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On the alternating groups II

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Introduction.

Let $\mathfrak{A}_m$ be the alternating group on m letters $\{1, 2, \cdots, m\}$. Put $m=4n+r$, where n is a positive integer and $0\leqq r\leqq 3$. Let $\tilde{\alpha}_n$ be an involution of $\mathfrak{A}_m$ which has a cycle decomposition

$$(1, 2)(3, 4)\cdots(4n-3, 4n-2)(4n-1, 4n).$$

$\tilde{\alpha}_n$ is contained in the center of a 2-Sylow subgroup of $\mathfrak{A}_m$. For $r=1$, 2 and 3, we denote by $\tilde{H}(n, r)$ the centralizer in $\mathfrak{A}_m$ of $\tilde{\alpha}_n$. In the present paper, we shall prove the following two theorems.

THEOREM I. *Let $G(n, r)$ be a finite group with the following properties:*

(1) *$G(n, r)$ has no subgroup of index 2, and*

(2) *$G(n, r)$ contains an involution α_n in the center of a 2-Sylow subgroup of $G(n, r)$ whose centralizer $C_{G(n,r)}(\alpha_n)$ is isomorphic to $\tilde{H}(n, r)$.*

Then if $r=2$ or 3, $G(n, r)$ is isomorphic to $\mathfrak{A}_{4n+r}$ except for the case $n=1$ and $r=2$ where $G(1, 2)\cong\mathfrak{A}_6$ or PSL (2, 7).

For the case $r=1$, the author has not obtained the analogous result. But we can prove much weaker result. We note that $\tilde{H}(n, 1)$ has a unique elementary abelian subgroup $\tilde{S}$ of order 2^{2n} up to conjugacy (cf. Appendix, Proposition 5). Then we have

THEOREM II[0]. *Let $G(n, 1)$ be a finite group containing an involution whose centralizer $H(n, 1)$ is isomorphic to $\tilde{H}(n, 1)$. Let S be an elementary abelian snbgroup of order 2^{2n} of $H(n, 1)$. Assume that there exists a one-to-one mapping θ from $\tilde{H}(n, 1)\cup N_{\mathfrak{A}_m}(\tilde{S})$ (the set theoretic union in $\mathfrak{A}_m$) onto $H(n, 1)\cup N_{G(n,1)}(S)$ such that θ induces an isomorphism between $\tilde{H}(n, 1)$ (resp. $N_{\mathfrak{A}_m}(\tilde{S})$) and $H(n, 1)$ (resp. $N_{G(n,1)}(S)$).*

Then $G(n, 1)$ is isomorphic to $\mathfrak{A}_{4n}$ or $\mathfrak{A}_{4n+1}$.

The proof of Theorem I depends on Theorem A of the author's previous paper [**9**] which was proved only in the case $r=2$ or 3. But we have not obtained such result for the case $r=1$. This is the reason why the stronger condition is necessary for the case $r=1$. However, we note: Theorem II shows that, if we can prove a result in the case $r=1$ similar to Theorem A of [**9**], we shall be able to at once obtain a characterization of $\mathfrak{A}_{4n}$ and $\mathfrak{A}_{4n+1}$ under a weaker condition[0]. The special cases $n\leqq 3$ of Theorems I and II were obtained by M. Suzuki [**10**], D. Held [**4**], [**5**], T. Kondo [**7**] and H. Yamaki [**12**].

The main work of the present paper is to determine the structure of the centralizer of every involution of $G(n, r)$. The arguments depend on Theorem A of [**9**] (the condition of Theorem II in the case $r=1$) and the knowledge of conjugacy classes of involutions of the wreath product $\boldsymbol{Z}_2 \wr \mathfrak{S}_{2n}$. The latter is summarized in § 1. In § 3, we determine the precise structure of the normalizer of an elementary abelian 2-subgroup of $G(n, r)$. § 4 is the collection of technical lemmas. In § 5 and § 6, we determine the structure of the centralizer of every involution of $G(n, r)$. Especially, the argument in § 6 is due to H. Yamaki [**12**] who proved the special case $n=3$ of Theorems I and II (a slightly better result for the case $n=3$ and $r=1$ than Theorem II). The final step of the proof is an application of a theorem of [8].

The author wishes to express his hearty thanks to Professor M. Suzuki for leading him to this work.

Notation and Terminology

x^y	$y^{-1}xy$
$[x, y]$	$x^{-1}y^{-1}xy$
$x\sim y$ in X	x is conjugate to y in a group X
$x: y\rightarrow z$	$y^x=z$
$\langle \cdots \mid \cdots \rangle$	a group generated by $\cdots$ with the conditions $\cdots$.
$\{\cdots\}$	a set consisting of elements $\cdots$
$X \wr Y$	the wreath product of a group X by a permutation group Y
$O^2(X)$	the smallest normal subgroup of X such that $X/O^2(X)$ is a 2-group
$O_2(X)$	the largest normal 2-subgroup of X
$Z(X)$	the center of X
$\mathfrak{S}_n$	the symmetric group of degree n
$\mathfrak{A}_n$	the alternating group of degree n
$\boldsymbol{Z}_n$	a cyclic group of order n.

The other notations are standard.

Let X be a group isomorphic to $\mathfrak{S}_l$. X is generated by $l-1$ elements $x_1, x_2, \cdots, x_{l-1}$ subject to the relations; $x_1^2=\cdots=x_{l-1}^2=(x_ix_{i+1})^3=(x_jx_k)^2=1$ $(1\leqq i\leqq l-2,\ 1\leqq j,\ k\leqq l-1$ and $|j-k|>1)$. We call an ordered set of such generators of X *a set of canonical generators of* X [**1**; p. 287]. Let Y be a group isomorphic to $\mathfrak{A}_l$. Y is generated by $l-2$ elements $y_1, y_2, \cdots, y_{l-2}$ subject to the relations:

0) Cf. (2.4) and Lemma A′ in (3.1).

$$y_1^2 = y_2^2 = \cdots = y_{l-2}^2 = (y_i y_{i+1})^3 = (y_j y_k)^2 = 1$$
$$(1 \leqq i \leqq l-3,\ 1 \leqq j,\ k \leqq l-2 \text{ and } |j-k| > 1).$$

We call an ordered set of such generators of Y *a set of canonical generators of* Y (cf. [**1**; p. 289]).

§ 1. Some properties of the wreath product $Z_2 \wr \mathfrak{S}_{2n}$.

(1.1) Let $\mathfrak{W}_n$ be a finite group isomorphic to the wreath product of a group of order 2 by the symmetric group of degree $2n$. In this section, we shall give some properties of $\mathfrak{W}_n$ which are necessary for the proof of Theorems I, II.

Let $\mathfrak{X}_n$ be an elementary abelian group of order 2^{2n} with the set of generators x_i $(1 \leqq i \leqq 2n)$ and $\mathfrak{Y}_n$ be a group isomorphic to $\mathfrak{S}_{2n}$ with $\{y_1, z_1, y_2, z_2, \cdots, z_{n-1}, y_n\}$ as a set of canonical generators of $\mathfrak{Y}_n$. Define the action on $\mathfrak{X}_n$ of $\mathfrak{Y}_n$ as follows:

$$(*) \qquad \begin{aligned} &x_{2i-1}^{y_i} = x_{2i}, \quad [x_j, y_i] = 1 \quad (1 \leqq i \leqq n, j \neq 2i-1, 2i) \\ &x_{2i}^{z_i} = x_{2i+1}, \quad [x_j, z_i] = 1 \quad (1 \leqq i \leqq n-1, j \neq 2i, 2i+1). \end{aligned}$$

Thus $\mathfrak{Y}_n$ can be regarded as the symmetric group on the set $\{x_1, x_2, \cdots, x_{2n}\}$. Construct the semi-direct product $\mathfrak{W}_n = \mathfrak{X}_n \cdot \mathfrak{Y}_n$. Then $\mathfrak{W}_n$ is isomorphic to the wreath product $Z_2 \wr \mathfrak{S}_{2n}$. Further we define a subgroup $\mathfrak{W}_n^*$ of $\mathfrak{W}_n$ of index 2. Put

$$\mathfrak{X}_n^* = \langle x_1 x_2, x_2 x_3, \cdots, x_{2n-1} x_{2n} \rangle.$$

Then $\mathfrak{X}_n^*$ is an elementary abelian group of order 2^{2n-1}, normal in $\mathfrak{W}_n$ and $\mathfrak{Y}_n \cap \mathfrak{X}_n^* = 1$. Put

$$\mathfrak{W}_n^* = \mathfrak{X}_n^* \cdot \mathfrak{Y}_n.$$

Further put

$$\widehat{\mathfrak{W}}_k^* = \mathfrak{X}_n^* (\langle y_1, z_1, \cdots, y_k \rangle \times \langle y_{k+1}, z_{k+1}, \cdots, y_n \rangle) \qquad (1 \leqq k \leqq n)$$

and

$$\xi_k = x_{2k-1} x_{2k} \qquad (1 \leqq k \leqq n).$$

We note that $\widehat{\mathfrak{W}}_n^* = \mathfrak{W}_n^*$.

(1.2) LEMMA. *The orbit of ξ_1 under the action on $\mathfrak{X}_n^*$ of $\mathfrak{Y}_n$ generates $\mathfrak{X}_n^*$.*

PROOF. Since $\mathfrak{Y}_n$ operates doubly transitively on $\{x_1, x_2, \cdots, x_n\}$ and $\mathfrak{X}_n^* = \langle x_i x_j \mid 1 \leqq i < j \leqq n \rangle$, our lemma is obvious.

(1.3) LEMMA[1]. *The representatives of conjugacy classes of involutions of*

1) In W. Specht [**11**] the conjugacy classes of elements, not necessarily involutions, of $\mathfrak{W}_n$ were determined. We note that this lemma (also the next lemma) was used in the author's previous papers [**8**; (1.3)] and [**9**; (5.2)] with the omission of the proof.

$\mathfrak{W}_n$ *are as follows:*

$$y_1y_2\cdots y_s\Big(\prod_{i=1}^{t}\xi_{s+i}\Big)\qquad(0<s+t\leqq n)\,,$$

and

$$y_1y_2\cdots y_s\Big(\prod_{i=s+1}^{s+t}\xi_i\Big)x_{2n-1}\qquad(0\leqq s+t\leqq n-1)\,.$$

PROOF. Let xy ($x\in\mathfrak{X}_n$ and $y\in\mathfrak{Y}_n$) be an involution of $\mathfrak{W}_n$. Then it is easy to see that x and y are involutions of $\mathfrak{X}_n$ and $\mathfrak{Y}_n$ respectively. Since $\mathfrak{Y}_n$ is isomorphic to $\mathfrak{S}_{2n}$ and $\{y_1, z_1, \cdots, y_n\}$ is a set of canonical generators of $\mathfrak{Y}_n$, the representatives of the conjugacy classes of involutions of $\mathfrak{Y}_n$ are $y_1y_2\cdots y_s$ ($1\leqq s\leqq n$). Therefore we may assume $y=y_1y_2\cdots y_s$ by taking a suitable conjugate of xy by an element of $\mathfrak{Y}_n$. Then, from the fact that $yx=y_1y_2\cdots y_sx$ is an involution, it follows that $x=\prod_{i=1}^{s}\xi_i^{\delta_i}\cdot\prod_{j>2s}x_j^{\delta_j}$ where δ_i, $\delta_j=0$ or 1. Since $(y_i\xi_i)^{x_{2i-1}}=y_i$ and $[y_j, x_{2i-1}]=1$ $(j\neq i)$, we may assume that $yx=y_1y_2\cdots y_s\prod_{j>2s}x_j^{\delta_j}$. Since $\langle y_{s+1}, z_{s+1}, \cdots, y_n\rangle\cong\mathfrak{S}_{2(n-s)}$ operates multiply-transitively on the set $\{x_{2s+1}, \cdots, x_{2n}\}$ and centralizes $y_1, y_2, \cdots, y_s$, a suitable conjugate of yx by an element of $\langle y_{s+1}, z_{s+1}, \cdots, y_n\rangle$ becomes $y_1y_2\cdots y_s\Big(\prod_{i=1}^{t}\xi_{s+i}\Big)$ or $y_1y_2\cdots y_s\Big(\prod_{i=1}^{t}\xi_{s+i}\Big)x_{2n-1}$. This completes the proof of our lemma.

(1.4) LEMMA. *The representatives of conjugacy classes of involutions of* $\widehat{\mathfrak{W}}_k^*$ *are as follows:*

(i) $$\Big(\prod_{i=1}^{s}y_i\Big)\cdot\Big(\prod_{i=1}^{s'}\xi_{s+i}\Big)\cdot\Big(\prod_{j=1}^{t}y_{k+j}\Big)\cdot\Big(\prod_{j=1}^{t'}\xi_{k+t+j}\Big)\quad\begin{pmatrix}0\leqq s+s'\leqq k,\ k\leqq k+t+t'\leqq n\\ \text{and } 0<s+s'+t+t'\end{pmatrix}$$

$$\Big(\prod_{i=1}^{s}y_i\Big)\cdot\Big(\prod_{i=1}^{s'}\xi_{s+i}\Big)\cdot\Big(\prod_{j=1}^{t}y_{k+j}\Big)\cdot\Big(\prod_{j=1}^{t'}\xi_{k+t+j}\Big)x_{2k-1}x_{2n-1}$$

$$(0\leqq s+s'<k,\ k\leqq k+t+t'<n)\,,$$

and $y_1y_2\cdots y_n\xi_n$.

In particular those of $\mathfrak{W}_n^*$ *are* $\prod_{i=1}^{s}y_i\cdot\prod_{i=1}^{t}\xi_{s+i}$ $(0<s+t\leqq n)$ *and* $y_1y_2\cdots y_n\xi_n$.

PROOF. Let yx ($y\in\mathfrak{Y}_n$ and $x\in\mathfrak{X}_n^*$) be an involution of $\mathfrak{W}_n^*$. As in the proof of (1.3), we may assume

$$yx=\prod_{i=1}^{s}y_i\prod_{i=1}^{s}\xi_i^{\delta_i}\prod_{2k\geqq j>2s}x_j^{\delta_j}\cdot\prod_{i=1}^{t}y_{k+i}\cdot\prod_{i=1}^{t}\xi_{k+i}^{\delta_{k+i}}\prod_{2(n-k)\geqq j>2(k+t)}x_{j+k+t}^{\delta_{j+k+t}}\,.$$

Note that $\sum_{2k\geqq j>2s}\delta_j+\sum_{2(n-k)\geqq j>2(k+t)}\delta_{j+k+t}\equiv 0 \bmod 2$ since $x\in\mathfrak{X}_n^*$. Firstly suppose that $s=k$ and $t=n-k$. Then we have

$$yx=\prod_{i=1}^{k}y_i\prod_{i=1}^{k}\xi_i^{\delta_i}\prod_{i=1}^{n-k}y_{k+i}\prod_{i=1}^{n-k}\xi_{k+i}^{\delta_{k+i}}\,.$$

By transforming by elements of $\mathfrak{X}_n^*$ with the form $x_{2i-1}x_{2j-1}$ $(1\leqq i<j\leqq n)$, we

easily see that a suitable conjugate of yx becomes $y_1 y_2 \cdots y_n$ or $y_1 y_2 \cdots y_n \xi_n$. Secondly suppose that $0 \leqq t < n-k$. (We can work similarly also in the case $0 \leqq s < k$.) By transforming by elements of $\mathfrak{X}_n^*$ with the form $x_{2i-1}x_{2n-1}$ $(1 \leqq i \leqq n-1)$, a suitable conjugate of yx becones

$$\prod_{i=1}^{s} y_i \cdot \prod_{2k \geqq j > 2s} x_j^{\delta_j} \prod_{i=1}^{t} y_{k+i} \prod_{2(n-k) \geqq j > 2(k+t)} x_{j+k+t}^{\delta_{j+k+t}} .$$

Since $\langle y_{s+1}, z_{s+1}, \cdots, y_k \rangle$ and $\langle y_{(k+t)+1}, z_{(k+t)+1}, \cdots, y_n \rangle$ operates multiply transitively on $\{x_{2s+1}, \cdots, x_{2k}\}$ and $\{x_{2(k+t)+1}, \cdots, x_n\}$ respectively, and centralizes $\prod_{i=1}^{s} y_i$ and $\prod_{i=1}^{t} y_{k+i}$ respectively, yx is conjugate to

$$\Big(\prod_{i=1}^{s} y_i\Big)\Big(\prod_{i=1}^{s'} \xi_{s+i}\Big) x_{2k-1} \Big(\prod_{i=1}^{t} y_{k+i}\Big)\Big(\prod_{i=1}^{t'} \xi_{k+t+i}\Big) x_{2n-1}$$

or

$$\prod_{i=1}^{s} y_i \prod_{i=1}^{s'} \xi_{s+i} \cdot \prod_{i=1}^{t} y_{k+i} \prod_{i=1}^{t'} \xi_{k+t+i}$$

according to whether $\sum_{i=1}^{s'} \delta_i \not\equiv 0 \bmod 2$ or $\sum_{i=1}^{s'} \delta_i \equiv 0 \bmod 2$. This completes the proof of our lemma.

(1.5) LEMMA. *Let z' be a non-identity element of $\mathfrak{W}_n$ (resp. $\mathfrak{W}_n^*$) with the following properties:*

(i) $[z_k^{-1} z', \mathfrak{X}_n] = 1$ *(resp.* $[z_k^{-1} z', \mathfrak{X}_n^*] = 1$) *and*

(ii) $(y_k z')^3 = (z' y_{k+1})^3 = 1$,

where k is a fixed integer such that $1 \leqq k \leqq n-1$.

Then we have $z' = z_k$ or $z_k x_{2k} x_{2k+1}$, and $\bar{\mathfrak{Y}} = \langle y_1, z_1, \cdots, y_k, z', y_{k+1}, \cdots, y_n \rangle$ is isomorphic to $\mathfrak{S}_{2n}$, and for the action on $\mathfrak{X}_n$ (resp. $\mathfrak{X}_n^*$) of $\bar{\mathfrak{Y}}$, the same relations as (*) *of* (1.1) *hold by replacing z_k by z'.*

PROOF. We shall prove the first statement. Then the second and the third statements are obvious. Since $\mathfrak{X}_n$ (resp. $\mathfrak{X}_n^*$) is selfcentralizing in $\mathfrak{W}_n$ (resp. $\mathfrak{W}_n^*$), it follows from (i) that $z' = z_k \prod_{i=1}^{2n} x_i^{\delta_i}$ where $\delta_i = 0$ or 1. Then we have, by using the relations (*),

$$\begin{aligned}
1 = (y_k z')^3 &= (y_k z_k \prod x_i^{\delta_i})(y_k z_k \prod x_i^{\delta_i}) y_k z' \\
&= y_k z_k y_k \Big(\Big(\prod_{i \neq 2k-1, 2k} x_i^{\delta_i}\Big) x_{2k-1}^{\delta_{2k}} x_{2k}^{\delta_{2k-1}}\Big) z_k \Big(\prod_i x_i^{\delta_i}\Big) y_k z' \\
&= (y_k z_k)^2 \Big(\prod_{|i-2k|>1} x_i^{\delta_i}\Big)(x_{2k-1}^{\delta_{2k}} x_{2k}^{\delta_{2k+1}} x_{2k+1}^{\delta_{2k-1}}) \Big(\prod_i x_i^{\delta_i}\Big) y_k z' \\
&= (y_k z_k)^2 (x_{2k-1}^{\delta_{2k}+\delta_{2k-1}} \cdot x_{2k}^{\delta_{2k+1}+\delta_{2k}} x_{2k+1}^{\delta_{2k-1}+\delta_{2k+1}}) y_k z' \\
&= (y_k z_k)^2 y_k (x_{2k-1}^{\delta_{2k}+\delta_{2k+1}} x_{2k}^{\delta_{2k-1}+\delta_{2k}} x_{2k+1}^{\delta_{2k-1}+\delta_{2k+1}}) z_k (\prod x_i^{\delta_i})
\end{aligned}$$

$$=(y_kz_k)^3(x_{2k-1}^{\delta_{2k}+\delta_{2k-1}}x_{2k}^{\delta_{2k-1}+\delta_{2k+1}}x_{2k+1}^{\delta_{2k}+\delta_{2k-1}})(\prod_i x_i^{\delta_i})$$

$$=(x_{2k-1}x_{2k}x_{2k+1})^{\delta}\prod_{|i-2k|>1}x_i^{\delta_i}$$

where $\delta=\delta_{2k-1}+\delta_{2k}+\delta_{2k+1}$. This yields that

$$\text{(1)}\qquad \begin{cases} \delta_i=0 & |i-2k|>1 \\ \delta=\delta_{2k-1}+\delta_{2k}+\delta_{2k+1}\equiv 0 & \mod 2\,. \end{cases}$$

Similarly, from $(z'y_{k+1})^3=1$, we get

$$\text{(2)}\qquad \begin{array}{ll} \delta_i=0 & |i-2k-1|>1 \\ \delta_{2k}+\delta_{2k+1}+\delta_{2k+2}\equiv 0 & \mod 2\,. \end{array}$$

Then (1) and (2) yields the first statement of our lemma.

(1.6) LEMMA. *Assume that $\mathfrak{W}_n$ is a subgroup of a finite group $\mathfrak{G}$ and 2-Sylow subgroup of $\mathfrak{W}_n$ is that of $\mathfrak{G}$. If $x_1\sim y_1\sim x_1x_2$ in $\mathfrak{G}$, $\mathfrak{G}$ has no subgroup of index* 2.

PROOF. We have $O^2(\mathfrak{W}_n)=\langle y_iy_j,\ y_iz_j,\ x_ix_j \mid 1\leqq i<j\leqq n\rangle$ and $\mathfrak{W}_n=\langle x_1, y_1\rangle O^2(\mathfrak{W}_n)$. Assume that $\mathfrak{G}$ has a subgroup $\mathfrak{G}_0$ of index 2. Then we have $\mathfrak{G}_0\supseteqq O^2(\mathfrak{W}_n)$. Since $x_1\sim y_1\sim x_1x_2$ in $\mathfrak{G}$ and $x_1x_2\in O^2(\mathfrak{W}_n)\subseteqq\mathfrak{G}_0$, we get x_1 and $y_1\in\mathfrak{G}_0$.

This implies $[\mathfrak{G}:\mathfrak{G}_0]=$ odd because of the assumption of our lemma. This is a contradiction.

§ 2. The groups $H(n, r)$ and $G(n, r)$.

(2.1) Here we shall define the groups $H(n, r)$ for a positive integer n and $r=1, 2, 3$. Firstly we define $H(n, 2)$. Let $H(n, 2)$ be a finite group with a set $\{\lambda_i, \pi_i, \pi_i', \sigma_j' \mid 1\leqq i\leqq n,\ 1\leqq j\leqq n-1\}$ of generators subject to the following relations:

(o) $\langle\lambda_1, \pi_1, \cdots, \lambda_n, \pi_n\rangle$ is an elementary abelian 2-group of order 2^{2n},

(i) $L_n=\langle\pi_1', \sigma_1', \pi_2', \cdots, \sigma_{n-1}', \pi_n'\rangle$ is isomorphic to $\mathfrak{S}_{2n}$ and the ordered set $\{\pi_1', \sigma_1', \cdots, \sigma_{n-1}', \pi_n'\}$ is a set of canonical generators of L_n,

(ii) $\lambda_i^{\pi_i'}=\lambda_i\pi_i$ $(1\leqq i\leqq n)$ and $(\lambda_i\pi_i)^{\sigma_i'}=\lambda_{i+1}$ $(1\leqq i\leqq n-1)$,

(iii) $[\lambda_j\pi_j, \pi_i']=[\lambda_j, \pi_i']=1$ $(i\neq j)$ and $[\lambda_{i+1}, \sigma_j']=[\lambda_i\pi_i, \sigma_j']=1$ $(i\neq j)$.

Then $H(n, 2)$ is isomorphic to $\mathfrak{W}_n$ defined in (1.1) by the correspondence $\pi_i'\leftrightarrow y_i$, $\sigma_j'\leftrightarrow z_j$, $\lambda_i\leftrightarrow x_{2i-1}$ and $\lambda_i\pi_i\leftrightarrow x_{2i}$. Put

$$H(n, 1)=L_n\cdot\langle\pi_1, \pi_2, \cdots, \pi_n, \lambda_1\lambda_2, \lambda_2\lambda_3, \cdots, \lambda_{n-1}\lambda_n\rangle\,.$$

Then $H(n, 1)$ is a subgroup of $H(n, 2)$ of index 2 and isomorphic to $\mathfrak{W}_n^*$ defined in (1.1). We have $H(n, 2)=\langle\lambda_1\rangle H(n, 1)$. Finally we define $H(n, 3)$. By definition, $H(n, 3)$ is a group generated by $H(n, 2)$ and an element ν subject to the following relations:

$$\nu^3=1,\ [H(n, 1), \nu]=1 \quad \text{and} \quad \nu^{\lambda_1}=\nu^{-1}.$$

We note that each element of $H(n, 2)-H(n, 1)$ inverts ν. If there is no confusion, we frequently write $H=H(n, r)$.

(2.2) For $r=1, 2, 3$, $H(n, r)$ is isomorphic to the centralizer in $\mathfrak{A}_{4n+r}$ of an involution

$$\tilde{\alpha}_n=(1, 2)(3, 4)\cdots(4n-3, 4n-2)(4n-1, 4n)$$

by the following mapping θ_n:

$$\theta_n:\begin{cases}\tilde{\pi}_i=(4i-3,\ 4i-2)(4i-1,\ 4i) & \longrightarrow \pi_i\\ \tilde{\pi}'_i=(4i-3,\ 4i-1)(4i-2,\ 4i) & \longrightarrow \pi'_i\\ \tilde{\lambda}_i=(4i-3,\ 4i-2)(4n+1,\ 4n+2) & \longrightarrow \lambda_i\\ \tilde{\sigma}'_i=(4i-1,\ 4i+1)(4i,\ 4i+2) & \longrightarrow \sigma'_i\\ \tilde{\nu}\ =(4n+1,\ 4n+2,\ 4n+3) & \longrightarrow \nu\end{cases}$$

For later use, besides $\tilde{\pi}_i, \tilde{\pi}'_i, \cdots$, etc., we define some elements of $\mathfrak{A}_{4n+r}$ as follows:

$$\tilde{\alpha}_i=\tilde{\pi}_1\tilde{\pi}_2\cdots\tilde{\pi}_i \qquad (1\leqq i\leqq n),$$

$$\tilde{\sigma}_j=(\tilde{\pi}'_j\tilde{\pi}'_{j+1})^{\tilde{\sigma}'_j} \qquad (1\leqq j\leqq n-1),$$

$$\tilde{\beta}_i=(4i-3,\ 4i-2,\ 4i-1) \qquad (1\leqq i\leqq n)$$

$$\tilde{S}=\langle\tilde{\pi}_1, \tilde{\pi}'_1, \tilde{\pi}_2, \tilde{\pi}'_2, \cdots, \tilde{\pi}_n, \tilde{\pi}'_n\rangle.$$

Further we introduce some notations:

$$\alpha_i=\pi_1\pi_2\cdots\pi_i \qquad (1\leqq i\leqq n),$$

$$\sigma_j=(\pi'_j\pi'_{j+1})^{\sigma'_j} \qquad (1\leqq j\leqq n-1),$$

$$S=S_1\times S_2\times\cdots\times S_n, \qquad S_i=\langle\pi_i, \pi'_i\rangle,$$

$$P=\langle\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\rangle,$$

$$M=\begin{cases}\langle\pi_1, \pi_2, \cdots, \pi_n, \lambda_1\lambda_2, \cdots, \lambda_{n-1}\lambda_n\rangle & \text{if } r=1\\ \langle\pi_1, \lambda_1, \pi_2, \lambda_2, \cdots, \pi_n, \lambda_n\rangle & \text{if } r\geqq 2.\end{cases}$$

Then we have $\theta_n(\tilde{\alpha}_i)=\alpha_i$, $\theta_n(\tilde{\sigma}_i)=\sigma_i$ and $\theta_n(\tilde{S})=S$.

S is an elementary abelian group of order 2^{2n} and M is an elementary abelian group of order 2^{2n-1} or 2^{2n} according to whether $r=1$ or $r\geqq 2$. P is isomorphic to $\mathfrak{S}_n$ and $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$ is a set of canonical generators of P. If $r=1$, any elementary abelian subgroup of $H(n, 1)$ of order 2^{2n} is conjugate in $H(n, 1)$ to S and S is normal in a 2-Sylow subgroup of $H(n, 1)$ containing it (see Appendix). So every 2-Sylow subgroup of $H(n, 1)$ has the only one elementary abelian subgroup of order 2^{2n}.

(2.3) *Let $G(n, r)$ be a finite group with the following properties:*

(i) *$G(n, r)$ contains $H(n, r)$ as a subgroup in such a way that $H(n, r)$ is the centralizer in $G(n, r)$ of an involution α_n in the center of a 2-Sylow subgroup of $G(n, r)$, and*

(ii) *if $r \geqq 2$, $G(n, r)$ has no subgroup of index 2, and if $r=1$, there exists a one-to-one mapping θ from $C_{\mathfrak{A}_{4n+1}}(\tilde{\alpha}_n) \cup N_{\mathfrak{A}_{4n+1}}(\tilde{S})$ onto $H(n, 1) \cup N_{G(n,1)}(S)$ such that θ induces an isomorphism from $C_{\mathfrak{A}_{4n+1}}(\tilde{\alpha}_n)$ (resp. $N_{\mathfrak{A}_{4n+1}}(\tilde{S})$) onto $H(n, 1)$ (resp. $N_{G(n,1)}(S)$).*

(2.4) REMARK. Suppose that $r=1$. Then the assumption that α_n is a central involution of $G(n, 1)$ is not necessary. In fact, if $\tilde{D}$ is a 2-Sylow subgroup of $C_{\mathfrak{A}_{4n+1}}(\tilde{\alpha}_n) \cap N_{\mathfrak{A}_{4n+1}}(\tilde{S})$, $\tilde{D}$ is that of $C_{\mathfrak{A}_{4n+1}}(\tilde{\alpha}_n)$ and so $\theta(\tilde{D})$ is a 2-Sylow subgroup of $H(n, 1)$ containing S. Denote by D_1 a 2-Sylow subgroup of $G(n, 1)$ with $\theta(\tilde{D}) \subseteqq D_1$. If $\theta(\tilde{D}) < D_1$, we have $N_{D_1}(S) > \theta(\tilde{D})$ since S is the unique elementary abelian subgroup of $\theta(\tilde{D})$ of order 2^{2n} as remarked in the last paragraph of (2.2). This contradicts $\theta(N_{\mathfrak{A}_{4n+1}} | (\tilde{S})) = N_G(S)$. Further we remark that, if S_1 is an arbitrary elementary abelian group of $H(n, 1)$ of order 2^{2n}, the condition (ii) for the specified S holds also for S_1, because S_1 is conjugate in $H(n, 1)$ to S. Therefore we may assume without loss of generality that the restriction of θ to $C_{\mathfrak{A}_{4n+1}}(\tilde{\alpha}_n)$ coincides with θ_n where θ_n is the isomorphism defined in (2.2). So we shall assume $\theta_n = \theta | C_{\mathfrak{A}_{4n+1}}(\alpha_n)$ throughout the present paper. For the sake of brevity, if there is no confusion, we frequently write $G=G(n, r)$ and $H=H(n, r)$.

(2.5) LEMMA. *If D is a 2-subgroup of G containing S, D normalizes S. We have $C_G(S)=S$ or $S \times \langle \nu \rangle$ according to whether $r \leqq 2$ or $r=3$.*

PROOF. The first statement follows from the uniqueness of S if $r=1$ and [**9**; (2.6)] if $r \geqq 2$. The second follows from the structure of $H(n, r)$.

§ 3. The structure of $N_G(S)$.

(3.1) In [**9**], we have proved the following result for $G=G(n, r)$ where $r=2$ or 3.

THEOREM A. (i) *G has exactly n classes of involutions with the representatives $\alpha_1, \alpha_2, \cdots, \alpha_n$, and*

(ii) *there exist $2n$ elements β_s and γ_s $(1 \leqq s \leqq n)$ with the following properties:*

(1) *β_s and γ_s are of odd order, $\beta_s \in N_G(S)$ and $\gamma_s \in N_G(M)$,*

(2) *$\beta_s: \pi_s \rightarrow \pi_s' \rightarrow \pi_s \pi_s'$ and $[\beta_s, \pi_t] = [\beta_s, \pi_t'] = [\beta_s, \lambda_t] = 1$ $(1 \leqq s, t \leqq n, s \neq t)$,*

(3) $\gamma_s : \pi_s \to \lambda_s \to \lambda_s \pi_s$ *and* $[\gamma_s, \pi_t] = [\gamma_s, \lambda_t \pi_t] = [\gamma_s, \pi'_t] = 1$ $(1 \leqq s, t \leqq n, s \neq t)$.

In particular we have

(4) $\pi'_1 \pi'_2 \cdots \pi'_s \pi_{s+1} \cdots \pi_{s+t} \sim \alpha_{s+t}$,

(5) $\lambda_1 \lambda_2 \cdots \lambda_{2s-1} \pi_{2s} \pi_{2s+1} \cdots \pi_{2s-1+t} \sim \lambda_1 \lambda_2 \cdots \lambda_{2s} \pi_{2s+1} \cdots \pi_{2s+t} \sim \alpha_{s+t}$,

(6) $\pi'_1 \pi'_2 \cdots \pi'_s \pi_{s+1} \cdots \pi_{s+t} \lambda_n \sim \alpha_{s+t+1}$ $(1 \leqq s+t < n)$.

In the case $r=1$, we have not obtained the analogous result. But $G(n, 1)$ satisfies a stronger condition (ii) of (2.3) than the case $r \geqq 2$. This yields the following lemma.

LEMMA A′. (i) *$G = G(n, 1)$ has exactly n classes of involutions with the representatives $\alpha_1, \alpha_2, \cdots, \alpha_n$.* (ii) *there exist n elements β_s $(1 \leqq s \leqq n)$ of $N_G(S)$ with the following properties:*

(1) *β_s is of order* 3,

(2) $\beta_s : \pi_s \to \pi'_s \to \pi_s \pi'_s$ *and* $[\beta_s, \pi_t] = [\beta_s, \pi'_t] = [\beta_s, \lambda_t \lambda_u] = 1$ $(s \neq t, u)$.

In particular we have

(3) $\pi'_1 \pi'_2 \cdots \pi'_s \pi_{s+1} \cdots \pi_{s+t} \sim \alpha_{s+t}$,

(4) $\pi'_1 \pi'_2 \cdots \pi'_n \pi_n \sim \alpha_n$, *and*

(5) $\lambda_1 \lambda_2 \cdots \lambda_{2s} \pi_{2s+1} \cdots \pi_{2s+t} \sim \alpha_{s+t}$.

PROOF. Let $\tilde{\beta}_s$ $(1 \leqq s \leqq n)$ be elements defined in (2.2). Then we have $\tilde{\beta}_s \in N_{\mathfrak{A}_{4n+1}}(\tilde{S})$, $\tilde{\beta}_s : \tilde{\pi}_s \to \tilde{\pi}'_s \to \tilde{\pi}_s \tilde{\pi}'_s$ and $[\tilde{\beta}_s, \tilde{\pi}_t] = [\tilde{\beta}_s, \tilde{\pi}'_t] = [\tilde{\beta}_s, \tilde{\lambda}_t \tilde{\lambda}_u] = 1$ $(s \neq t, u)$. Put $\beta_s = \theta(\tilde{\beta}_s)$. Then the β_s $(1 \leqq s \leqq n)$ have the properties (1) and (2). By (1.4) the representatives of conjugacy classes of involutions of $H(n, 1)$ are $\pi'_1 \pi'_2 \cdots \pi'_s \pi_{s+1} \cdots \pi_{s+t}$ $(0 < s+t \leqq n)$ and $\pi'_1 \pi'_2 \cdots \pi'_n \pi_n$. Since any one of these is conjugate to α_k for some k by (2), any involution of G is conjugate to one of $\alpha_1, \alpha_2, \cdots, \alpha_n$ which yields (i) of our lemma. (3) and (4) follow from (2), while (5) follows from the structure of $H(n, 1)$ (cf. (1.4)).

Theorem A and Lemma A′ are fundamental for the proof of Theorems I, II. We shall refer to these as (*A*) throughout this paper.

(3.2) LEMMA. *Suppose that $r=1$. $N_G(S)$ is generated by S, β_s $(1 \leqq s \leqq n)$, $\lambda_t \lambda_{t+1}$ $(1 \leqq t \leqq n-1)$, σ_u $(1 \leqq u \leqq n-1)$ which satisfy the following relations besides* (1) *and* (2) *of Lemma A′:*

$$\beta_s^{\lambda_s \lambda_t} = \beta_s^{-1} \ (s \neq t), \quad \beta_s^{\sigma_s} = \beta_{s+1} \quad \textit{and} \quad [\beta_s, \sigma_t] = 1 \ (t \neq s, s+1),$$

where the σ_u $(1 \leqq u \leqq n-1)$ are elements defined in (2.2).

PROOF. We easily see that $N_{\mathfrak{A}_{4n+1}}(\tilde{S}) = \tilde{S} \cdot \tilde{M} \langle \tilde{\beta}_s, \tilde{\sigma}_u \mid 1 \leqq s \leqq n, 1 \leqq u \leqq n-1 \rangle$ (cf. Appendix, Lemma 4). Then our lemma follows from the existence of the mapping θ.

(3.3) In (3.4)$\sim$(3.6) of this section, we shall determine the precise structure of $N_G(S)$ for the case $r \geqq 2$. We assume that $r=2$ or 3 in (3.4)$\sim$(3.6). It is convenient to put $\nu = 1$ if $r=2$. So we have $C_G(S) = S \times \langle \nu \rangle$ in both cases $r = 2, 3$.

(3.4) LEMMA. *Put* $K=\langle\beta_1, \beta_2, \cdots, \beta_n, C_G(S)\rangle$ *and* $K_s=\langle\lambda_s, \beta_s, \pi_s, \pi'_s\rangle$ $(1\leqq s \leqq n)$. *Then the followings hold:* (i) $\langle\beta_1, \beta_2, \cdots, \beta_n, \nu\rangle$ *is a* 3-*Sylow subgroup of* K *and is an elementary abelian subgroup of order* 3^{n+r-2}, *and* (ii) K_s *is isomorphic to* $\mathfrak{S}_4$.

PROOF. From the action of β_s and λ_s $(1\leqq s\leqq n)$ on S, it follows that $\beta_s^3\in C_G(S)$, $\beta_s^{\lambda_s}\equiv\beta_s^{-1} \bmod C_G(S)$ and $[\beta_s, \beta_t]\in C_G(S)$. Since β_s is of odd order and $\beta_s^3\in C_G(S)=S\times\langle\nu\rangle$, we have $\beta_s^3=\nu^i$ $(i=0, 1$ or $2)$. By using the fact that $\beta_s^3=(\beta_s^3)^{\lambda_t}=(\nu^i)^{\lambda_t}=\nu^{-i}$ $(s\neq t)$ (cf. Theorem A (2)), we get $\nu^i=1$. Thus β_s is of order 3. From $[\beta_s, \beta_t]\in C_G(S)$, it follows that $\beta_t^{-1}\beta_s\beta_t=\beta_s x\nu^i$ where $x\in S$ and $i=0, 1$ or 2. Since β_s is of odd order, we get $x\in\langle\pi_s, \pi'_s\rangle$ by (2) of Theorem A. From $\beta_s^{-1}\beta_t\beta_s=x\nu^i\beta_t^{-1}$, we get $x\in\langle\pi_t, \pi'_t\rangle$ by the same reason. Hence we get $x\in\langle\pi_s, \pi'_s\rangle\cap\langle\pi_t, \pi'_t\rangle$ and so $x=1$. Then we have $\nu^{-i}=(\nu^i)^{\lambda_u}=[\beta_s, \beta_t]^{\lambda_u}=[\beta_s, \beta_t]=\nu^i$ for $u\neq s, t$[2]. Hence we get $\nu^i=1$ and so $[\beta_s, \beta_t]=1$. Since $\langle\nu\rangle\lhd K$, we have $[\beta_s, \nu]=1$ $(1\leqq s\leqq n)$ and so $\langle\beta_1, \beta_2, \cdots, \beta_n, \nu\rangle$ is an elementary abelian group. This proves (i). From the fact that $\beta_s^{\lambda_s}=\beta_s^{-1} \bmod C_G(S)$, we have $\beta_s^{\lambda_s}=\beta_s^{-1}x\nu^i$ for some $x\in S$ and $i=0, 1$ or 2. Since β_s is of odd order, we have $x\in\langle\pi_s, \pi'_s\rangle$. Further if $s\neq t$, we have $\nu^{-i}=(\nu^i)^{\lambda_t}=(x\beta_s\beta_s^{\lambda_s})^{\lambda_t}=x\beta_s\beta_s^{\lambda_s}=\nu^i$ by (2) of Theorem A. Hence we get $\beta_s^{\lambda_s}=\beta_s^{-1}x$. This implies that K_s is isomorphic to $\mathfrak{S}_4$. This completes the proof of our lemma.

(3.5) LEEMA. *We may assume that the* β_s $(1\leqq s\leqq n)$ *have the following additional properties besides* (1) *and* (2) *of Theorem A.*

$$\beta_s^{\lambda_s}=\beta_s^{-1},$$

$$\beta_s^{\sigma_s}=\beta_{s+1} \quad \text{and} \quad [\sigma_s, \beta_t]=1 \qquad (1\leqq s\leqq n-1,\ t\neq s,\ s+1),$$

and

$$[\beta_s, \beta_t]=1 \qquad (1\leqq s<t\leqq n),$$

where the σ_s $(1\leqq s\leqq n-1)$ *are elements defined in* (2.3).

PROOF. Since $K_1=\langle\lambda_1, \beta_1, \pi_1, \pi'_1\rangle$ is isomorphic to $\mathfrak{S}_4$ by (3.4), we may assume that $\beta_1^{\lambda_1}=\beta_1^{-1}$ by interchanging β_1 by $\beta_1 x$ for suitable $x\in\langle\pi_1, \pi'_1\rangle$ if necessary. Put $\rho_s=\sigma_1\sigma_2\cdots\sigma_{s-1}$ $(1\leqq s\leqq n$ and $\rho_1=1)$ and $\beta'_s=\beta_1^{\rho_s}$ $(1\leqq s\leqq n$ and $\beta'_1=\beta_1)$. We shall show that the β'_s $(1\leqq s\leqq n)$ have all the required properties. It is obvious from the definition that $\beta_s'^{\sigma_s}=\beta'_{s+1}$. Further from $\lambda_s\beta'_s\lambda_s=(\lambda_1\beta'_1\lambda_1)^{\rho_s}$ and $\beta_1'^{\lambda_1}=\beta_1'^{-1}$, we get $\beta_s'^{\lambda_s}=\beta_s'^{-1}$. Similarly we have $\beta'_s: \pi_s\rightarrow\pi'_s \rightarrow\pi_s\pi'_s$. Since we have

$$\pi_t^{\rho_s^{-1}}=\begin{cases}\pi_t & \text{if } t>s\\ \pi_1 & \text{if } t=s\\ \pi_{t+1} & \text{if } t<s,\end{cases}$$

2) If $n\leqq 2$, we can not choose u such that $u\neq s, t$. However, in the case $n\leqq 2$, our Theorem I were proved (cf. Introduction). So we may assume that $n\geqq 3$. It is easy to prove directly that $[\beta_s, \beta_t]=1$ also in the case $n\leqq 2$.

we get $[\beta_s', \pi_t] = [\beta_1', \pi_t^{\rho_s^{-1}}]^{\rho_s} = [\beta_1', \pi_t]^{\rho_s}$ or $[\beta_1', \pi_{t+1}]^{\rho_s}$ according to whether $t>s$ or $t<s$. Thus, if $s \neq t$, we get $[\beta_s', \pi_t]=1$ from the fact that $\beta_1'=\beta_1$ and (2) of Theorem A. Similarly, if $s \neq t$, we have $[\beta_s', \pi_t'] = [\beta_s', \lambda_t] = 1$. Thus the β_s' $(1 \leqq s \leqq n)$ have the properties (1) and (2) of Theorem A and so we have $[\beta_s', \beta_t'] = 1$ by (3.4). It remains to show that $[\sigma_s, \beta_t'] = 1$ $(t \neq s, s+1)$. Suppose that $t \neq s, s+1$. If $t<s$, we have $\rho_t \sigma_s = \sigma_s \rho_t$ and so $\beta_t'^{\sigma_s} = \beta_1'^{\rho_t \sigma_s} = \beta'^{\sigma_s \rho_t} = \beta_1'^{\rho_t} = \beta_t'$. If $s+1<t$, we have

$$\begin{aligned}
\rho_t \sigma_s &= (\sigma_1 \cdots \sigma_s \sigma_{s+1} \cdots \sigma_{t-1}) \sigma_s \\
&= \sigma_1 \cdots (\sigma_s \sigma_{s+1} \sigma_s) \cdots \sigma_{t-1} \\
&= \sigma_1 \cdots (\sigma_{s+1} \sigma_s \sigma_{s+1}) \cdots \sigma_{t-1} \qquad \text{by } (\sigma_s \sigma_{s+1})^3 = 1 \\
&= \sigma_{s+1} (\sigma_1 \cdots \sigma_{s-1} \sigma_s \sigma_{s+1} \cdots \sigma_{t-1}) \\
&= \sigma_{s+1} \rho_t .
\end{aligned}$$

This yields that $\beta_t'^{\sigma_s} = \beta_1'^{\rho_t \sigma_s} = \beta_1'^{\sigma_{s+1} \rho_t} = \beta_1'^{\rho_t} = \beta_t'$. Thus we have verified that the β_s' $(1 \leqq s \leqq n)$ have all the required properties.

(3.6) LEMMA. *$N_G(S)$ is generated by ν, $K_1 \times K_2 \times \cdots \times K_n$ and P. $(K_1 \times \cdots \times K_n)P$ is a complement of $N_G(S)$ over $\langle \nu \rangle$ and is isomorphic to the wreath product $\mathfrak{S}_3 \wr \mathfrak{S}_n$. The structure of $N_G(S)$ is completely determined.*

PROOF. From (3.4) and (3.5), we see that K_s $(1 \leqq s \leqq n)$ is isomorphic to $\mathfrak{S}_4$ and $[K_s, K_t] = 1$ if $s \neq t$. Let N be a subgroup of $N_G(S)$ generated by ν, $K_1 \times \cdots \times K_n$ and $P = \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \rangle$ (cf. (2.2)). Then we have $[N : C_G(S)] = 3^n \cdot n! \cdot 2^n$. On the other hand, we know by [**9**; (4.5)] that $[N_G(S) : C_G(S)] = 3^n \cdot n! \cdot 2^n$. This yields $N = N_G(S)$. This proves the first statement of our lemma. The other follows from (3.4) and (3.5).

(3.7) LEMMA. *Put*

$$N_1 = \begin{cases} \langle \lambda_i \lambda_j, \beta_s, \sigma_t \mid 2 \leqq i < j \leqq n,\ 2 \leqq s \leqq n,\ 2 \leqq t \leqq n-1 \rangle & \text{if } r=1, \\ \langle \lambda_i, \beta_s, \sigma_t \mid 2 \leqq i \leqq n,\ 2 \leqq s \leqq n,\ 2 \leqq t \leqq n-1 \rangle & \text{if } r=2, \\ \langle \lambda_i, \beta_s, \sigma_t, \nu \mid 2 \leqq i \leqq n,\ 2 \leqq s \leqq n,\ 2 \leqq t \leqq n-1 \rangle & \text{if } r=3. \end{cases}$$

Then we have

$$N_G(S) \cap C_G(\pi_1, \pi_1') = \langle \pi_1, \pi_1' \rangle \times (S_2 \times S_3 \times \cdots \times S_n) N_1 .$$

In particular, a 2-Sylow subgroup of $N_G(S) \cap C_G(\pi_1, \pi_1')$ splits over $\langle \pi_1, \pi_1' \rangle$.

PROOF. This follows from (3.2), (3.5) and (3.6).

§ 4. Technical Lemmas.

(4.1) In this section, we collect some technical lemmas which will be used in the proof of lemmas in § 6. The arguments depend on (A) and lemmas in

§1. Firstly we introduce some notations.

$$E_k=\langle\pi_1,\pi_2,\cdots,\pi_k,\lambda_1\lambda_2,\lambda_1\lambda_3,\cdots,\lambda_1\lambda_k\rangle \qquad (n\geqq k\geqq 1)$$

$$T_k=E_k\cdot\langle\pi_1',\sigma_1',\cdots,\sigma_{k-1}',\pi_k'\rangle \qquad (n\geqq k\geqq 1)$$

$$U_k=\langle\pi_2,\cdots,\pi_k,\lambda_2\lambda_3,\cdots,\lambda_2\lambda_k\rangle\langle\pi_2',\sigma_2',\cdots,\sigma_{k-1}',\pi_k'\rangle \qquad (n-1\geqq k\geqq 2)$$

$$V_k=\langle\pi_{k+1},\cdots,\pi_n,\lambda_{k+1}\lambda_{k+2},\cdots,\lambda_{k+1}\lambda_n\rangle\langle\pi_{k+1}',\sigma_{k+1}',\cdots,\pi_n'\rangle \qquad (n-1\geqq k\geqq 1)$$

$$W_k=\begin{cases}(U_k\times V_k)\langle\lambda_k\lambda_n\rangle & \text{if } r=1\\ (U_k\times V_k)\langle\lambda_k,\lambda_n\rangle & \text{if } r\geqq 2.\end{cases}$$

E_k is an elementary abelian group of order 2^{2k-1}. U_k, V_k and T_k are isomorphic to $\mathfrak{W}_{k-1}^*$, $\mathfrak{W}_{n-k}^*$ and $\mathfrak{W}_k^*$ respectively. W_k is isomorphic to a subgroup $\widehat{\mathfrak{W}_k^*}$ of $\mathfrak{W}_{n-1}^*$ if $r=1$, and a group $\mathfrak{W}_{k-1}\times\mathfrak{W}_{n-k}$ if $r=2$. So we can apply lemmas in §1 to these groups.

It is convenient to say that, *if an involution x of G is conjugate in G to α_k, x is of length k.*

(4.2) LEMMA. *Suppose that $z\in\langle\pi_1,\pi_1'\rangle\times W_k$ and $\pi_1'z\sim\pi_1$ and $\pi_1'\alpha_k z\sim\alpha_{k-1}$ in G. Then we have $z^y=\pi_1'\pi_1$ or $\pi_1'\pi_2$ for some $y\in U_k$.*

PROOF. Suppose that $z=v_1'v_2'$ where $v_1'\in\langle\pi_1,\pi_1'\rangle$ and $v_2'\in W_k$. From (*A*) and the assumption of our lemma it follows that the v_i' $(1\leqq i\leqq 2)$ are at most of length 1. Put $C_1=\{1,\pi_1,\pi_1',\pi_1\pi_1'\}$ and $C_2=\{1,\pi_2,\pi_2',\pi_{k+1},\pi_{k+1}',\lambda_k\lambda_n\}$ or $\{1,\pi_2,\pi_2',\pi_{k+1},\pi_{k+1}',\lambda_k\lambda_n,\lambda_k,\lambda_n\}$ according to whether $r=1$ or $r\geqq 2$. Then C_1 and C_2 are the sets of the representatives of conjugacy classes of involutions with at most length 1 of $\langle\pi_1,\pi_1'\rangle$ and W_k respectively. This follows by applying (1.3) and (1.4) with $\pi_{i+1},\pi_{i+1}',\lambda_{i+1},\sigma_{j+1}'$ in place of ξ_i,y_i,x_{2i-1},z_j $(1\leqq i\leqq n-1$, $1\leqq j\leqq n-2)$. Therefore we can find an element $y\in W_k$ such that $z^y=v_1'v_2'^y$ with $v_2'^y\in C_2$. Put $v_1=v_1'$ and $v_2=v_2'^y$. So we have $z^y=v_1v_2$ with $v_1\in\langle\pi_1,\pi_1'\rangle$ and $v_2\in C_2$. If $v_1=1$, we have $\pi_1\sim\pi_1'z^y=\pi_1'v_2$ and $v_2\neq 1$ because of $[y,\pi_1']=1$ and the assumption of our lemma. This is impossible because $\pi_1'v_2$ is at least of length 2 if $1\neq v_2\in C_2$. If $v_1=\pi_1$, we must have $v_2=1$ by (*A*) because $\pi_1\sim\pi_1'z^y=\pi_1'\pi_1v_2$ in G. Then we get $\alpha_{k-1}\sim\pi_1'\alpha_k z^y=\pi_1'\alpha_k\pi_1\sim\alpha_k$ which is impossible by (*A*). If $v_1=\pi_1'\pi_1$, we must have $v_2=1$ and so $z^y=\pi_1'\pi_1$. Then we may assume that $y\in U_k$ since $[V_k\langle\lambda_k,\lambda_n\rangle,\pi_1'\pi_1]=1$. This proves our lemma in this case. Finally suppose that $v_1=\pi_1'$. Then we have $\pi_1\sim v_2$ and $\alpha_k v_2\sim\alpha_{k-1}$ in G by the assumption of our lemma. If $v_2\neq\pi_2$, we see from (*A*) that "$\alpha_k v_2\sim\alpha_{k-1}$ in G" is violated. Hence we get $z^y=\pi_1'\pi_2$. Again we may assume that $y\in U_k$ since $[V_k\langle\lambda_k,\lambda_n\rangle,\pi_1'\pi_2]=1$. Our lemma is complete.

(4.3) LEMMA. *Suppose that $1\neq z\in\langle\pi_1,\pi_1'\rangle\times W_k$ and $\pi_1'\sim\pi_1'z$ and $\alpha_k\pi_1'\sim\alpha_k\pi_1'z$ in G. Then we have $z^y=\pi_1$, $\pi_1'\pi_2'$, $\pi_1'\lambda_k\lambda_n$ or $\pi_1'\lambda_k$ for some $y\in W_k$ where $\pi_1'\lambda_k$ appears only in the case $r\geqq 2$. In particular, if $z\neq\pi_1$, we have $z\sim\alpha_2$ in*

G and $z \neq \alpha_2$.

PROOF. Let C_1 and C_2 be the sets defined in the proof of (4.2). Then we can find $y \in W_k$ such that $z^y = v_1 v_2$ where $v_i \in C_i$ $(1 \leqq i \leqq 2)$. Since $\pi_1' \sim \pi_1' z^y$ in G, the v_i must be at most of length 1. If $v_1 = 1$, $\pi_1' z^y = \pi_1' v_2$ would be at least of length 2, which is impossible because $\pi_1' \sim \pi_1' z^y$ in G and so $\pi_1' z^y$ is of length 1. If $v_1 = \pi_1$, we get $v_2 = 1$ from the fact that $\pi_1' \sim \pi_1' z^y = \pi_1' \pi_1 v_2$ in G. Then we have $z^y = \pi_1$. If $v_1 = \pi_1' \pi_1$, we must have $v_2 = 1$ since $\pi_1' \sim \pi_1' z^y = \pi_1 v_2$. Then we get $\alpha_k \sim \alpha_k \pi_1' \sim \alpha_k \pi_1' z^y = \alpha_k \pi_2 \sim \alpha_{k-1}$ which is impossible by (*A*). Finally suppose that $v_1 = \pi_1'$. If $v_2 = \pi_2$, we have $\alpha_k \sim \alpha_k \pi_1' \sim \alpha_k \pi_1' z^y = \alpha_k \pi_2 \sim \alpha_{k-1}$ which is impossible. Similarly "$v_2 = \pi_{k+1}, \pi_{k+1}'$ or λ_n" is impossible. Thus we must have $v_2 = \pi_2'$, $\lambda_k \lambda_n$ or λ_k. This completes the proof of our lemma.

(4.4) LEMMA. *Put*

$$\hat{W}_k = \begin{cases} (T_k \times V_k)\langle \lambda_1 \lambda_{k+1} \rangle & \text{if } r = 1 \\ (T_k \times V_k)\langle \lambda_1, \lambda_{k+1} \rangle & \text{if } r = 2. \end{cases}$$

Suppose that $x \in C_G(\alpha_k)$ *and* $\pi_1^x \in \hat{W}_k$. *Then we have* $\pi_1^x \in E_k$.

PROOF. Put $C = \{\pi_1, \pi_1', \pi_{k+1}, \pi_{k+1}', \lambda_k \lambda_n\}$ or $\{\pi_1, \pi_1', \pi_{k+1}, \pi_{k+1}', \lambda_k \lambda_n, \lambda_k, \lambda_n\}$ according to whether $r = 1$ or $r \geqq 2$. Then C is the set of the representatives of conjugacy classes of involutions with length 1 of $\hat{W}_k$ by (1.3) and (1.4). (Remark that $\hat{W}_k$ is isomorphic to a subgroup $\hat{\mathfrak{W}}_k^*$ of $\mathfrak{W}_n^*$, or a group $\mathfrak{W}_k \times \mathfrak{W}_{n-k}$ according to whether $r = 1$ or $r \geqq 2$.) Therefore we can find an element $y \in \hat{W}_k$ such that $\pi_1^{xy} \in C$. Since $x, y \in C_G(\alpha_k)$, we have $\alpha_{k-1} \sim (\alpha_k \pi_1)^{xy} = \alpha_k \pi_1^{xy}$. Therefore if $\pi_1^{xy} \neq \pi_1$, we would obtain $\alpha_{k-1} \sim \alpha_k$ or α_{k+1} which is impossible by (*A*). Thus we have obtained $\pi_1^{xy} = \pi_1$. Since $\pi_1 \in E_k$ and $E_k \lhd \hat{W}_k$, we get $\pi_1^x \in E_k$. This completes the proof of our lemma.

§ 5. The structure of $C_G(\alpha_1)$ and $N_G(E_k)$ $(k \geqq 1)$.

(5.1) The proof of our Theorems I, II proceeds by induction on n. In this section, we shall determine the structure of $C_G(\alpha_1)$ and $N_G(E_k)$ by using the inductive hypothesis.

(5.2) LEMMA. (i) *A 2-Sylow subgroup of* $N_G(S) \cap C_G(\alpha_k)$ $(1 \leqq k \leqq n)$ *is that of* $C_G(\alpha_k)$. (ii) *A 2-Sylow sucgroup of* $N_G(S) \cap C_G(\pi_1, \pi_1')$ *is that of* $C_G(\pi_1, \pi_1')$.

PROOF. (i) We have $C_G(\alpha_k) \supseteqq S$. Denote by D_k a 2-Sylow subgroup of $C_G(\alpha_k)$ with $S \subseteqq D_k \subseteqq C_G(\alpha_k)$. Since $D_k \rhd S$ by (2.5), we have $D_k \subseteqq N_G(S) \cap C_G(\alpha_k)$. This proves (i). Also the proof of (ii) is quite similar.

(5.3) LEMMA. $C_G(\pi_1, \pi_1') = \langle \pi_1, \pi_1' \rangle \times X_1$ *where* $X_1 \cong \mathfrak{A}_{4n-4}$ *or* $\mathfrak{A}_{4n-3}$ *if* $r = 1$ *and* $X_1 \cong \mathfrak{A}_{4n+r-4}$ *if* $r \geqq 2$.

PROOF. From (3.7) and (5.2), we see that 2-Sylow subgroup of $C_G(\pi_1, \pi_1')$ splits over $\langle \pi_1, \pi_1' \rangle$. Then a theorem of Gaschütz [**6**; p. 121] yields that $C_G(\pi_1, \pi_1') = \langle \pi_1, \pi_1' \rangle \times X_1$ for some subgroup X_1 of $C_G(\pi_1, \pi_1')$. We shall determine the structure of X_1. From the structure of $C_G(\alpha_n)=H(n, r)$, we see that

$$(\sharp) \qquad C_G(\pi_1, \pi_1')\cap C_G(\alpha_n)=\begin{cases} \langle\pi_1, \pi_1'\rangle\times V_1 & \text{if } r=1 \\ \langle\pi_1, \pi_1'\rangle\times V_1\langle\lambda_2, \nu\rangle & \text{if } r\geqq 2, \end{cases}$$

where V_1 is the subgroup defined in (4.1). Firstly suppose that $r=1$. Since β_s is of odd order and $\beta_s\in C_G(\pi_1, \pi_1')$ $(2\leqq s\leqq n)$, we have $\beta_s\in X_1$ $(2\leqq s\leqq n)$. Then we get $X_1\ni\pi_s, \pi_s'$ $(2\leqq s\leqq n)$ because $\beta_s: \pi_s\to\pi_s'\to\pi_s'\pi_s$ and so each of $\pi_s, \pi_s', \pi_s'\pi_s$ is commutator. If we put $\rho_k=\pi_k'\sigma_k'$ $(2\leqq k\leqq n)$, ρ_k is of order 3 and ρ_k is contained in $C_G(\pi_1, \pi_1')$ (cf. (2.1)). So we have $\sigma_k', \lambda_k\lambda_{k+1}\in X_1$ $(2\leqq k\leqq n-1)$ because $\pi_k', \pi_k\in X_1$ and $\rho_k: \pi_k\to\lambda_k\lambda_{k+1}$. Then we see from (3.7) and ($\sharp$) that X_1 satisfies the inductive hypothesis for Theorem II with $n-1$, $\pi_2\pi_3\cdots\pi_n$ and $S_2\times S_3\times\cdots\times S_n$ in place of n, α_n and S respectively. This yields that $X_1\cong\mathfrak{A}_{4n-4}$ or $\mathfrak{A}_{4n-3}$. Secondly suppose that $r\geqq 2$. Since β_s and γ_s $(2\leqq s\leqq n)$ are contained in $C_G(\pi_1, \pi_1')$, we have $\beta_s, \gamma_s\in X_1$ $(2\leqq s\leqq n)$. Then from the fact that $\beta_s: \pi_s\to\pi_s'\to\pi_s\pi_s'$ and $\gamma_s: \pi_s\to\lambda_s\to\lambda_s\pi_s$, it follows that π_s, π_s' and $\lambda_s\in X_1$ $(2\leqq s\leqq n)$. Since $\pi_k'\sigma_k'$ is of order 3 and is contained in $C_G(\pi_1, \pi_1')$ if $k\geqq 2$, we have $C_{X_1}(\pi_2\cdots\pi_n)=V_1\langle\nu\rangle$ by ($\sharp$). Further we see from (1.6) that X_1 has no subgroup of index 2. This implies that X_1 satisfies the inductive hypothesis for Theorem I, and so we get $X_1\cong\mathfrak{A}_{4n+r-4}$. This completes the proof of our lemma.

(5.4) In the preceding arguments, we have considered three groups $G=G(n, r)$ $(r=1, 2, 3)$. But it is convenient to put $G=G(n, 0)$ and $H=H(n, 0)$ if $r=1$ and $X_1\cong\mathfrak{A}_{4n-4}$. So hereafter we have $r=0$, 1, 2, or 3. We note that $H(n, 0)=H(n, 1)$.

(5.5) Let D_1 be a 2-Sylow subgroup of V_1 or $V_1\langle\lambda_2\rangle$ according to $r\leqq 1$ or $r\geqq 2$, which normalizes S (cf. (3.7)). Here we remark that v_1 normalizes M where M is the group defined in (2.2).

Put

$$D_0=\langle\pi_1\rangle D_1\,,$$

and

$$D=\langle\lambda_1\lambda_2,\ \pi_1,\ \pi_1'\rangle D_1\,.$$

Then D is a 2-group by the above remark. We note that $D_0=\langle\pi_1\rangle\times D_1$ and, if $r\geqq 2$, $D=\langle\lambda_1, \pi_1, \pi_1'\rangle\times D_1$. Further we have $D=\langle\lambda_1\lambda_2, \pi_1'\rangle D_1$, D/D_1 is isomorphic to a dihedral group of order 8 and D is a 2-Sylow subgroup of $C_G(\alpha_1)$ by (5.2).

(5.6) LEMMA. *Any element of* $\langle\lambda_1\lambda_2\pi_1'\rangle D_0-D_0$ *is at least of order* 4 *and* $\lambda_1\lambda_2\pi_1'$ *is not conjugate in* $C_G(\alpha_1)$ *to any element of* D_0.

PROOF. Since $\langle\lambda_1\lambda_2\pi_1'\rangle D_0=\langle\lambda_1\lambda_2\pi_1'\rangle D_1$ and $(\lambda_1\lambda_2\pi_1')^2=\pi_1$, the first statement is obvious. Suppose that $(\lambda_1\lambda_2\pi_1')^z=x$ where $x\in D_0$ and $z\in C_G(\alpha_1)$. Then we have $\pi_1^z = x^2$ and $x^2 \in D_1$ by taking the squares of both sides of the equation $(\lambda_1\lambda_2\pi_1')^z = x$. This is impossible because $\pi_1^z = \pi_1$ and $\pi_1 \notin D_1$.

(5.7) LEMMA. *$\lambda_1\lambda_2$ is not conjugate in $C_G(\alpha_1)$ to any element of $\langle\lambda_1\lambda_2\pi_1'\rangle D_0$.*

PROOF. Suppose that $(\lambda_1\lambda_2)^z = x$ for some $x \in \langle\lambda_1\lambda_2\pi_1'\rangle D_0$ and $z \in C_G(\alpha_1)$. Then, by (5.6), we have $x \in D_0$. Write $x = \pi_1^\delta y$ where $y \in D_1$ and $\delta = 0$ or 1. If $y = 1$, we have $x = \pi_1$ which is obviously impossible since $z \in C_G(\alpha_1)$. If $y \neq 1$, we can find an element z' of X_1 such that $y^{z'} = \pi_2\pi_3 \cdots \pi_k$ $(2 \leqq k \leqq n)$ because the $\pi_2\pi_3 \cdots \pi_l$ $(2 \leqq l \leqq n)$ are the representatives of conjugacy classes of involutions of X_1. Then we have $(\lambda_1\lambda_2)^{zz'} = \pi_1^\delta\pi_2 \cdots \pi_k$, and so $(\lambda_1\lambda_2)^{zz'} = \pi_2$ because $\lambda_1\lambda_2 \sim \alpha_1$ in G by (*A*). Since $zz' \in C_G(\alpha_1)$, we get $(\lambda_1\lambda_2\pi_1)^{zz'} = \pi_1\pi_2$ which is impossible because $\lambda_1\lambda_2\pi_1 \sim \alpha_1$ and $\pi_1\pi_2 = \alpha_2$. This completes the proof.

(5.8) LEMMA. *There exists a subgroup K_1 of $C_G(\alpha_1)$ of index 2 such that K_1 does not contain $\lambda_1\lambda_2$ and a 2-Sylow subgroup of K_1 is $\langle\lambda_1\lambda_2\pi_1'\rangle \cdot D_1$ or $\langle\pi_1, \pi_1'\rangle \cdot D_1$.*

PROOF. From (5.7) and a lemma of Thompson[3] it follows that $C_G(\alpha_1)$ has a subgroup K_1 of index 2, which does not contain $\lambda_1\lambda_2$. Obviously we have $K_1 \supset \langle\pi_1\rangle \times X_1 \supset \langle\pi_1\rangle \times D_1$. From this our lemma follows.

(5.9) LEMMA. *K_1 has a subgroup K_2 of index 2 with $\langle\pi_1\rangle \times D_1$ as a 2-Sylow subgroup.*

PROOF. Firstly suppose that $\langle\lambda_1\lambda_2\pi_1'\rangle D_1$ is a 2-Sylow subgroup of K_1. From (5.6) and a lemma of Thompson[3] it follows that K_1 has a subgroup K_2 of index 2 which does not contain $\lambda_1\lambda_2\pi_1'$. Since $K_2 \supseteqq X_1 \supseteqq D_1$, we must have $K_2 \supset \langle\pi_1\rangle \times D_1$. Secondly suppose that $\langle\pi_1, \pi_1'\rangle \times D_1$ is a 2-Sylow subgroup of K_1. If $\pi_1'^z = \pi_1^\delta y$ for $y \in D_1$ and $z \in C_G(\alpha_1)$ $(\delta = 0$ or $1)$, we must have $\delta = 0$ and $y \neq 1$. Then there is an element z' of X_1 such that $y^{z'} = \pi_2$, and so $\pi_1'^{zz'} = \pi_2$. By multiplying both sides of $\pi_1'^{zz'} = \pi_2$ by π_1, we get $(\pi_1\pi_1')^{zz'} = \pi_1\pi_2$ which is impossible because $\pi_1\pi_1' \sim \alpha_1$ and $\pi_1\pi_2 = \alpha_2$. Thus we have proved that π_1' is not conjugate in $C_G(\alpha_1)$ to any element of $\langle\pi_1\rangle \times D_1$. Then a lemma of Thompson[3] yields that K_1 has a subgroup of index 2 which does not contain π_1'. This implies that $C_G(\alpha_1)$ has a normal subgroup of K_2 of index 4. But clearly K_2 must contain $\langle\pi_1\rangle \times X_1$ and so $\langle\pi_1\rangle \times D_1$. This completes the proof of our lemma.

(5.10) LEMMA. $K_2 = \langle\pi_1\rangle \times X_1$.

PROOF. Since a 2-Sylow subgroup of K_2 is $\langle\pi_1\rangle \times D_1$ and π_1 is in the center of K_2, a theorem of Gaschütz [**6**; p. 121] yields that $K_2 = \langle\pi_1\rangle \times Y_1$ for some subgroup Y_1 of K_2. Obviously we have $Y_1 \supseteqq X_1$ and $C_{Y_1}(\pi_2 \cdots \pi_n) \supseteqq C_{X_1}(\pi_2 \cdots \pi_n)$. Further we have

3) See [**2**; p. 265 Exercise 3] or [**3**; Lemma 16]. The latter is a slight generalization of the former. For the first application in (5.9) [**3**; Lemma 16] should be used.

$$C_G(\alpha_1)\cap C_G(\pi_2\cdots\pi_n)=C_G(\alpha_1)\cap C_G(\alpha_n)=\begin{cases}\langle\lambda_1\lambda_2,\ \pi_1'\rangle V_1 & \text{if } r\leqq 1\\ \langle\lambda_1\lambda_2,\ \pi_1'\rangle V_1\langle\lambda_2,\ \nu\rangle & \text{if } r\geqq 2,\end{cases}$$

and

$$C_G(\alpha_1)\cap N_G(S)=\langle\lambda_1\lambda_2,\ \pi_1'\rangle(S_2\times\cdots\times S_n)N_1,$$

where N_1 is the groups defined in (3.7).

From these, it follows that $C_{Y_1}(\pi_2\cdots\pi_n)=C_{X_1}(\pi_2\cdots\pi_n)$ and $N_{Y_1}(S_2\times\cdots\times S_n)=N_{X_1}(S_2\times\cdots\times S_n)$. If $r\geqq 2$, by the inductive hypothesis, we must have $Y_1\cong\mathfrak{A}_{4n+r-4}$ and so $Y_1=X_1$. (Note that $Y_1=O^2(Y_1)$ because $X_1=O^2(X_1)$ and $[Y_1:X_1]=\text{odd}$). If $r\leqq 1$, the induction hypothesis yields that $Y_1\cong\mathfrak{A}_{4n-4}$ or $\mathfrak{A}_{4n-3}$. So if $r=1$, we get $Y_1=X_1$ since $X_1\cong\mathfrak{A}_{4n-3}$ in this case. But we must have $Y_1=X_1$ also in the case $r=0$ because $C_G(\alpha_1)=\langle\lambda_1\lambda_2,\ \pi_1'\rangle K_2\supseteqq Y_1\supseteqq X_1$ and $[C_G(\alpha_1):X_1]=8$. This completes the proof.

(5.11) LEMMA. *$C_G(\alpha_1)=(\langle\pi_1,\ \pi_1'\rangle\times X_1)\langle\lambda_1\lambda_2\rangle$ and $X_1\langle\lambda_1\lambda_2\rangle\cong\mathfrak{S}_{4n+r-4}$.*

PROOF. The first statement is obvious from $C_G(\alpha_1)=\langle\lambda_1\lambda_2,\ \pi_1'\rangle K_2$ and (5.3). We shall prove $X_1\langle\lambda_1\lambda_2\rangle\cong\mathfrak{S}_{4n+r-4}$. Suppose false. Then there is an element x of X_1 such that $[\lambda_1\lambda_2x,\ X_1]=1$. Put $F=\langle\pi_2,\cdots,\pi_n,\ \lambda_2\lambda_3,\cdots,\lambda_2\lambda_n\rangle$ or $\langle\pi_2,\cdots,\pi_n,\ \lambda_2,\ \lambda_3,\cdots,\lambda_n\rangle$ according to whether $r\leqq 1$ or $r\geqq 2$. Since $[\lambda_1\lambda_2,\ F]=1$ and $F\subset X_1$, we get $[x,\ F]=1$. Since F is self-centralizing in X_1, we see that $x\in F$. On the other hand, we can find one of the β_s $(2\leqq s\leqq n)$ such that $[\lambda_1\lambda_2y,\ \beta_s]\neq 1$ for any nonidentity element y of F. In particular, we have $[\lambda_1\lambda_2x,\ \beta_k]\neq 1$ for some k $(2\leqq k\leqq n)$. This is a contradiction because $\beta_k\in X_1$ and $[\lambda_1\lambda_2x,\ X_1]=1$.

(5.12) LEMMA (H. Nagao). *Let X be a group isomorphic to $\mathfrak{S}_l$ and Y be a subgroup of X which is of the form $S^{(1)}\times S^{(2)}\times\cdots S^{(l')}\times S^{(l'+1)}$ where $S^{(i)}\cong\mathfrak{S}_4$ $(1\leqq i\leqq l')$ and $S^{(l'+1)}\cong\mathfrak{S}_k$ $(k=0,\ 1,\ 2$ or $3)$. Assume that*

(i) *$l-1=4l'+k$ and $l\neq 6,\ 7$,*

(ii) *$S^{(i)}$ is conjugate in X to $S^{(j)}$ $(1\leqq i,\ j\leqq l')$ and $S^{(l'+1)}$ is contained in a subgroup conjugate in X to $S^{(i)}$ for every i $(1\leqq i\leqq l')$, and*

(iii) *$S^{(i)}\nsubseteq X'$ (= the alternating subgroup of X). Then each member of a set of canonical generators of $S^{(i)}$ $(1\leqq i\leqq l'+1)$ is a transposition in X*[4].

PROOF. This is a reformulation of [8; (1.8)], the proof of which is due to Professor H. Nagao.

(5.13) LEMMA. *There are $n-2$ involutions $\delta_j^{(1)}$ $(2\leqq j\leqq n-1)$ or $n-1$ involutions $\delta_j^{(1)}$ $(2\leqq j\leqq n)$ of $X_1\langle\lambda_1\lambda_2\rangle$ according to whether $r=0$ or $r\geqq 1$ such that the following ordered set is a set of canonical generators of $X_1\langle\lambda_1\lambda_2\rangle\cong\mathfrak{S}_{4n+r-4}$:*

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4) We say that an involution of X is a transposition in X if, for a fixed isomorphism θ from $\mathfrak{S}_l$ to X, $x=\theta(y)$ for some transposition y of $\mathfrak{S}_l$. It is well known that, if $l\neq 6$, this definition does not depend on the choice of an isomorphism θ.

$$\bigcup_{j=2}^{n-1}\{\lambda_1\lambda_j,\ \lambda_1\lambda_j\beta_j,\ \lambda_1\lambda_j\pi_j,\ \delta_j^{(1)}\}\cup\{\lambda_1\lambda_n,\ \lambda_1\lambda_n\beta_n,\ \lambda_1\lambda_n\pi_n\}\qquad \textit{if } r=0$$

$$\bigcup_{j=2}^{n}\{\lambda_1\lambda_j,\ \lambda_1\lambda_j\beta_j,\ \lambda_1\lambda_j\pi_j,\ \delta_j^{(1)}\}\qquad \textit{if } r=1\,,$$

$$\bigcup_{j=2}^{n}\{\lambda_1\lambda_j,\ \lambda_1\lambda_j\beta_j,\ \lambda_1\lambda_j\pi_j,\ \delta_j^{(1)}\}\cup\{\lambda_1\}\qquad \textit{if } r=2$$

and

$$\bigcup_{j=2}^{n}\{\lambda_1\lambda_j,\ \lambda_1\lambda_j\beta_j,\ \lambda_1\lambda_j\pi_j,\ \delta_j^{(1)}\}\cup\{\lambda_1,\ \lambda_1\nu\}\qquad \textit{if } r=3\,.$$

PROOF. We shall prove the case $r=3$. Also in any other cases, the proof is quite similar. Put $S^{(i)}=\langle\lambda_1\lambda_{i+1},\ \lambda_1\lambda_{i+1}\beta_{i+1},\ \lambda_1\lambda_{i+1}\pi_{i+1}\rangle$ $(1\leqq i\leqq n-1)$ and $S^{(n)}=\langle\lambda_1\rangle$. Then it is easy to see that $S^{(i)}\cong\mathfrak{S}_4$ $(1\leqq i\leqq n-1)$ and $\{\lambda_1\lambda_{i+1},\ \lambda_1\lambda_{i+1}\beta_{i+1},\ \lambda_1\lambda_{i+1}\pi_{i+1}\}$ is a set of canonical generators of $S^{(i)}$. Since $\sigma_{i+1}: \lambda_1\lambda_{i+1}\to\lambda_1\lambda_{i+2}$, $\beta_{i+1}\to\beta_{i+2}$ and $\pi_{i+1}\to\pi_{i+2}$ by (3.5), and $\gamma_i: \lambda_1\lambda_i\to\lambda_1$ by $(\boldsymbol{A})$, we see that the $S^{(i)}$ $(1\leqq i\leqq n)$ satisfy the condition (ii) of (5.12) with $X_1\langle\lambda_1\lambda_2\rangle$, $4n-1$, $n-1$ and 2 in place of X, l, l' and k in (5.12) respectively. The other conditions can be checked easily. Then (5.12) yields that $\lambda_1\lambda_j$, $\lambda_1\lambda_j\beta_j$, $\lambda_1\lambda_j\pi_j$ and λ_1 $(2\leqq j\leqq n)$ are transposition in $X_1\langle\lambda_1\lambda_2\rangle\cong\mathfrak{S}_{4n-1}$. From this our lemma follows.

(5.14) Put $m=4n+r$. From (5.11), we see that $C_G(\alpha_1)$ is isomorphic to $C_{\mathfrak{A}_m}(\tilde{\alpha}_1)$. Further, (5.13) implies that we can find an isomorphism θ_1 from $C_{\mathfrak{A}_m}(\tilde{\alpha}_1)$ to $C_G(\alpha_1)$ as follows:

$$\begin{array}{lll} \tilde{\delta}_i^{(1)}=(1,2)(4i,4i+1)\longrightarrow\delta_i^{(1)} & & (2\leqq i\leqq n)\\ \tilde{\lambda}_j\longrightarrow\lambda_j & & \\ \tilde{\pi}_j\longrightarrow\pi_j & & (1\leqq j\leqq n)\\ \tilde{\pi}'_j\longrightarrow\pi'_j & & \\ \tilde{\nu}\longrightarrow\nu\,, & & \end{array}$$

where the elements of the left hand side were defined in (2.2) except for the $\tilde{\delta}_i^{(1)}$.

Define the set Ω_k $(1\leqq k\leqq n-1)$ of $m-4k$ elements as follows:

$$\Omega_k=\{4k+1,\ 4k+2,\ \cdots,\ m\}\,.$$

Then we note that θ_1 induces an isomorphism from the alternating group $\mathfrak{A}_{\Omega_1}$ on the set Ω_1 onto X_1.

Put

$$X_k=\theta_1(\mathfrak{A}_{\Omega_k})\qquad(1\leqq k\leqq n)\,,$$

and

$$\delta_i^{(k)}=\lambda_k\lambda_{k+1}\delta_i^{(k-1)}\qquad(k+1\leqq i\leqq n)\,,$$

where the $\delta_i^{(k)}$ are defined inductively. Then the ordered set

$$\{\beta_{k+1}, \pi_{k+1}, \delta_{k+1}^{(k)}\} \cup \Big(\bigcup_{i=k+2}^{n} \{\lambda_{k+1}\lambda_i, \lambda_{k+1}\lambda_i\beta_i, \lambda_{k+1}\lambda_i\pi_i, \delta_i^{(k)}\}\Big) \cup \{\lambda_1, \lambda_1\nu\}$$

is a set of canonical generators of X_k. Here we remark that the last $3-r$ generators do not appear in the above set. Moreover we easily see by using the isomorphism θ_1 that

$$C_G(E_k) = E_k \times X_k \qquad (2 \leqq k \leqq n),$$

where the E_k are elementary abelian groups defined in (4.1).

(5.15) LEMMA. $[\sigma_1', X_2] = 1$, *where* σ_1' *is defined in* (2.1).

PROOF. We know that $C_G(E_2) = E_2 \times X_2$. Since E_2 is normalized by $\rho_1 = \pi_1\sigma_1'$ (cf. (2.1)), so does X_2. Since ρ_1 is of order 3, ρ_1 induces an inner automorphism of $X_2 \cong \mathfrak{A}_{m-8}$ $(m = 4n+r)$. So we can find an element $v \in X_2$ such that $[\rho_1 v, X_2] = 1$. Put $F = \langle \pi_3, \cdots, \pi_n, \lambda_3\lambda_4, \cdots, \lambda_3\lambda_n\rangle$ or $\langle \pi_3, \pi_4, \cdots, \pi_n, \lambda_3, \lambda_4, \cdots, \lambda_n\rangle$ according to whether $r \leqq 1$ or $r \geqq 2$ when $n \geqq 4$, and $F = \langle \pi_3, \pi_3'\rangle$ or $\langle \pi_3\lambda_3\rangle$ according to whether $r \leqq 1$ or $r \geqq 2$ when $n = 3$. Then we easily see that F is an elementary abelian group and self-centralizing in X_2. Since $[\rho_1, F] = 1 = [\rho_1 v, F] = 1$, we have $[v, F] = 1$. Therefore we get $v \in F$ and then $v = 1$ because v is of odd order. This yields that $[\rho_1, X_2] = 1$. Then we must have $[\sigma_1', X_2] = 1$ because π_1' centralize X_1 and $X_1 \supset X_2$. The proof is complete.

(5.16) LEMMA. *Without loss of generality, we may assume* $\sigma_i' = (\lambda_i\pi_i\lambda_{i+1})^{\delta_i^{(1)}}$ $(2 \leqq i \leqq n-1)$.

PROOF. Put $\sigma_i'' = (\lambda_i\pi_i\lambda_{i+1})^{\delta_i^{(1)}}$ $(2 \leqq i \leqq n-1)$. It is easy to see that $[\sigma_k'^{-1}\sigma_k'', M] = 1$ and $(\pi_k'\sigma_k'')^3 = (\sigma_k''\pi_k')^3 = 1$. (Compute directly by using the isomorphism θ_1. For the definition of M, see (2.2)). Then we can apply (1.5) with $\pi_1', \sigma_1', \sigma_k''$ in place of y_i, z_i, z in (1.5). Our lemma follows from the third statement of (1.5).

(5.17) *The definition of a mapping* φ. In (2.2), we defined an isomorphism θ_n from $C_{\mathfrak{A}_m}(\tilde{\alpha}_n)$ to $H(n, r) = C_G(\alpha_n)$. Then (5.16) yields that the restrictions of θ_1 and θ_n to $C_{\mathfrak{A}_m}(\tilde{\alpha}_n) \cap C_{\mathfrak{A}_m}(\tilde{\alpha}_1)$ coincide. Therefore we can define a one-to-one mapping φ from $C_{\mathfrak{A}_m}(\tilde{\alpha}_n) \cup C_{\mathfrak{A}_m}(\tilde{\alpha}_1)$ to $C_G(\alpha_n) \cup C_G(\alpha_1)$ such that $\varphi | C_{\mathfrak{A}_m}(\tilde{\alpha}_n) = \theta_n | C_{\mathfrak{A}_m}(\tilde{\alpha}_n)$ and $\varphi | C_{\mathfrak{A}_m}(\tilde{\alpha}_1) = \theta_1 | C_{\mathfrak{A}_m}(\tilde{\alpha}_1)$.

(5.18) LEMMA. $N_G(E_k) = (T_k \times X_k)\langle \lambda_1\lambda_{k+1}\rangle$ $(2 \leqq k \leqq n)$ *where* T_k *is defined in* (4.1).

PROOF. From (5.14), (5.15) and (5.16) it follows that $[T_k, X_k] = 1$. Put $Y_k = (T_k \times X_k)\langle \lambda_1\lambda_{k+1}\rangle$. Clearly we have $Y_k \subseteqq N_G(E_k)$. Denote by B_k the set of elements of E_k which are conjugate to α_1 in G. Then it follows from (*A*) that B_k is the orbit of α_1 under conjugation of T_k. This implies that Y_k operates transitively on B_k by conjugation. Further, by (1.2), E_k is generated by elements of B_k. Therefore, in order to see $Y_k = N_G(E_k)$, it is sufficient to see that $C_G(\alpha_1)\cap N_G(E_k)\subseteqq Y_k$. But from (5.14) we see that $C_G(\alpha_1)\cap N_G(E_k)=(\langle\pi_1, \pi_1'\rangle\times(V_k\times X_k))\langle\lambda_1\lambda_{k+1}\rangle$. This completes the proof of our lemma.

(5.19) REMARK. In the next section, we shall prove that $N_G(E_k)=C_G(\alpha_k)$ $(2\leqq k\leqq n)$. Then, roughly speaking, the mapping φ induces an isomorphism from $C_{\mathfrak{A}_m}(\tilde{\alpha}_k)$ to $N_G(E_k)$ as seen from (5.14), (5.16) and (5.18), and φ and n involutions $\alpha_1, \alpha_2, \cdots, \alpha_n$ of G will satisfy the conditions of a theorem of [8] which yields that G is isomorphic to $\mathfrak{A}_m$. In § 7, we shall describe more strictly these situations.

§ 6. The structure of $C_G(\alpha_k)$ $(k\geqq 2)$.

(6.1) In this section, we shall use lemmas proved in § 4 and notations introduced in (4.1).

(6.2) LEMMA. (i) *We have* $C_G(\pi_1')\cap C_G(\alpha_k)=C_G(\pi_1'\pi_1)\cap C_G(\alpha_k)=\langle\pi_1, \pi_1'\rangle\times(U_k\times X_k)\langle\lambda_2\lambda_{k+1}\rangle$. (ii) $\langle\pi_1, \pi_1'\rangle\times W_k$ *contains a 2-Sylow subgroup of* $C_G(\pi_1')\cap C_G(\alpha_k)$. *In particular, if v is an involution of $C_G(\pi_1')\cap C_G(\alpha_k)$, there is an element $y\in X_k$ such that $v^y\in\langle\pi_1, \pi_1'\rangle\times W_k$.*

PROOF. From (5.3) and (5.11), we know that $C_G(\pi_1, \pi_1')=\langle\pi_1, \pi_1'\rangle\times X_1$ and $C_G(\pi_1)=(\langle\pi_1, \pi_1'\rangle\times X_1)\langle\lambda_1\lambda_2\rangle$. Since $\beta_1: \pi_1\to\pi_1'\to\pi_1\pi_1'$, β_1 normalizes X_1 and so $C_G(\pi_1')=(\langle\pi_1, \pi_1'\rangle\times X_1)\langle(\lambda_1\lambda_2)^{\beta_1}\rangle$ and $C_G(\pi_1'\pi_1)=(\langle\pi_1, \pi_1'\rangle\times X_1)\langle(\lambda_1\lambda_2)^{\beta_1^2}\rangle$. If $y\in\langle\pi_1, \pi_1'\rangle\times X_1$, $y(\lambda_1\lambda_2)^{\beta_1}$ does not centralize α_k. In fact, if $[\alpha_k, y(\lambda_1\lambda_2)^{\beta_1}]=1$, we should have $1=[\alpha_k, (\lambda_1\lambda_2)^{\beta_1}][\alpha_k, y]^{(\lambda_1\lambda_2)^{\beta_1}}$ which yields $\pi_1'=[\alpha_k, y]$. This is impossible since $[\alpha_k, y]\in\langle\pi_1\rangle\times X_1$. Therefore we get $C_G(\pi_1')\cap C_G(\alpha_k)=\langle\pi_1, \pi_1'\rangle\times C_{X_1}(\pi_2\cdots\pi_k)$. Similarly we have $C_G(\pi_1'\pi_1)\cap C_G(\alpha_k)=\langle\pi_1, \pi_1'\rangle\times C_{X_1}(\pi_2\cdots\pi_k)$. On the other hand, from (5.14) we see that $C_{X_1}(\pi_2\cdots\pi_k)=(U_k\times X_k)\langle\lambda_2\lambda_{k+1}\rangle$. This proves (i). Then (ii) is obvious from the definition of W_k.

(6.3) LEMMA. *Suppose that $x\in C_G(\alpha_k)$ and $\pi_1^x\in C_G(\alpha_1)\cap C_G(\alpha_k)$. Then we have $\pi_1^x\in E_k$.*

PROOF. From (5.11) and (5.14) it follows that $C_G(\alpha_1)\cap C_G(\alpha_k)=(\langle\pi_1, \pi_1'\rangle\times(U_k\times X_k)\langle\lambda_2\lambda_{k+1}\rangle)\langle\lambda_1\lambda_2\rangle$. Let $\hat{W}_k$ be a group defined in (4.4). Clearly $\hat{W}_k$ contains a 2-Sylow subgroup of $C_G(\alpha_1)\cap C_G(\alpha_k)$. Therefore we can find an element y of X_k such that $\pi_1^{xy}\in\hat{W}_k$. Then (4.4) yields $\pi_1^{xy}\in E_k$. Since $[y, E_k]=1$, we get $\pi_1^x\in E_k$.

(6.4) LEMMA. *If $x\in C_G(\alpha_2)$, we have $\pi_1^x\in E_2$.*

PROOF. We have $\pi_1^x\not\sim\pi_1'$ in $C_G(\alpha_2)$ for any $x\in C_G(\alpha_2)$. In fact, if $\pi_1^{xy}=\pi_1'$ for some $y\in C_G(\alpha_2)$, we must have $\pi_2^{xy}=\alpha_2\pi_1^{xy}=\pi_1\pi_2\pi_1'$ which is impossible by (*A*). Put $\mathfrak{D}_x=\langle\pi_1^x, \pi_1'\rangle$. As is well known [2; Chap. 9, p. 301], $\mathfrak{D}_x$ is a dihedral group with the non trivial center of order 2. Denote by z_x the nonidentity element of $Z(\mathfrak{D}_x)$. Then we have $\pi_1^x\sim\pi_1'z_x$ or $\pi_1'\sim\pi_1'z_x$ in $\mathfrak{D}_x\subseteqq C_G(\alpha_2)$.

Case (i). Suppose that $\pi_1^x\sim\pi_1'z_x$ in $\mathfrak{D}_x$. Then we have $\pi_2^x\sim\pi_1'\pi_1\pi_2z_x$ in

$C_G(\alpha_2)$ by multiplying both sides of $\pi_1^x \sim \pi_1' z_x$ by α_2. Since $z_x \in C_G(\pi_1') \cap C_G(\alpha_2)$, we can find an element $u_x \in X_2$ such that $z_x^{u_x} \in \langle \pi_1, \pi_1' \rangle \times W_2$ by (6.2; (ii)). Further we have $\pi_1^x \sim \pi_1' z_x^{u_x}$ and $\pi_2^x \sim \pi_1' \alpha_2 z_x^{u_x}$ because $[u_x, \pi_1'] = [u_x, \alpha_2] = 1$. Then (4.2) yields that $z_x^{u_x y} = \pi_1' \pi_1$ or $\pi_1' \pi_2$ for some $y \in U_k$, and so $z_x = \pi_1' \pi_1$ or $\pi_1' \pi_2$ by $[u_x y, \pi_1' \pi_1] = [u_x y, \pi_1' \pi_2] = 1$. In any cases, we get $\pi_1^x \in C_G(\pi_1' \pi_1) \cap C_G(\alpha_2) = \langle \pi_1, \pi_1' \rangle \times (\text{something})$ by (6.2; (i)) and then $[\pi_1^x, \pi_1'] = 1$. This implies that $\pi_1^x = \pi_1$ or π_2. Thus we get $\pi_1^x \in E_2$.

Case (ii). Suppose that $\pi_1' \sim \pi_1' z_x$. Then we have $\pi_1' \alpha_2 \sim \pi_1' \alpha_2 z_x$ in $C_G(\alpha_2)$. Since $z_x \in C_G(\pi_1') \cap C_G(\alpha_2)$, we can find an element $u_x \in X_2$ such that $z_x^{u_x} \in \langle \pi_1, \pi_1' \rangle \times W_2$ by (6.2). Then we have $\pi_1' \sim \pi_1' z_x^{u_x}$ and $\pi_1' \alpha_2 \sim \pi_1' \alpha_2 z_x^{u_x}$. (4.3) yields that, if $z_x \neq \pi_1$, we have $z_x \sim \alpha_2$ in G but $z_x \neq \alpha_2$. On the other hand, we have $\pi_1^{x y_x} = \pi_1^x z_x$ for some $y_x \in \mathfrak{D}_x \subseteqq C_G(\alpha_2)$. Since $[\pi_1^{x y_x}, \pi_1^x] = [\pi_1^{x y_x}, \pi_1 \pi_2] = 1$ and $[\pi_1 \pi_2, x] = 1$, we have $\pi_1^{x y_x} \in C_G(\pi_1^x) \cap C_G(\pi_2^x)$ and so $\pi_1^{x y_x x^{-1}} \in C_G(\pi_1) \cap C_G(\pi_2)$. From (6.3) we get $E_2 \ni \pi_1^{x y_x x^{-1}} = \pi_1 z_x^{x^{-1}}$. Since $\pi_1^{x y_x x^{-1}} = \pi_1, \pi_2, \lambda_1 \lambda_2, \lambda_1 \pi_1 \lambda_2, \lambda_1 \lambda_2 \pi_2$ or $\lambda_1 \pi_1 \lambda_2 \pi_2$ (which are the totality of elements of E_2 conjugate to α_1 in G) and $z_x \sim \alpha_2$ but $z_x \neq \alpha_2$, we must have $z_x \sim \pi_1$ which is a contradiction. Therefore we get $z_x = \pi_1$ which yields $\pi_1^x \in C_G(\pi_1) \cap C_G(\pi_1 \pi_2) = C_G(\pi_1) \cap C_G(\pi_2)$. Then it follows from (6.3) that $\pi_1^x \in E_2$. The proof is complete.

(6.5) LEMMA. $C_G(\alpha_2) = N_G(E_2)$.

PROOF. From (A) it follows that α_2 is the only one involution of E_2 conjugate in G to α_2. This yields $C_G(\alpha_2) \supseteqq N_G(E_2)$. On the other hand, (1.2) and (6.4) implies $E_2 \lhd C_G(\alpha_2)$. The proof is complete.

(6.5) We introduce some notations:

$$B_k = \{\pi_s, \lambda_t \lambda_u x_{tu} \mid 1 \leqq s \leqq k,\ 1 \leqq t < u \leqq k,\ x_{tu} \in \langle \pi_t, \pi_u \rangle\} ,$$

$$\hat{B}_k = \{\pi_s, \lambda_t \lambda_u x_{tu} \mid 2 \leqq s \leqq k,\ 2 \leqq t < u \leqq k,\ x_{tu} \in \langle \pi_t, \pi_u \rangle\} .$$

Then $\hat{B}_k$ is the orbit of π_2 under the action on E_k of U_k as easily seen from the structure of U_k. Further from (A) we see that B_k is the set of elements of E_k which are conjugate in G to α_1.

(6.7) LEMMA. *We have* $Z(C_G(\pi_1' x) \cap C_G(\alpha_k)) \ni \pi_1'$ *for any* $x \in \{\pi_1\} \cup \hat{B}_k$.

PROOF. If $x = \pi_1$, our assertions follow from (6.2). By (5.18) and (6.5) we know that $C_G(\alpha_2) = (T_2 \times X_2)\langle \lambda_1 \lambda_3 \rangle$. From this we easily see that $C_G(\alpha_2) \cap C_G(\pi_1' \alpha_k) = \langle \pi_1, \pi_1' \rangle \times (\langle \pi_2, \pi_2' \rangle \times C_{X_2}(\pi_3 \cdots \pi_k)) \langle \lambda_2 \lambda_3 \rangle$. Since $\alpha_2^{\beta_1} = \pi_1' \pi_2$, $(\pi_1' \alpha_k)^{\beta_1} = \alpha_k$ and β_1 normalizes $\langle \pi_1, \pi_1' \rangle$, we get $C_G(\pi_1' \pi_2) \cap C_G(\alpha_k) = \langle \pi_1, \pi_1' \rangle \times (\text{something})$. This implies $\pi_1' \in Z(C_G(\pi_1' \pi_2) \cap C_G(\alpha_k))$. Further for each $x \in \hat{B}_k$, we can find an element σ of $U_k \subseteqq C_G(\alpha_k)$ such that $\pi_2^\sigma = x$ (cf. (6.6)). This yields $\pi_1' \in Z(C_G(\pi_1' x) \cap C_G(\alpha_k))$.

(6.8) LEMMA. $C_G(\alpha_k) = N_G(E_k)$.

PROOF. We have $C_G(\alpha_k) \supseteqq N_G(E_k)$ because α_k is the only one element of E_k conjugate in G to α_k by (A). So we shall show $C_G(\alpha_k) \subseteqq N_G(E_k)$. By (1.2) and the fact that $T_k \subseteqq C_G(\alpha_k)$, it is sufficient to see that $\pi_1^x \in E_k$ for any

$x \in C_G(\alpha_k)$. We have $\pi_1^x \not\sim \pi_1'$ in $C_G(\alpha_k)$ for any $x \in C_G(\alpha_k)$. In fact, if $\pi_1^{xy} = \pi_1'$ for $y \in C_G(\alpha_k)$, we have $(\alpha_k \pi_1)^{xy} = \alpha_k \pi_1'$ which contradicts (*A*). So $\mathfrak{D}_x = \langle \pi_1^x, \pi_1' \rangle$ has the non trivial center $\langle z_x \rangle$, and $\pi_1^x \sim \pi_1' z_x$ or $\pi_1' \sim \pi_1' z_x$ in $\mathfrak{D}_x \subseteqq C_G(\alpha_k)$.

Case (i). Suppose that $\pi_1^x \sim \pi_1' z_x$ in $\mathfrak{D}_x$. Since $z_x \in C_G(\pi_1') \cap C_G(\alpha_k)$, we can find $u_x \in X_k$ such that $z_x^{u_x} \in \langle \pi_1, \pi_1' \rangle \times W_k$ by (6.1). Then we have $\pi_1^x \sim \pi_1' z_x^{u_x}$ and $(\alpha_k \pi_1)^x \sim \alpha_k \pi_1' z_x^{u_x}$. By (4.2) we get $z_x^{u_x y} = \pi_1' \pi_1$ or $\pi_1' \pi_2$ for some $y \in U_k$. From (6.6) it follows that $z_x = \pi_1' v$ where $v \in \{\pi_1\} \cup \hat{B}_k$. Then (6.7) yields that $[\pi_1^x, \pi_1'] = 1$ and then $z_x = \pi_1^x \pi_1'$. Hence we get $\pi_1^x \in E_k$.

Case (ii). Suppose that $\pi_1' \sim \pi_1' z_x$ in $\mathfrak{D}_x$. Since $z_x \in C_G(\pi_1') \cap C_G(\alpha_k)$, there is $u_x \in X_k$ such that $z_x^{u_x} \in \langle \pi_1, \pi_1' \rangle \times W_k$ by (6.1). From (4.3) we get $z_x^{u_x y} = \pi_1$, $\pi_1' \pi_2'$, $\pi_1' \lambda_k \lambda_n$ or $\pi_1' \lambda_k$ for some $y \in W_k$. We shall show $z_x = \pi_1$. Assume by way of contradiction that $z_x \neq \pi_1$. Then we have $z_x^{u_x y} = \pi_1' \pi_2'$, $\pi_1' \lambda_k \lambda_n$ or $\pi_1' \lambda_k$ and so $\alpha_k z_x \sim \alpha_k$ in G since $\alpha_k \pi_1' \pi_2' \sim \alpha_k \pi_1' \lambda_k \sim \alpha_k \pi_1' \lambda_k \lambda_n \sim \alpha_k$ in G by (*A*). On the other hand, we have $\pi_1^{x y_x} = \pi_1^x z_x$ for some $y_x \in \mathfrak{D}_x$ and so $\pi_1^{x y_x} \in C_G(\pi_1^x) \cap C_G(\alpha_k)$. Since $\pi_1^{x y_x x^{-1}} \in C_G(\pi_1) \cap C_G(\alpha_k)$ and $x y_x x^{-1} \in C_G(\alpha_k)$, (4.4) implies $\pi_1^{x y_x x^{-1}} \in E_k$. Then (6.6) yields $\pi_1^{x y_x} = v^x$ where $v \in B_k$ and so $z_x = (\pi_1 v)^x$. If $v = \lambda_1 \lambda_j x_{1j}$ ($x_{1j} \in \langle \pi_1, \pi_j \rangle$), we have $z_x = (\pi_1 \lambda_1 \lambda_j x_{1j})^x \sim \alpha_1$ in G by (*A*) which yields $z_x = \pi_1$, a contradiction. If $v = \pi_s$ or $\lambda_i \lambda_j x_{ij}$ ($2 \leqq s \leqq k$, $2 \leqq i < j \leqq k$), we get $\alpha_k z_x \sim \alpha_{k-2}$ or α_{k-1} in G which contradicts the fact that $\alpha_k z_x \sim \alpha_k$ in G. Thus we have proved that $z_x = \pi_1$. Hence we get $\pi_1^x \in C_G(\pi_1) \cap C_G(\alpha_k)$. Then (6.3) yields $\pi_1^x \in E_k$. This completes the proof of our lemma.

§ 7. The final step of the proof of Theorems I, II.

(7.1) The following lemma is almost obvious.

LEMMA. *For $k=1$ or 2, let $H_i^{(k)}$ ($i = 1, 2, \cdots, n$) be subgroups of a group $G^{(k)}$ with a set $\mathcal{M}_i^{(k)}$ of generators which have the following properties:*

(1) *$\mathcal{M}_i^{(k)} \cap \mathcal{M}_j^{(k)}$ is a set of generators of $H_i^{(k)} \cap H_j^{(k)}$ for $1 \leqq i, j \leqq n$,*

(2) *there exists a one-to-one mapping ϕ from the subset $\bigcup_{i=1}^{n} \mathcal{M}_i^{(1)}$ of $G^{(1)}$ onto $\bigcup_{i=1}^{n} \mathcal{M}_i^{(2)}$ such that for each i, $\phi(\mathcal{M}_i^{(1)}) = \mathcal{M}_i^{(2)}$ and $\phi_i = \phi | \mathcal{M}_i^{(1)}$ can be extended to an isomorphism from $H_i^{(1)}$ onto $H_i^{(2)}$. (Of course, the extension is unique.)*

Then there exists a one to one mapping φ from the subset $\bigcup_{i=1}^{l} H_i^{(1)}$ of $G^{(1)}$ onto $\bigcup_{i=1}^{l} H_i^{(2)}$ of $G^{(2)}$ such that the restrictions of φ to $H_i^{(1)}$ induces an isomorphism from $H_i^{(1)}$ onto $H_i^{(2)}$.

PROOF. Let ϕ_i ($i = 1, 2, \cdots, n$) be an isomorphism from $H_i^{(1)}$ onto $H_i^{(2)}$ obtained by the condition (2). Then (1) yields that $\phi_i | H_i^{(1)} \cap H_j^{(1)} = \phi_j | H_i^{(1)} \cap H_j^{(1)}$. This implies the existence of φ with the required property.

(7.2) Define the subset $\mathcal{M}$ of $G = G(n, r)$ as follows:

$$\mathcal{M}=\begin{cases}\{\pi_s, \pi'_s, \lambda_t\lambda_{t+1}, \beta_u, \delta_v^{(1)}, \sigma'_1 \mid 1\leqq s\leqq n, 1\leqq t\leqq n-1, 2\leqq u\leqq n, 2\leqq v\leqq n-1\} & \text{if } r=0, \\ \{\pi_s, \pi'_s, \lambda_t\lambda_{t+1}, \beta_u, \delta_v^{(1)}, \sigma'_1 \mid 1\leqq s\leqq n, 1\leqq t\leqq n-1, 2\leqq u, v\leqq n\} & \text{if } r=1, \\ \{\pi_s, \pi'_s, \lambda_t, \beta_u, \delta_v^{(1)}, \sigma'_1 \mid 1\leqq s, t\leqq n, 2\leqq u, v\leqq n\} & \text{if } r=2, \\ \{\pi_s, \pi'_s, \lambda_t, \beta_u, \delta_v^{(1)}, \sigma'_1, \nu \mid 1\leqq s, t\leqq n, 2\leqq u, v\leqq n\} & \text{if } r=3.\end{cases}$$

Let $\tilde{\mathcal{M}}$ be the subset of $\mathfrak{A}_m$ consisting of the corresponding elements with the tilde sign (cf. (2.1) and (5.14)). Further put

$$\mathcal{M}_1=\mathcal{M}-\{\sigma'_1\},\ \mathcal{M}_k=\mathcal{M}-\{\beta_u\ (2\leqq u\leqq k),\ \delta_k^{(1)}\}\ (2\leqq k\leqq n).$$

Again let $\tilde{\mathcal{M}}_k$ $(1\leqq k\leqq n)$ be the set of the corresponding elements of $\mathfrak{A}_m$. Then we have $\mathcal{M}=\bigcup_{i=1}^{n}\mathcal{M}_i$ and $\tilde{\mathcal{M}}=\bigcup_{i=1}^{n}\tilde{\mathcal{M}}_i$. From (2.1), (5.14), (5.18) and (6.8) we see that $\mathcal{M}_k$ (resp. $\tilde{\mathcal{M}}_k$) is a set of generators of $C_G(\alpha_k)$ (resp. $C_{\mathfrak{A}_m}(\tilde{\alpha}_k)$). Clearly the present situation is sufficient to apply the above lemma (7.1), which yields that there is a one-to-one mapping φ from $\bigcup_{k=1}^{n}C_{\mathfrak{A}_m}(\tilde{\alpha}_k)$ onto $\bigcup_{k=1}^{n}C_G(\alpha_k)$ such that φ induces an isomorphism from $C_{\mathfrak{A}_m}(\tilde{\alpha}_k)$ onto $C_G(\alpha_k)$ for each k. Then a theorem of [8] implies that G is isomorphic to $\mathfrak{A}_m$ $(m=4n+r)$. This completes the proof of our Theorems I, II.

Appendix. Abelian 2-subgroups of the symmetric groups.

Let m be a positive and even integer. Put $m=4n+r$. So we have $r=0$ or 2. Let $\mathfrak{S}_m$ be the symmetric group on the set $\{1, 2, \cdots, m\}$. Each abelian 2-subgroup of $\mathfrak{S}_m$ is at most of order $2^{\frac{m}{2}}$ (cf. the proof of [8; (1.4)]). Denote by $\mathcal{A}$ the set of abelian 2-subgroups of $\mathfrak{S}_m$ of order $2^{\frac{m}{2}}$.

LEMMA 1. *Let L be an abelian group contained in $\mathcal{A}$. Then L has a basis*

$$\{u_1, u_2, \cdots, u_a, v_1, v'_1, \cdots, v_b, v'_b, w_1, w_2, \cdots, w_c\}$$

with the following properties:

(1) *the u_i, v_j and v'_j are involutions and the w_k are of order* 4,

(2) *the u_i are transpositions, the v_i and v'_i are products of two transpositions, and the w_k are cycles of length* 4,

(3) *any two of the u_i, v_j and w_k do not displace a common letter, and*

(4) *for every j, the letters which v_j and v'_j displace are the same.*

PROOF. This proceeds by induction on m. If $m\leqq 4$, our assertion can be checked easily. So we assume that $m\geqq 6$. If every involution of L has no fixed points, L is semi-regular and so $|L|\leqq m<2^{\frac{m}{2}}$, which is impossible. Therefore we can find an involution x of $\mathfrak{S}_m$ which has a fixed point. Then we have $C_{\mathfrak{S}_m}(x)=U\times V$, where, by taking a suitable conjugate in $\mathfrak{S}_m$ of x, U is a subgroup of the symmetric group on the set $\{1, 2, \cdots, 2k\}$ and V is the symmetric group on the set $\{2k+1, 2k+2, \cdots, m\}$.

We note that U contains a 2-Sylow subgroup of the symmetric group on the set $\{1, 2, \cdots, 2k\}$. Put $L_1=U\cap L$ and $L_2=V\cap L$. Then from the maximality of L we easily see that $L=L_1\times L_2$. The inductive hypothesis yields our lemma, q. e. d.

We call *a canonical basis* of L a basis as in Lemma 1. Define elements of $\mathfrak{S}_m$ as follows:

$$\pi_i=(4i-3,\ 4i-2)(4i-1,\ 4i),$$

$$\pi_i'=(4i-3,\ 4i-1)(4i-2,\ 4i), \qquad (1\leqq i\leqq n)$$

$$\mu_i=(4i-3,\ 4i-2),$$

$$\mu_{n+1}=\begin{cases}1 & \text{if } r=0\\ (4n+1,\ 4n+2) & \text{if } r=2,\end{cases}$$

$$\sigma_j=(4j-3,\ 4j+1)(4j-2,\ 4j+2)(4j-1,\ 4j+3)(4j,\ 4j+4) \qquad (1\leqq j\leqq n-1).$$

Put

$$L_{a,b}=\langle\mu_1, \mu_1\pi_1, \cdots, \mu_a, \mu_a\pi_a, \pi_{a+1}, \pi_{a+1}', \cdots, \pi_{a+b}, \pi_{a+b}', \mu_{a+b+1}\pi_{a+b+1}', \cdots, \mu_n\pi_n', \mu_{n+1}\rangle \qquad (0<a+b\leqq n),$$

$$P_m=\langle\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\rangle,$$

$$\alpha_n=\pi_1\pi_2\cdots\pi_n,$$

$$H_m=C_{\mathfrak{S}_m}(\pi_1\pi_2\cdots\pi_n),$$

$$J_m=\langle\pi_i, \pi_i', \mu_j \mid 1\leqq i\leqq n, 1\leqq j\leqq n+1\rangle.$$

We note that $L_{a,b}\in\mathcal{A}$.

LEMMA 2. *If $L\in\mathcal{A}$, L is conjugate in $\mathfrak{A}_m$ to one of the $L_{a,b}$.*

PROOF. This is obvious from Lemma 1, q. e. d.

REMARK. It is easy to see that we can choose an element x of $\mathfrak{A}_m$ such that $L^x=L_{a,b}$.

LEMMA 3. *Put $L_0=L_{0,n}$. Then we have $N_{H_m}(L_0)=J_m\cdot P_m$.*

PROOF. Put $\Pi_k=\{4k-3, 4k-2, 4k-1, 4k\}$ $(1\leqq k\leqq n)$. $N_{H_m}(L_0)$ operates transitively on the set $\{\Pi_1, \Pi_2, \cdots, \Pi_n\}$ of n elements and J_m is the kernel of this permutation representation of $N_{H_m}(L_0)$. From the fact that $J_m\cap P_m=1$ and $P_m\cong\mathfrak{S}_n$, our lemma follows.

LEMMA 4. *If $L\in\mathcal{A}$ and $L\in N_{H_m}(L_0)$, L is contained in J_m.*

PROOF. Take a canonical basis of L. Let x be a member of such basis of L. Since x satisfies the conditions of Lemma 1, we see that $\Pi_k^x=\Pi_k$ for each k. From the proof of Lemma 3, we get $x \in J_m$. This proves our lemma.

PROPOSITION 5. *Let $H(n, 1)$ and $H(n, 2)$ be as in §2.* (i) *$H(n, 1)$ has a unique abelian group of order 2^{2n} up to conjugacy in $H(n, 1)$, which is normal in a 2-Sylow subgroup of $H(n, 1)$ containing it.* (ii) *if S and M are subgroups of $H(n, 2)$ defined in (2.2), $J = S \cdot M$ is the Thompson subgroup of a 2-Sylow subgroup of $H(n, 2)$ containing it.*

PROOF. (i) $C_{\mathfrak{A}_{4n}}(\alpha_n)$ is isomorphic to $H(n, 1)$. Since $J_m \cap \mathfrak{A}_{4n} = L_0\langle \mu_1\mu_2, \mu_2\mu_3, \cdots, \mu_{n-1}\mu_n \rangle$. Our assertion follows from Lemma 4. (Note that $N_{\mathfrak{A}_{4n}}(L_0)$ contains a 2-Sylow subgroup of $\mathfrak{A}_{4n}$.) (ii) $C_{\mathfrak{A}_{4n+2}}(\alpha_n)$ is isomorphic to $H(n, 2)$ and J_m corresponds to J. Then our lemma follows from Lemma 4.

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References

[1] L. E. Dickson, Linear groups with an exposition of the Galois field theory, Dover, 1958.
[2] D. Gorenstein, Finite Groups, Haper & Row, 1967.
[3] K. Harada, Finite simple groups with short chains of subgroups, J. Math. Soc. Japan, **20** (1968), 655–672.
[4] D. Held, A characterization of the alternating groups of degree eight and nine, J. Algebra, **7** (1967), 218–237.
[5] D. Held, A characterization of some multiply transitive permutation groups I, Illinois J. Math., (to appear).
[6] B. Huppert, Endlichen Gruppen I, Springer, Berlin, 1967.
[7] T. Kondo, On a characterization of the alternating groups of degree eleven, Illinois J. Math., (to appear).
[8] T. Kondo, On the alternating groups, J. Fac. Sci., Univ. Tokyo, **15** (1968), 87–97.
[9] T. Kondo, On finite groups with a 2-Sylow subgroup isomorphic to that of the symmetric group of degree $4n$, J. Math. Soc. Japan, **20** (1968), 695–713.
[10] M. Suzuki, On finite groups containing an element of order four which commutes only with its powers, Illinois J. Math., **3** (1959), 255–271.
[11] W. Specht, Eine Verallgemeinerung der symmetrischen Gruppe, Schr. Math. Sem. Berlin, **1** (1932), 1–32.
[12] H. Yamaki, A characterization of the alternating groups of degrees 12, 13, 14, 15, J. Math. Soc. Japan, **20** (1968), 673–694.

Added in Proof. Recently the author has also proved Theorem I for the case $r=1$. Namely, if $n \geqq 4$ and $G(n, 1)$ is a finite group satisfying the conditions of Theorem I for $r=1$, $G(n, 1)$ is isomorphic to $\mathfrak{A}_{4n}$ or $\mathfrak{A}_{4n+1}$. This will be published elsewhere.