sets P and Q of $X-C$ such that

$$S\cup G\subset P,\ T\cup H\subset Q,\ \bar{P}\cap\bar{Q}\subset C.$$

Since $P\cap(F_{\alpha}-K)$ and $Q\cap(F_{\alpha}-K)$ constitute a disjoint relatively closed pair in $F_{\alpha}-K$, there exist, by (e), sets G', C' and H' satisfying the following conditions:

(7) $F_{\alpha}-K=G'\cup C'\cup H'$.

(8) $\{G', C', H'\}$ is disjoint.

(9) G' and H' are open in $F_{\alpha}-K$.

(10) $\operatorname{Ind} C'\leqq n-1$.

Setting

$$G_{\alpha}=G\cup G',$$

$$C_{\alpha}=C\cup C',$$

$$H_{\alpha}=H\cup H',$$

let us prove (P_{α}). The conditions (1), (2), (4), (5), where β's are replaced by α's, are evidently true. Since $G_{\alpha}-K=G'$, $G_{\alpha}-K$ is open in $F_{\alpha}-K$ by (9) and hence open in K_{α}. Since

$$G_{\alpha}=(G_{\alpha}-K)\cup(P\cap K_{\alpha}),$$

then G_{α} is open in K_{α}. Analogously H_{α} is also open in K_{α}. Thus the condition (3) is satisfied for $\beta=\alpha$. Since $\operatorname{Ind} C\leqq n-1$ and $\operatorname{Ind} C'\leqq n-1$ by (10), we get at once

$$\operatorname{Ind} C_{\alpha}\leqq n-1$$

by C. H. Dowker [**1**, 2.1]. Thus the condition (6) for $\beta=\alpha$ is also satisfied. Now by transfinite induction (P_{α}) is true for each α. Especially (P_{η}) is true. (P_{η}) implies the existence of a closed set C_{η}, with $\operatorname{Ind} C_{\eta}\leqq n-1$, separating S and T. Thus $\operatorname{Ind} X\leqq n$ and the induction on n is completed. The proof is finished.

By this theorem and (e) we get the following theorem without any change of the proof given in K. Nagami [**9**, Lemma 2].

THEOREM 2. *In a totally normal space X the following are equivalent:*

i) $\operatorname{Ind} X\leqq n$.

ii) *Every binary open covering of X is refined by a σ-locally finite open covering $\mathfrak{U}$ such that for each element U of $\mathfrak{U}$*

$$\operatorname{Ind} B(U)\leqq n-1,$$

where $B(U)$ indicates the boundary of U.

According to K. Morita [8], a space X is a $P(\mathfrak{m})$-*space* if the following condition is satisfied:

If $\{G(\alpha_1 \cdots \alpha_i): \alpha_1, \cdots, \alpha_i \in \Omega\}$ is a collection of open sets of X such that the power of Ω, $|\Omega|$, is at most $\mathfrak{m}$ and such that

$$G(\alpha_1 \cdots \alpha_i) \subset G(\alpha_1 \cdots \alpha_i \alpha_{i+1})$$

for each sequence $\alpha_1, \alpha_2, \cdots$, then there exists a collection of closed sets

$$\{H(\alpha_1 \cdots \alpha_i): \alpha_1, \cdots, \alpha_i \in \Omega\}$$

such that

$$\bigcup_i H(\alpha_1 \cdots \alpha_i) = X \quad \text{whenever} \quad \bigcup_i G(\alpha_1 \cdots \alpha_i) = X .$$

If a space X is a $P(\mathfrak{m})$-space for each power $\mathfrak{m}$, then X is a *P-space.*

According to K. Nagami [**10**], a space X is a *Σ-space* if X has a sequence

$$\{\mathfrak{F}_i = \{F_{i\alpha}: \alpha \in A_i\},\ i = 1, 2, \cdots\}$$

of locally finite closed coverings of X which satisfies the following condition: If $K_1 \supset K_2 \supset \cdots$ is a sequence of non-empty closed sets of X such that

$$K_i \subset C(x, \mathfrak{F}_i)^{2)}$$

for some point x and for each i, then $\cap K_i \neq \emptyset$.

The above sequence is a *Σ-net* of X. If $|A_i| \leqq \mathfrak{m}$ for each i, then such a Σ-net is a *$\Sigma(\mathfrak{m})$-net.* A space is a *$\Sigma(\mathfrak{m})$-space* if it has a $\Sigma(\mathfrak{m})$-net. A *spectral Σ-net* is a Σ-net $\{\mathfrak{F}_i\}$ such that

i) each $\mathfrak{F}_i$ is indexed as

$$\mathfrak{F}_i = \{F(\alpha_1 \cdots \alpha_i): \alpha_1, \cdots, \alpha_i \in \Omega\},$$

ii) $F(\alpha_1 \cdots \alpha_i) = \cup \{F(\alpha_1 \cdots \alpha_i \alpha_{i+1}): \alpha_{i+1} \in \Omega\}$ for every sequence $\alpha_1, \cdots, \alpha_i, \alpha_{i+1}$,

iii) each $\mathfrak{F}_i$ is (finitely) multiplicative,

iv) for each point x in X there exists a sequence $\alpha_1, \alpha_2, \cdots \in \Omega$ such that

$$C(x, \mathfrak{F}_i) = F(\alpha_1 \cdots \alpha_i)$$

for each i.

LEMMA. *A $\Sigma(\mathfrak{m})$-space has a spectral Σ-net with $|\Omega| \leqq \mathfrak{m}$.*

This is K. Nagami [**10**, Lemma 1-4].

THEOREM 3. *Let $X(\neq \emptyset)$ be a $P(\mathfrak{m})$-space and Y a $\Sigma(\mathfrak{m})$-space such that $X \times Y$ is a hereditarily paracompact space*[3]. *Then*

$$\operatorname{Ind}(X \times Y) \leqq \operatorname{Ind} X + \operatorname{Ind} Y .$$

PROOF (by induction on $\operatorname{Ind} X + \operatorname{Ind} Y$). When $\operatorname{Ind} X$ or $\operatorname{Ind} Y$ is infinite,

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2) $C(x, \mathfrak{F}_i)$ denotes the intersection of all elements F of $\mathfrak{F}_i$ with $x \in F$.

3) Our proof needs merely the paracompactness of X and Y and the total normality of $X \times Y$. But this condition automatically implies that $X \times Y$ is hereditarily paracompact.

the theorem is trivially true. Hence we prove the theorem for the case: $\operatorname{Ind} X = m < \infty$, $\operatorname{Ind} Y = n < \infty$. When $m+n=-1$, Y is empty. Hence the theorem is true. Assume that the theorem is true for the case when $\operatorname{Ind} X + \operatorname{Ind} Y < k$. Let $m+n=k$. To apply Theorem 2 let $\mathfrak{G}$ be an arbitrary binary open covering of $X \times Y$. Let

$$\{\mathfrak{F}_i = \{F(\alpha_1 \cdots \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega\}\}$$

be a spectral Σ-net of Y such that $|\Omega| \leqq \mathfrak{m}$. By K. Morita [8, Lemma to Theorem 3.2, p. 22] there exists for each i a locally finite open covering

$$\mathfrak{D}_i = \{D(\alpha_1 \cdots \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega\}$$

of Y such that

$$F(\alpha_1 \cdots \alpha_i) \subset D(\alpha_1 \cdots \alpha_i)$$

for each sequence $\alpha_1, \alpha_2, \cdots$. Let

$$\mathfrak{W} = \{U_\lambda \times V_\lambda : \lambda \in \Lambda(\alpha_1 \cdots \alpha_i)\}$$

be the collection of all possible $U_\lambda \times V_\lambda$ which satisfy the following conditions:

(1) Each U_λ is an open set of X.

(2) Each V_λ is an open set of Y which admits a finite collection $\mathfrak{V}_\lambda$ of open sets of Y whose sum is V_λ such that for each element V in $\mathfrak{V}_\lambda$

$$\operatorname{Ind} B(V) \leqq n-1 .$$

(3) Each $U_\lambda \times \mathfrak{V}_\lambda <$ (refines) $\mathfrak{G}$.

(4) $F(\alpha_1 \cdots \alpha_i) \subset V_\lambda \subset D(\alpha_1 \cdots \alpha_i)$ for each $\lambda \in \Lambda(\alpha_1 \cdots \alpha_i)$.

Set

$$U(\alpha_1 \cdots \alpha_i) = \cup \{U_\lambda : \lambda \in \Lambda(\alpha_1 \cdots \alpha_i)\} .$$

Then

$$U(\alpha_1 \cdots \alpha_i) \subset U(\alpha_1 \cdots \alpha_i \alpha_{i+1})$$

for each sequence $\alpha_1, \alpha_2, \cdots$. Since X is a normal $P(\mathfrak{m})$-space, there exist cozero-sets $C(\alpha_1 \cdots \alpha_i)$ such that

(5) $C(\alpha_1 \cdots \alpha_i) \subset U(\alpha_1 \cdots \alpha_i)$,

(6) $\bigcup_i U(\alpha_1 \cdots \alpha_i) = X$ implies $\bigcup_i C(\alpha_1 \cdots \alpha_i) = X$.

Let $C_j(\alpha_1 \cdots \alpha_i)$, $j=1, 2, \cdots$, be open sets of X such that

(7) $\bigcup_j C_j(\alpha_1 \cdots \alpha_i) = C(\alpha_1 \cdots \alpha_i)$,

(8) $B(C_j(\alpha_1 \cdots \alpha_i)) \subset C(\alpha_1 \cdots \alpha_i)$ for each j,

(9) $\operatorname{Ind} B(C_j(\alpha_1 \cdots \alpha_i)) \leqq m-1$ for each j.

Set

$$\mathfrak{U}(\alpha_1 \cdots \alpha_i) = \{U_\lambda : \lambda \in \Lambda(\alpha_1 \cdots \alpha_i)\} .$$

Let

$$\mathfrak{U}'(\alpha_1 \cdots \alpha_i) = \{U'_\lambda : \lambda \in \Lambda(\alpha_1 \cdots \alpha_i)\}$$

be a locally finite (in $U(\alpha_1 \cdots \alpha_i)$) open covering of $U(\alpha_1 \cdots \alpha_i)$ such that

(10) each U'_λ is contained in U_λ,

(11) $\operatorname{Ind} B'(U'_\lambda) \leqq m-1$, for each λ, where $B'(U'_\lambda)$ is the relative boundary of U'_λ in $U(\alpha_1 \cdots \alpha_i)$.

Set

(12) $U^j_\lambda = U'_\lambda \cap C_j(\alpha_1 \cdots \alpha_i)$,

(13) $\mathfrak{U}_j(\alpha_1 \cdots \alpha_i) = \mathfrak{U}'(\alpha_1 \cdots \alpha_i) | C_j(\alpha_1 \cdots \alpha_i) = \{U^j_\lambda : \lambda \in \Lambda(\alpha_1 \cdots \alpha_i)\}$.

Then $\mathfrak{U}_j(\alpha_1 \cdots \alpha_i)$ is locally finite in X. Moreover

(14) $\operatorname{Ind} B(U) \leqq m-1$

for each element U in $\mathfrak{U}_j(\alpha_1 \cdots \alpha_i)$. Set

$$\mathfrak{H}^j_\lambda = U^j_\lambda \times \mathfrak{V}_\lambda,\ \lambda \in \Lambda(\alpha_1 \cdots \alpha_i)\,,$$

$$\mathfrak{H}^j(\alpha_1 \cdots \alpha_i) = \cup \{\mathfrak{H}^j_\lambda : \lambda \in \Lambda(\alpha_1 \cdots \alpha_i)\}\,,$$

$$\mathfrak{H}^j_i = \cup \{\mathfrak{H}^j(\alpha_1 \cdots \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega\}\,,$$

$$\mathfrak{H} = \cup \{\mathfrak{H}^j_i : i, j = 1, 2, \cdots\}\,.$$

Since each $\mathfrak{H}^j_i$ is locally finite in $X \times Y$, $\mathfrak{H}$ is a σ-locally finite open collection refining $\mathfrak{G}$. Moreover

$$\operatorname{Ind} B(H) \leqq k-1$$

for each element H of $\mathfrak{H}$ by induction assumption.

To prove $\mathfrak{H}$ covers $X \times Y$ let (x, y) be an arbitrary point of $X \times Y$. Set

$$C(y) = \cap C(y, \mathfrak{F}_i)\,.$$

Since $C(y)$ is compact, there exist an open neighborhood D of x and a finite collection

$$\mathfrak{E} = \{E_1, \cdots, E_s\}$$

of open sets of Y such that

(15) $\mathfrak{E}$ covers $C(y)$,

(16) $D \times \mathfrak{E} < \mathfrak{G}$.

Set $E = \cup E_i$. Let $\beta_1, \beta_2, \cdots \in \Omega$ be a sequence such that

$$F(\beta_1 \cdots \beta_i) = C(y, \mathfrak{F}_i)$$

for each i. Let t be an integer such that

$$F(\beta_1 \cdots \beta_t) \subset E\,.$$

Let $L_1, \cdots, L_s$ be open sets of Y such that

(17) $L_i \subset E_i$, $i = 1, \cdots, s$,

(18) $F(\beta_1 \cdots \beta_t) \subset \cup L_i \subset E \cap D(\beta_1 \cdots \beta_t)$,

(19) $\operatorname{Ind} B(L_i) \leqq n-1$ for each i.

Set

$$L = L_1 \cup \cdots \cup L_s\,.$$

Then $D\times L$ is an element of $\mathfrak{W}$ and $D\subset U(\beta_1\cdots\beta_t)$. Since x was arbitrary,

$$\bigcup_i U(\beta_1\cdots\beta_i)=X.$$

Hence for some $u\geqq t$ and some j

$$x\in C_j(\beta_1\cdots\beta_u).$$

Let $M_1,\cdots,M_s$ be open sets of Y such that

(20) $F(\beta_1\cdots\beta_u)\subset\cup M_i\subset D(\beta_1\cdots\beta_u)$,

(21) $M_i\subset L_i,\ i=1,\cdots,s$,

(22) $\operatorname{Ind} B(M_i)\leqq n-1,\ i=1,\cdots,s$.

Set

$$M=\cup M_i.$$

Then there exists a $\mu\in\Lambda(\beta_1\cdots\beta_u)$ such that

(23) $D=U_\mu$,

(24) $M=V_\mu$,

(25) $\{M_1,\cdots,M_s\}=\mathfrak{B}_\mu$.

Thus

$$(x,y)\in C_j(\beta_1\cdots\beta_u)\times V_\mu.$$

Let ν be an element of $\Lambda(\beta_1\cdots\beta_u)$ such that $x\in U_\nu^j$. Then $x\in U_\nu^j$ by (12). Since (x,y) is now contained in an element of $\mathfrak{H}_u^j$, $\mathfrak{H}$ is a covering of $X\times Y$. Since i) $\mathfrak{H}$ is already σ-locally finite, ii) $\mathfrak{H}<\mathfrak{G}$ and iii) $\operatorname{Ind} B(H)\leqq k-1$ for each $H\in\mathfrak{H}$, then $\operatorname{Ind}(X\times Y)\leqq k$ by Theorem 2. Thus the induction is completed and the proof is finished.

COROLLARY 1. *Let X be a P-space and Y a Σ-space such that $X\times Y$ is a hereditarily paracompact space. If X or Y is not empty, then*

$$\operatorname{Ind}(X\times Y)\leqq\operatorname{Ind} X+\operatorname{Ind} Y.$$

The coincidence of the large and small inductive dimension, $\operatorname{Ind} X=\operatorname{ind} X$, has been proved for the following cases:

(α) Metric spaces with the star-finite property (K. Morita [**6**]).

(β) Hereditarily paracompact spaces with the star-finite property (Y. Katsuta [**4**]).

(γ) Totally paracompact metric spaces (R. Ford [**3**]).

(δ) Order totally paracompact metric spaces (B. Fitzpatrick, Jr. and R. Ford [**2**]).

Case (β) is a generalization of Case (α). Case (δ) is a generalization of Cases (α) and (γ). A slight modification of total paracompactness yields the concept of σ-total paracompactness, which is a generalization of Cases (α), (β) and (γ). Then we shall prove the equality between two inductive dimensions for such class of spaces which are also totally normal.

DEFINITION. A space X is *σ-totally paracompact* if for every open base $\mathfrak{G}$ of X there exists a σ-locally finite open covering $\mathfrak{H}$ of X having the following property:

For each element H of $\mathfrak{H}$ there exists an element G of $\mathfrak{G}$ such that $H \subset G$ and $B(H) \subset B(G)$.

THEOREM 4. *Let X be a totally normal, σ-totally paracompact space*[4]. *Then*

$$\operatorname{Ind} X = \operatorname{ind} X .$$

PROOF (by induction). It is almost evident that $\operatorname{ind} X \leqq \operatorname{Ind} X$. So we merely prove $\operatorname{ind} X \geqq \operatorname{Ind} X$. When $\operatorname{ind} X = \infty$, there is nothing to do. Consider the case when $\operatorname{ind} X = n < \infty$. Put the induction assumption that the desired inequality is true for dimension $< n$. To apply Theorem 2 let $\mathfrak{U}$ be an arbitrary binary open covering of X. Since $\operatorname{ind} X = n$, there exists an open base $\mathfrak{G}$ of X such that $\operatorname{ind} B(G) \leqq n-1$ for each element G of $\mathfrak{G}$. Then by induction assumption $\operatorname{Ind} B(G) \leqq n-1$ for each element G of $\mathfrak{G}$. Let $\mathfrak{H}$ be a σ-locally finite open covering of X such that for each element H of $\mathfrak{H}$ there exists an element G of $\mathfrak{G}$ with $H \subset G$ and with $B(H) \subset B(G)$. Then $\mathfrak{H}$ refines $\mathfrak{U}$ and $\operatorname{Ind} B(H) \leqq n-1$ for each element H of $\mathfrak{H}$. Thus $\operatorname{Ind} X \leqq n$ by Theorem 2 and the induction is completed. The proof is finished.

COROLLARY 2. *Let $X (\neq \emptyset)$ and $Y (\neq \emptyset)$ be spaces such that $X \times Y$ is totally normal and σ-totally paracompact. Then*

$$\operatorname{Ind}(X \times Y) \leqq \operatorname{Ind} X + \operatorname{Ind} Y .$$

PROOF. It is almost evident, by induction, that

$$\operatorname{ind}(X \times Y) \leqq \operatorname{ind} X + \operatorname{ind} Y .$$

Hence

$$\operatorname{Ind}(X \times Y) \leqq \operatorname{Ind} X + \operatorname{Ind} Y$$

by Theorem 4.

This is a generalization of Y. Katsuta [4, Theorem 2].

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4) Every open covering of a σ-totally paracompact space can be refined by a σ-locally finite open covering. Hence by E. Michael [5] every regular, σ-totally paracompact space is paracompact. Thus every totally normal, σ-totally paracompact space is hereditarily paracompact.

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