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Fractional powers of operators, IV
Potential operators

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A closed linear operator A in a Banach space X is said to be non-negative if $(0, \infty)$ is contained in the resolvent set of $-A$ and if $\lambda(\lambda+A)^{-1}$ is uniformly bounded for $0<\lambda<\infty$. This is a short supplement to the third paper of the author's series on fractional powers of non-negative operators A and mainly concerned with the potential operator associated with A, which is by definition the inverse A_-^{-1} of the restriction A_- of A to the closure $\overline{R(A)}$.

A typical result is the Abel and the Cesàro (the Cauchy) convergence of the integral formula

$$A_-^{-1}x=\int_0^\infty T_s x\, ds$$

when $-A$ is the infinitesimal generator of a bounded continuous (analytic resp.) semi-group T_t. A related integral formula of A_-^α with $\operatorname{Re}\alpha<0$ is also investigated.

§ 1. Potential operators.

Suppose that $-A$ generates a bounded continuous semi-group T_t in a Banach space X. Then for $\lambda>0$ we have

$$(1.1)\qquad (\lambda+A)^{-1}x=\int_0^\infty e^{-\lambda s}T_s x\, ds\,, \qquad x\in X\,.$$

Letting $\lambda\to 0$, we may expect that

$$(1.2)\qquad A^{-1}x=\int_0^\infty T_s x\, ds\,, \qquad x\in D(A^{-1})\,.$$

Of course, this is not true in general, for A need not be even one-to-one.

Yosida [4] proves, however, that A has a densely defined inverse A^{-1} if and only if

$$(1.3)\qquad \lambda(\lambda+A)^{-1}x\to 0 \qquad \text{as } \lambda\to 0$$

for all $x\in X$ and that if this is the case, then the potential operator A^{-1} is expressed as

$$A^{-1}x=\lim_{\lambda\to 0}\int_0^\infty e^{-\lambda s}T_s x\, ds\,, \qquad x\in D(A^{-1})\,. \tag{1.4}$$

In other words, (1.2) holds in the sense of Abel. The purpose of this paper is to obtain more results in this direction.

In spite of Yosida's definition, we define the potential operator associated with a non-negative operator A to be the inverse A_-^{-1} of its restriction A_- to $\overline{R(A)}$. We know that A_- is one-to-one by the Abelian ergodic theorem (Theorem 1.1 of [3]).

THEOREM 1.1. *Let A be a non-negative operator. If there is a sequence of positive numbers $\lambda_j\to 0$ such that*

$$y=\text{w-}\lim_{j\to\infty}(\lambda_j+A)^{-1}x \tag{1.5}$$

exists, then $x\in D(A_-^{-1})$ and $y=A_-^{-1}x$.

Conversely, if $x\in D(A_-^{-1})$, then

$$A_-^{-1}x=\text{s-}\lim_{\lambda\to 0}(\lambda+A)^{-1}x\,. \tag{1.6}$$

PROOF. Suppose that limit (1.5) exists for some $\lambda_j\to 0$. Then, $\lambda_j(\lambda_j+A)^{-1}x$ converges to zero strongly. Hence it follows from the Abelian ergodic theorem (Theorem 1.1 of [3]) that x belongs to $\overline{R(A)}$ and hence so does y. Since A is closed and $A(\lambda_j+A)^{-1}x=x-\lambda_j(\lambda_j+A)^{-1}x$ converges to x, we have $y\in D(A_-)=D(A)\cap\overline{R(A)}$ and $x=A_-y$.

Conversely suppose that $x=A_-y$ belongs to $D(A_-^{-1})$. Then applying the Abelian ergodic theorem to y, we have

$$\begin{aligned}(\lambda+A)^{-1}x&=(\lambda+A)^{-1}A_-y\\ &=y-\lambda(\lambda+A)^{-1}y\to y \qquad \text{as } \lambda\to 0\,.\end{aligned}$$

This completes the proof.

Remark that the theorem is also a special case of Theorem 3.1 of [1].

When $-A$ generates a bounded continuous semi-group T_t, (1.6) is written

$$\begin{aligned}A_-^{-1}x&=\text{s-}\lim_{\lambda\to\infty}\int_0^\infty e^{-\lambda s}T_s x\, ds\\ &=(\mathrm{A})\int_0^\infty T_s x\, ds\,.\end{aligned} \tag{1.7}$$

The following theorem shows that the integral converges in the sense of Cesàro also.

THEOREM 1.2. *Let $-A$ be the infinitesimal generator of a bounded continuous semi-group T_t and let σ be a positive number.*

If there is a sequence $t_j \to \infty$ *such that*

$$y = \text{w-}\lim_{j\to\infty} t_j^{-\sigma}\int_0^{t_j}(t_j-s)^\sigma T_s x\, ds \tag{1.8}$$

exists, then $x \in D(A_-^{-1})$ *and* $y = A_-^{-1}x$.

Conversely, if $x \in D(A_-^{-1})$, *then*

$$\begin{aligned} A_-^{-1}x &= \text{s-}\lim_{t\to\infty} t^{-\sigma}\int_0^t (t-s)^\sigma T_s x\, ds \\ &= (\mathrm{C}, \sigma)\int_0^\infty T_s x\, ds\,. \end{aligned} \tag{1.9}$$

PROOF. According to [3], we use the notation

$$I_t^{(\sigma)}x = \Gamma(\sigma)^{-1}\int_0^t (t-s)^{\sigma-1}T_s x\, ds\,.$$

Then,

$$t^{-\sigma}\int_0^t (t-s)^\sigma T_s x\, ds = \Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma+1)}x\,.$$

Therefore, if limit (1.8) exists, $\Gamma(\sigma+1)t^{-\sigma-1}I_t^{(\sigma+1)}x$ converges to zero. This implies by the Cesàro ergodic theorem (Theorem 1.4 of [3]) that x belongs to $\overline{R(A)}$. Since $\overline{R(A)}$ is invariant under T_t, y is also contained in $\overline{R(A)}$.

Now, applying [3], Lemma 1.2, we have $\Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma+1)}x \in D(A)$ and

$$A(\Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma+1)}x) = x - \Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma)}x\,.$$

The Cesàro ergodic theorem shows that the second term converges to zero as $t \to \infty$. Thus, we have $y \in D(A_-)$ and $A_-y = x$.

Conversely suppose that $x = A_-y$. Then

$$\Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma+1)}x = y - \Gamma(\sigma+1)t^{-\sigma}I_t^{(\sigma)}y$$

converges to y, for y is in $\overline{R(A)}$.

THEOREM 1.3. *Let* $-A$ *be the infinitesimal generator of a bounded analytic semi-group* T_t.

If there is a sequence $t_j \to \infty$ *such that*

$$y = \text{w-}\lim_{j\to\infty}\int_0^{t_j} T_s x\, ds \tag{1.10}$$

exists, then $x \in D(A_-^{-1})$ *and* $y = A_-^{-1}x$.

If $x \in D(A_-^{-1})$, *then*

$$A_-^{-1}x = \text{s-}\lim_{t\to\infty}\int_0^t T_s x\, ds\,. \tag{1.11}$$

PROOF. Write

$$I_t x = \int_0^t T_s x\, ds$$

as in [3]. Suppose that limit (1.10) exists. Then $t_j^{-1}I_{t_j}x$ converges to zero. Thus, by the mean ergodic theorem, x and y are contained in $\overline{R(A)}$. As is well known (see Lemma 1.2 of [3]), $I_{t_j}x \in D(A)$ and we have

$$AI_{t_j}x = x - T_{t_j}x \to x$$

by the simple ergodic theorem (Theorem 1.5 of [3]). Thus it follows that $y \in D(A_-)$ and $A_-y = x$.

In the same way, we have for $x = A_-y \in D(A_-^{-1})$

$$I_t x = y - T_t y \to y \qquad \text{as } t \to \infty .$$

§ 2. Negative powers.

When T_t decays exponentially, (1.2) is generalized as

$$A_-^{\alpha}x = \Gamma(-\alpha)^{-1}\int_0^{\infty} s^{-\alpha-1}T_s x\, ds , \qquad x \in X , \tag{2.1}$$

for $\operatorname{Re}\alpha < 0$ (see [1], Proposition 11.1). We discuss, in this section, the summability of (2.1) in the case where T_t is only uniformly bounded.

First, we prepare three propositions in order to prove the Abel summability in the case where A is a non-negative operator.

The first one is given in [1] as Proposition 6.7 without proof ($(\mu+A)_0^{\alpha}$ in [1] should be $(\mu+A)_-^{\alpha}$).

PROPOSITION 2.1. *Let A be a non-negative operator, let $\operatorname{Re}\alpha < 0$ and let $\mu > 0$.*

If $x \in \overline{R(A)}$, then $(A(\mu+A)^{-1})_0^{-\alpha}x \in D(A_-^{\alpha})$ and we have

$$A_-^{\alpha}(A(\mu+A)^{-1})_0^{-\alpha}x = (\mu+A)_-^{\alpha}x . \tag{2.2}$$

If $x \in D(A_-^{\alpha})$, then

$$\begin{aligned} A_-^{\alpha}(A(\mu+A)^{-1})_0^{-\alpha}x &= (A(\mu+A)^{-1})_0^{-\alpha}A_-^{\alpha}x \\ &= (\mu+A)_-^{\alpha}x . \end{aligned} \tag{2.3}$$

PROOF. First we note that

$$(\mu A)_*^{\alpha} = \mu^{\alpha}A_*^{\alpha} \tag{2.4}$$

holds for any non-negative operator A and any subscript $*$ ([1], Theorem 10.1). Since $(A(\mu+A)^{-1})_B = A_R(\mu+A_R)^{-1}$ (see [1] for the notation), we have

$$\begin{aligned} (A(\mu+A)^{-1})_0^{-\alpha} &= (A_R(\mu+A_R)^{-1})_0^{-\alpha} \\ &= (\mu^{-1}(\mu^{-1}+A_R^{-1})^{-1})_0^{-\alpha} \\ &= \mu^{\alpha}((\mu^{-1}+A_R^{-1})^{-1})_0^{-\alpha} \\ &= \mu^{\alpha}(\mu^{-1}+A_R^{-1})_0^{\alpha} . \end{aligned}$$

The last equality follows from the remark prior to Proposition 4.10 of [1]. We have also

$$A_{-}^{\alpha}=(A_{R}^{-1})_{+}^{-\alpha}$$

in the same way.

On the other hand, it follows that

$$(\mu+A_{R})_{-}^{\alpha}=((\mu+A_{R})^{-1})_{+}^{-\alpha}=\mu^{\alpha}(\mu(\mu+A_{R})^{-1})_{+}^{-\alpha}$$
$$=\mu^{\alpha}(A_{R}^{-1}(\mu^{-1}+A_{R}^{-1})^{-1})_{+}^{-\alpha}.$$

Since we have $(\mu+A)_{-}^{\alpha}x=(\mu+A_{R})_{-}^{\alpha}x$ for $x\in\overline{R(A)}$, the proposition reduces to [1], Proposition 6.3, where A and α are replaced by A_{R}^{-1} and $-\alpha$ respectively.

PROPOSITION 2.2. *If A is a non-negative operator and* $\mathrm{Re}\,\alpha<0$, *then for each $x\in\overline{R(A)}$ $(A(\lambda+A)^{-1})_{0}^{-\alpha}x$ converges strongly to x as $\lambda>0$ tends to zero.*

PROOF. In view of [1], Proposition 6.2 we have

$$(A(\lambda+A)^{-1})_{+}^{-\alpha}x=x+\alpha\lambda(\lambda+A)^{-1}x+\cdots$$
$$+\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}(\lambda(\lambda+A)^{-1})^{n}x \tag{2.5}$$
$$+\frac{\sin\pi\alpha}{\pi}\int_{0}^{\lambda}\tau^{-\alpha}(\lambda-\tau)^{\alpha+n}(\tau+A)^{-1}(\lambda+A)^{-n}x\,d\tau\,,$$

where n is an integer with $n+1+\mathrm{Re}\,\alpha>0$. Since $(\lambda(\lambda+A)^{-1})^{m}x$ converges strongly to zero as $\lambda\rightarrow 0$ by the Abelian ergodic theorem and since

$$\int_{0}^{\lambda}\tau^{-\mathrm{Re}\alpha-1}(\lambda-\tau)^{\mathrm{Re}\alpha+n}\lambda^{-n}d\tau$$

is uniformly integrable, the right hand side converges strongly to x as $\lambda\rightarrow 0$.

The following proposition is a generalization of the Abelian ergodic theorem.

PROPOSITION 2.3. *Let A be a non-negative operator and let* $\mathrm{Re}\,\alpha<0$. *If $x=x_{0}+x_{1}$ with $x_{0}\in N(A)$ and $x_{1}\in\overline{R(A)}$, then $\lambda^{-\alpha}(\lambda+A)_{-}^{\alpha}x$ converges strongly to x_{0} as $\lambda>0$ tends to zero.*

Conversely, if there is a sequence $\lambda_{j}\rightarrow 0$ such that $\lambda_{j}^{-\alpha}(\lambda_{j}+A)_{-}^{\alpha}x$ converges weakly, then x belongs to $N(A)+\overline{R(A)}$.

PROOF. We note that for $\lambda>0$

$$\lambda^{-\alpha}(\lambda+A)_{-}^{\alpha}=(\lambda(\lambda+A)^{-1})_{+}^{-\alpha} \tag{2.6}$$

and that $\tau(\tau+\lambda(\lambda+A)^{-1})^{-1}$ is uniformly bounded for $0<\lambda,\ \tau<\infty$. Thus in view of [1], Theorem 8.1 (or [2], Proposition 2.4) we have

$$\|\lambda^{-\alpha}(\lambda+A)_{-}^{\alpha}x\|\leqq C\|(\lambda(\lambda+A)^{-1})^{m}x\|^{-\mathrm{Re}\alpha/m}\|x\|^{1+\mathrm{Re}\alpha/m} \tag{2.7}$$

with a constant C independent of $0<\lambda<\infty$ when m is an integer greater than

$-\mathrm{Re}\,\alpha$.

If $x_0 \in N(A)$, we have easily $\lambda^{-\alpha}(\lambda+A)_-^{\alpha}x_0 = x_0$ by the definition of fractional powers. On the other hand, if $x_1 \in \overline{R(A)}$, $(\lambda(\lambda+A)^{-1})^m x_1$ converges strongly to zero as $\lambda \to 0$ as the Abelian ergodic theorem shows. Hence it follows from (2.7) that $\lambda^{-\alpha}(\lambda+A)_-^{\alpha}x_1$ converges strongly to zero.

(2.7) implies also that $\lambda^{-\alpha}(\lambda+A)_-^{\alpha}$ is uniformly bounded. On the other hand, (2.5) proves that $(A(\lambda+A)^{-1})_+^{-\alpha}$ is also uniformly bounded.

Now, suppose that $\lambda_j^{-\alpha}(\lambda_j+A)_-^{\alpha}x$ converges weakly to x_0 as a sequence $\lambda_j \to 0$. According to [1], Proposition 6.3, we have

$$\lambda_j^{-\alpha}(\lambda_j+A)_-^{\alpha}x = \lambda_j^{-\alpha}(\lambda_j+A)_-^{\alpha/2}(\lambda_j+A)_-^{\alpha/2}x \in D(A_+^{-\alpha/2})$$

and

$$A_+^{-\alpha/2}\lambda_j^{-\alpha}(\lambda_j+A)_-^{\alpha}x = \lambda_j^{-\alpha/2}(A(\lambda_j+A)^{-1})_+^{-\alpha/2}\lambda_j^{-\alpha/2}(\lambda_j+A)_-^{\alpha/2}x\,.$$

Since the right hand side converges to zero, we have $x_0 \in N(A_+^{-\alpha/2}) = N(A)$.

Let $x_1 = x - x_0$. Then $\lambda_j^{-\alpha}(\lambda_j+A)_-^{\alpha}x_1$ converges weakly to zero. We have by [2], (2.1)

$$\begin{aligned}&(\lambda_j(\lambda_j+A)^{-1})_+^{-\alpha}x_1 - x_1\\ &\quad= \frac{\Gamma(m)}{\Gamma(-\alpha)\Gamma(m+\alpha)}\int_0^\infty \tau^{-\alpha-1}\{(\lambda(\lambda+A)^{-1}(\tau+\lambda(\lambda+A)^{-1})^{-1})^m - (\tau+1)^{-m}\}x_1 d\tau\end{aligned}$$

where m is an integer greater than $-\mathrm{Re}\,\alpha$. As is easily shown, the integrand is in $R(A)$. Therefore x_1 belongs to $\overline{R(A)}$.

THEOREM 2.4. *Let A be a non-negative operator and let* $\mathrm{Re}\,\alpha < 0$. *If there is a sequence of positive numbers $\lambda_j \to 0$ such that*

$$y = \operatorname*{w-lim}_{j\to\infty}(\lambda_j+A)_-^{\alpha}x \tag{2.8}$$

exists, then $x \in D(A_-^{\alpha})$ and $y = A_-^{\alpha}x$.

If $x \in D(A_-^{\alpha})$, then

$$A_-^{\alpha}x = \operatorname*{s-lim}_{\lambda\to 0}(\lambda+A)_-^{\alpha}x\,. \tag{2.9}$$

PROOF. (2.9) follows easily from Propositions 2.1 and 2.2. Suppose that limit (2.8) exists. Then x belongs to $\overline{R(A)}$ by Proposition 2.3 and hence so does y. If n is an integer greater than $-\mathrm{Re}\,\alpha$ and $\mu > 0$, then $(A(\mu+A)^{-1})^n x \in D(A_-^{\alpha})$, so that we have

$$\begin{aligned}A_-^{\alpha}(A(\mu+A)^{-1})^n x &= \operatorname*{s-lim}_{j\to\infty}(\lambda_j+A)_-^{\alpha}(A(\mu+A)^{-1})^n x\\ &= (A(\mu+A)^{-1})^n \operatorname*{w-lim}_{j\to\infty}(\lambda_j+A)_-^{\alpha}x\\ &= (A(\mu+A)^{-1})^n y\,.\end{aligned}$$

Now let μ tend to zero. Then $(A(\mu+A)^{-1})^n x$ and $(A(\mu+A)^{-1})^n y$ converge to

x and y respectively. Since A_{-}^{α} is a closed operator, x belongs to $D(A_{-}^{\alpha})$ and $A_{-}^{\alpha}x=y$.

COROLLARY 2.5. *Let $-A$ be the infinitesimal generator of a bounded continuous semi-group T_t and let* $\operatorname{Re}\alpha<0$.

If there is a sequence of positive numbers $\lambda_j\to 0$ such that

$$y=\text{w-}\lim_{j\to\infty}\frac{1}{\Gamma(-\alpha)}\int_0^{\infty}e^{-\lambda_j s}s^{-\alpha-1}T_s x\,ds \tag{2.10}$$

exists, then $x\in D(A_{-}^{\alpha})$ and $y=A_{-}^{\alpha}x$.

Conversely, if $x\in D(A_{-}^{\alpha})$, then

$$\begin{aligned}A_{-}^{\alpha}x&=\text{s-}\lim_{\lambda\to 0}\frac{1}{\Gamma(-\alpha)}\int_0^{\infty}e^{-\lambda s}s^{-\alpha-1}T_s x\,ds\\&=\frac{1}{\Gamma(-\alpha)}(\mathrm{A})\int_0^{\infty}s^{-\alpha-1}T_s x\,ds\,.\end{aligned}\tag{2.11}$$

However, the Cesàro summability of (2.1) does not hold in general. We consider the case where $\alpha=-2$ and $\sigma=1$. We have

$$\int_0^t(t-s)sT_s x\,ds=-2I_t^{(3)}x+tI_t^{(2)}x\,.$$

Now substitute A^2y for x and apply [3], Lemma 1.2. Then we have

$$t^{-1}\int_0^t(t-s)sT_s x\,ds=(1+T_t-2t^{-1}I_t)y\,.$$

If $y\in\overline{R(A)}$, then $t^{-1}I_t y$ converges strongly to zero as $t\to\infty$. However, $T_t y$ does not converge in general.

Lastly we quote Theorem 6.3 of [3] for the sake of completeness.

THEOREM 2.6. *Let $-A$ be the infinitesimal generator of a bounded analytic semi-group T_t and let* $\operatorname{Re}\alpha<0$.

If there is a sequence $t_j\to\infty$ such that

$$y=\text{w-}\lim_{j\to\infty}\frac{1}{\Gamma(-\alpha)}\int_0^{t_j}s^{-\alpha-1}T_s x\,ds \tag{2.12}$$

exists, then $x\in D(A_{-}^{\alpha})$ and $y=A_{-}^{\alpha}x$.

If $x\in D(A_{-}^{\alpha})$, then

$$A_{-}^{\alpha}x=\text{s-}\lim_{t\to\infty}\frac{1}{\Gamma(-\alpha)}\int_0^{t}s^{-\alpha-1}T_s x\,ds\,. \tag{2.13}$$

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