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On restricted roots of semi-simple algebraic groups*)

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§ 0. Introduction and notation.

In this paper, we are interested in re-examining, under more general assumptions, some of the recent work of Satake, Tits and Borel concerning restricted roots of semi-simple algebraic groups, and the Weyl group associated to these roots ([1], [4]). Their work concentrates on the study of the system of k-roots of a connected semi-simple (or reductive) algebraic group G defined over a ground field k, and hence the Galois group $G(K/k)$, where K is a splitting field for a maximal torus of G defined over k, plays an important role. The initial question which led to this paper was "what is the importance of the maximal k-trivial torus and the Galois group in this study?" That is, are there more general assumptions on a subtorus of G under which much of the theory holds true, and can the Galois group be replaced by a more general automorphism group of the root system of G?

We will show that both of these questions have affirmative answers, and obtain necessary and sufficient conditions for a large class of tori (called *admissible* tori) to induce sets of restricted roots which possess many of the properties of k-roots. Since maximal k-trivial tori are a special case of all the admissible tori we consider, many of our theorems yield properties of maximal k-trivial tori. Only a few of these properties are not proved in [1], [4]; however, it is hoped that our method of proof indicates that many of these properties are equivalent, and depend on a minimum set of assumptions.

Throughout the paper, we will use the following standard notation (patterned after that in [4]).

G: a connected reductive algebraic group, (assumed semi-simple in § 2-§ 5)

T: a fixed maximal torus of G

$X=X(T)$: the group of rational characters of T

$\mathfrak{r}$: the root system of G with respect to T

W: the Weyl group of $\mathfrak{r}$

w_α: the element of W which is the reflection with respect to $\alpha \in \mathfrak{r}$.

We will denote by G_a and G_m the one-dimensional additive and multiplicative

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algebraic groups, respectively, of the universal domain, and denote by $\boldsymbol{Z}$, $\boldsymbol{Z}_+$, $\boldsymbol{Q}$, the integers, non-negative integers, and rationals, respectively. Finally, for subsets M and N of $\mathfrak{r}$, we denote by $M-N$ the set-theoretic complement of N in M.

In the first section, we restate, in more general terms, the definitions and some of the results on the set of restricted roots $\bar{\mathfrak{r}}$ and group $\overline{W}$ which are given in [4]. In § 2, we define an admissible torus of G, and for such a torus, obtain necessary and sufficient conditions for the group $\overline{W}$ to be generated by a set of reflections $\{r_{\bar{\gamma}}, \gamma \in \bar{\mathfrak{r}}\}$. These conditions are equivalent to the fact that the set of "reduced" restricted roots is a root system, having Weyl group $\overline{W}$. They also imply a structure theorem for $\overline{W}$. The "opposition automorphism" of segments of the Dynkin diagram of G is of key importance. In § 3, a special class of admissible tori which are an obvious generalization of maximal k-trivial tori is studied. These are the maximal subtori of T which are pointwise fixed under the action of a subgroup Γ of $\mathrm{Aut}(G, T)$ (the group of rational automorphisms of G leaving T invariant). In § 4, two actions of Γ on W are defined, and we show that $\overline{W}$ is isomorphic to a subgroup of W which is pointwise fixed by Γ. Finally, in § 5, we mention some applications of our results to the special case of a maximal k-trivial torus of G, where G is defined over a field k.

§ 1. Restricted roots and the group $\overline{W}$.

Most proofs are omitted in this section since Satake's arguments in [4] can be used in the more general setting, almost without change. For complete proofs and a more detailed discussion of the objects defined in this section, see [6], [5].

Throughout the section, S is a fixed subtorus of T, and we denote by X_0 the annihilator of S in X. It is well-known that X_0 is a co-torsion free submodule of X, and that X/X_0 is isomorphic to the group of rational characters of S, which we denote by Y. We will identify X/X_0 and Y, and denote by π the canonical homomorphism of X onto Y; that is, for each $\chi \in X$, $\pi(\chi)$ is the restriction of χ to S. Let $\mathfrak{r}_0 = \mathfrak{r} \cap X_0$, and put $\bar{\mathfrak{r}} = \pi(\mathfrak{r} - \mathfrak{r}_0)$. The subset $\bar{\mathfrak{r}}$ of Y will be called the set of *restricted roots of* $\mathfrak{r}$ *relative to* X_0 (or relative to S).

In order to talk of fundamental roots of $\bar{\mathfrak{r}}$, we need to define a linear order on X which is compatible with π. Thus we say that a linear order $>$ on X (which is compatible with addition) is an X_0-*linear order* if and only if the following condition is satisfied:

(1) if $\chi, \chi' \in X$, $\chi \notin X_0$, $\chi > 0$, and $\chi \equiv \chi' \pmod{X_0}$, then $\chi' > 0$.

From the definition, it is clear that an X_0-linear order on X induces linear orders on X_0 and Y, and conversely, it is easily shown that given linear orders on X_0 and Y, there is a unique X_0-linear order on X which induces these given orders (the order on Y satisfies the condition: for $\chi \notin X_0$, $\pi(\chi)>0$ if and only if $\chi>0$). An alternate characterization of an X_0-linear order is given in the following lemma.

LEMMA 1.1. *A linear order* $>$ *(compatible with addition) on* X *is an* X_0-*linear order if and only if the following condition is satisfied:*

$$\text{(2)} \qquad \textit{if } \chi_1, \chi_2 \notin X_0 \textit{ and } \chi_1, \chi_2>0, \textit{ then } \chi_1+\chi_2 \notin X_0 .$$

PROOF. Suppose a linear order $>$ on X satisfies (1), and suppose $\chi_1, \chi_2 \notin X_0$, and $\chi_1, \chi_2>0$. If $\chi_1+\chi_2 \in X_0$, then $\chi_1 \equiv -\chi_2 \pmod{X_0}$, which contradicts (1); thus (1)$\Rightarrow$(2). Conversely, suppose $>$ satisfies (2), and suppose $\chi_1, \chi_2 \in X$, $\chi_1 \notin X_0$, $\chi_1>0$, and $\chi_1 \equiv \chi_2 \pmod{X_0}$. Clearly $\chi_2 \notin X_0$ and $\chi_2 \neq 0$; if $\chi_2<0$, then $-\chi_2>0$ and $-\chi_2 \notin X_0$, and $\chi_1-\chi_2 \in X_0$, which contradicts (2). Thus $\chi_2>0$, and (2)$\Rightarrow$(1).

We will call the set of simple roots of $\mathfrak{r}$ with respect to an X_0-linear order on X an X_0-*fundamental system of* $\mathfrak{r}$. If Δ is any X_0-fundamental system of $\mathfrak{r}$, and we put $\Delta_0=\Delta \cap X_0$, then we call the set $\bar{\Delta}=\pi(\Delta-\Delta_0)$ a *restricted fundamental system of* $\mathfrak{r}$ (corresponding to Δ). The next proposition follows easily from Lemma 1.1 and the definitions of $\mathfrak{r}_0$, Δ_0, $\bar{\mathfrak{r}}$ and $\bar{\Delta}$.

PROPOSITION 1.2. *Let* Δ *be an* X_0-*fundamental system of* $\mathfrak{r}$.

(a) $\mathfrak{r}_0$ *is a root system with fundamental system* Δ_0.

(b) *If* $\bar{\Delta}=\{\gamma_1, \cdots, \gamma_\nu\}$, *the* γ_i *assumed mutually distinct, then every* $\gamma \in \bar{\mathfrak{r}}$ *can be written in the form*

$$\gamma=\pm \sum_{i=1}^{\nu} m_i \gamma_i, \quad m_i \in \boldsymbol{Z}_+ .$$

(c) *If* Δ' *is another* X_0-*fundamental system of* $\mathfrak{r}$, *and* $\Delta_0'=\Delta' \cap X_0$, $\bar{\Delta}'=\pi(\Delta'-\Delta_0')$, *then* $\Delta=\Delta'$ *if and only if* $\Delta_0=\Delta_0'$ *and* $\bar{\Delta}=\bar{\Delta}'$.

Let W_0 denote the subgroup of W generated by $\{w_\alpha, \alpha \in \mathfrak{r}_0\}$; then W_0 can be identified with the Weyl group of $\mathfrak{r}_0$. Define

$$\text{(3)} \qquad W_0'=\{w \in W \mid w(X_0)=X_0\} .$$

Clearly W_0' is a subgroup of W; in addition, W_0 is a normal subgroup of W_0', for if $\alpha \in \mathfrak{r}_0$, and $w \in W_0'$, then $w w_\alpha w^{-1}=w_{w\alpha}$, with $w\alpha \in \mathfrak{r}_0$. Each $w \in W_0'$ induces an automorphism $\bar{w}$ of Y which is defined by the following equation:

$$\text{(4)} \qquad \pi(w\chi)=\bar{w}(\pi(\chi)), \qquad \text{for all } \chi \in X .$$

We denote by $\overline{W}$ the group $\{\bar{w} \mid w \in W_0'\}$; it is clear from (4) that $\overline{W}$ leaves $\bar{\mathfrak{r}}$ invariant. Also, if $w \in W_0$, then $w\chi-\chi \in (\mathfrak{r}_0)_{\boldsymbol{Z}}$, for all $\chi \in X$, so (4) implies $\bar{w}=1$.

PROPOSITION 1.3. *Let Δ be an X_0-fundamental system of* $\mathfrak{r}$. *For any $w \in W'_0$, $w(\Delta)$ is an X_0-fundamental system of* $\mathfrak{r}$, *and $\bar{w}(\bar{\Delta})$ is the corresponding restricted fundamental system. One has $\bar{w}(\bar{\Delta}) = \bar{\Delta}$ if and only if $w \in W_0$.*

COROLLARY 1.4. $\overline{W} \cong W'_0/W_0$. *(Specifically, the homomorphism of W'_0 onto $\overline{W}$ given by $w \to \bar{w}$ has kernel W_0.)*

For any subset M of G, we denote by $N(M)$ and $Z(M)$ the normalizer and centralizer, respectively, of M in G.

PROPOSITION 1.5.

(a) $N(S) = (N(S) \cap N(T)) \cdot Z(S)$

(b) *If w_s is the element of W determined by $s \in N(T)$, then*

(i) *$w_s \in W'_0$ if and only if $s \in N(S)$*

(ii) *$w_s \in W_0$ if and only if $s \in Z(S)$.*

Using the second isomorphism theorem, Corollary 1.4, and Proposition 1.5, one obtains

COROLLARY 1.6. $\overline{W} \cong N(S)/Z(S)$.

The canonical homomorphism of $N(S)$ into $\mathrm{Aut}(Y)$ having kernel $Z(S)$ is $\varphi: s \to {}^t(I_s|S)^{-1}$, where $s \in N(S)$ and I_s is the inner automorphism of G defined by s. Thus it follows from (4) that for $s \in N(S) \cap N(T)$, one has $\bar{w}_s = \varphi(s)$. By Proposition 1.5(a), for each $s \in N(S)$, there is an element $s' \in N(S) \cap N(T)$ such that $\varphi(s) = \varphi(s') = \bar{w}_{s'}$. To simplify our notation, we make the following convention: for any $s \in N(S)$ (not necessarily belonging to $N(S) \cap N(T)$, we put $\bar{w}_s = \varphi(s)$.

Let $Y^*_{\boldsymbol{Q}}$ denote the dual space of $Y_{\boldsymbol{Q}}$, and for each $\eta \in Y$, define

$$H_\eta = \{\omega^* \in Y^* \mid \omega^*(\eta) = 0\}\,. \tag{5}$$

Thus H_η is the hyperplane in $Y^*_{\boldsymbol{Q}}$ defined by η. The elements $\bar{w} \in \overline{W}$ are extended to linear transformations of $Y_{\boldsymbol{Q}}$ in a natural manner, and then $\overline{W}$ becomes a group of linear transformations in $Y^*_{\boldsymbol{Q}}$ in defining $\bar{w}\omega^*$ for $\bar{w} \in \overline{W}$, $\omega^* \in Y^*_{\boldsymbol{Q}}$ by the following equation:

$$\bar{w}\omega^*(\bar{w}\eta) = \omega^*(\eta) \qquad \text{for all } \eta \in Y_{\boldsymbol{Q}}\,. \tag{6}$$

PROPOSITION 1.7. *Let $\gamma \in \bar{\mathfrak{r}}$, and let S_γ be the identity component of the annihilator of γ in S. For each $s \in N(S)$, one has $s \in Z(S_\gamma)$ if and only if $\bar{w}_s$ leaves H_γ elementwise fixed.*

For each $\eta \in Y$, we will denote by r_η the reflection in $Y^*_{\boldsymbol{Q}}$ with respect to η. Thus r_η is a linear transformation in $Y^*_{\boldsymbol{Q}}$ which is characterized by the properties: $r_\eta \neq 1$, $r_\eta^2 = 1$, and r_η leaves pointwise fixed the hyperplane H_η.

PROPOSITION 1.8. *Let $s \in N(S)$. The element $\bar{w}_s \in \overline{W}$ is the reflection in $Y^*_{\boldsymbol{Q}}$ with respect to $\gamma \in \bar{\mathfrak{r}}$ if and only if $s \in Z(S_\gamma)$, $s \notin Z(S)$.*

PROOF. By Propositions 1.5 and 1.7, we see that $\bar{w}_s \neq 1$ and $\bar{w}_s$ leaves H_γ elementwise fixed if and only if $s \notin Z(S)$ and $s \in Z(S_\gamma)$. Moreover, if that is so, one has clearly $\bar{w}_s^2=1$, since $\bar{w}_s$ is of finite order.

REMARK. Although at the beginning of this paper we fixed T (and hence X and $\mathfrak{r}$), the definition of restricted roots $\bar{\mathfrak{r}}$ with respect to S depends only on S. That is, if T' is another maximal torus of G containing S, and we define the corresponding objects X', X'_0, $\mathfrak{r}'$, $\mathfrak{r}'_0$, Y', $\bar{\mathfrak{r}}'$, then the fact that T and T' are conjugate by an element of $Z(S)$ implies that $\bar{\mathfrak{r}}=\bar{\mathfrak{r}}'$ in the identification of X/X_0 and X'/X'_0 induced by this conjugation.

§2. Admissible tori.

We now assume that G is a connected *semi-simple* algebraic group; all other notations remain the same.

Under the assumptions in §1, if Δ is an X_0-fundamental system of $\mathfrak{r}$, the distinct elements of $\bar{\Delta}$ are not always linearly independent over $\boldsymbol{Q}$. In fact, an easy example shows that for one X_0-fundamental system Δ of $\mathfrak{r}$, $\bar{\Delta}$ can be a linearly independent set, while for another X_0-fundamental system Δ' of $\mathfrak{r}$, $\bar{\Delta}'$ can be a linearly dependent set. Take G a simple group of type A_3, with fundamental system of roots $\Delta=\{\alpha_1, \alpha_2, \alpha_3\}$, and let S be the subtorus of T whose annihilator X_0 is generated by $\alpha_2-\alpha_3$. Clearly Δ is an X_0-fundamental system of $\mathfrak{r}$, and if we let $\pi(\alpha_1)=\gamma_1$, $\pi(\alpha_2)=\pi(\alpha_3)=\gamma_2$, then $\bar{\Delta}=\{\gamma_1, \gamma_2\}$ is a linearly independent set over $\boldsymbol{Q}$. However, $\Delta'=\{-\alpha_1, \alpha_1+\alpha_2, \alpha_3\}$ is also an X_0-fundamental system of $\mathfrak{r}$, and $\bar{\Delta}'=\{-\gamma_1, \gamma_1+\gamma_2, \gamma_2\}$ is a linearly dependent set over $\boldsymbol{Q}$. (One can verify directly that Δ' is an X_0-fundamental system, or see the remark after Lemma 2.2, later in this section.)

We are only interested in studying the case where S is a subtorus of T such that every restricted fundamental system of $\mathfrak{r}$ with respect to S consists of linearly independent elements.

DEFINITION. A subtorus S of T whose annihilator in X is X_0 is called *admissible* if, for each X_0-fundamental system Δ of $\mathfrak{r}$, the distinct elements of $\bar{\Delta}$ are linearly independent over $\boldsymbol{Q}$.

If S is an admissible subtorus of T, then part (b) of Proposition 1.2 can be strengthened:

PROPOSITION 1.2 (b′). *If $\bar{\Delta}=\{\gamma_1, \cdots, \gamma_\nu\}$, the γ_i assumed mutually distinct, then each $\gamma\in\bar{\mathfrak{r}}$ can be written uniquely in the form*

$$\gamma=\pm\sum_{i=1}^{\nu} m_i\gamma_i, \quad m_i\in \boldsymbol{Z}_+ .$$

An alternate criterion for a subtorus of T to be admissible is given in the next proposition.

PROPOSITION 2.1. *A subtorus S of T is admissible if and only if for each X_0-fundamental system Δ of $\mathfrak{r}$, the module X_0 is generated over $\boldsymbol{Q}$ by Δ_0 and* *elements of the form* $\alpha-\alpha'$, *where* $\alpha, \alpha' \in \Delta-\Delta_0$, *and* $\alpha \equiv \alpha' \pmod{X_0}$.

PROOF. Let G have rank l (i.e., $\dim X_{\boldsymbol{Q}}=l$), and let $\bar{\Delta}=\{\gamma_1, \cdots, \gamma_\nu\}$, the elements assumed mutually distinct. Clearly $\dim Y_{\boldsymbol{Q}} \leqq \nu$. By reordering subscripts if necessary, we may assume that $\alpha_1, \cdots, \alpha_\nu \in \Delta-\Delta_0$ satisfy $\pi(\alpha_i)=\gamma_i$, $1 \leqq i \leqq \nu$. Then the elements of Δ_0, together with the non-zero elements of the form $\alpha_i-\alpha$, where $\alpha \in \Delta$ and $\alpha \equiv \alpha_i \pmod{X_0}$ are all in X_0, and are linearly independent over $\boldsymbol{Q}$. Thus $\dim X_{0\boldsymbol{Q}} \geqq l_0+(l-l_0-\nu)=l-\nu$, where $l_0=|\Delta_0|$, and $l-l_0-\nu$ is the number of differences $\alpha-\alpha_i$. Since $l=\dim X_{\boldsymbol{Q}}=\dim Y_{\boldsymbol{Q}}+\dim X_{0\boldsymbol{Q}}$, we see that $\dim Y_{\boldsymbol{Q}}=\nu$ (i.e., $\gamma_1, \cdots, \gamma_\nu$ are linearly independent) if and only if $\dim X_{0\boldsymbol{Q}}=l-\nu$ (i.e., X_0 is generated over $\boldsymbol{Q}$ by Δ_0 and the differences $\alpha-\alpha'$ with $\alpha \equiv \alpha' \pmod{X_0}$).

For the rest of this section, we will assume that S is an admissible subtorus of T.

Fix $\Delta=\{\alpha_1, \cdots, \alpha_l\}$, an X_0-fundamental system of $\mathfrak{r}$ with corresponding restricted fundamental system $\bar{\Delta}=\{\gamma_1, \cdots, \gamma_{\bar{l}}\}$ (the γ_i assumed mutually distinct). For each $i, 1 \leqq i \leqq \bar{l}$, denote $\Delta^i=\Delta \cap \pi^{-1}(\gamma_i)$; then $\Delta=\Delta^1 \cup \cdots \cup \Delta^{\bar{l}} \cup \Delta_0$, a disjoint union. Denote $\Delta_i=\Delta^i \cup \Delta_0$; then the set $\mathfrak{r}_i=\mathfrak{r} \cap (\Delta_i)_{\boldsymbol{Z}}$ is a closed subsystem of $\mathfrak{r}$, having Δ_i as fundamental system. In fact, $\mathfrak{r}_i$ is the root system of the connected reductive group $Z(S_{\gamma_i})$. For, $Z(S_{\gamma_i})$ is generated by T and the one-dimensional unipotent subgroups $P_\alpha (\alpha \in \mathfrak{r})$ which are contained in it, and $P_\alpha \subset Z(S_{\gamma_i})$ if and only if $\pi(\alpha)=c\gamma_i$ for some $c \in \boldsymbol{Q}$. (This last assertion follows from the well-known condition on roots: $tx_\alpha(\xi)t^{-1}=x_\alpha(\alpha(t)\xi)$ for all $t \in T$, $\xi \in \boldsymbol{G}_a$, where x_α is the isomorphism of $\boldsymbol{G}_a$ onto P_α.) Now the root system of $Z(S_{\gamma_i})$ is, by definition, the set $\{\alpha \in \mathfrak{r} \mid P_\alpha \subset Z(S_{\gamma_i})\}$, which we have just shown coincides with the set $\{\alpha \in \mathfrak{r} \mid \pi(\alpha)=c\gamma_i,\ c \in \boldsymbol{Q}\}$. But by Proposition 1.2(b′), this last set coincides with $\mathfrak{r}_i$.

For each $i, 1 \leqq i \leqq \bar{l}$, let W_i be the subgroup of W generated by $\{w_\alpha, \alpha \in \mathfrak{r}_i\}$; W_i can be identified with the Weyl group of $\mathfrak{r}_i$. Since $\mathfrak{r}_0 \subset \mathfrak{r}_i$, W_0 is a subgroup of W_i. It is clear from our definitions and discussion above that all of the results in §1 hold when G is replaced by $Z(S_{\gamma_i})$, $\mathfrak{r}$ by $\mathfrak{r}_i$, Δ by Δ_i, W by W_i, etc., since $Z(S_{\gamma_i})$ is a connected reductive algebraic group containing T and S, and Δ_i is an X_0-fundamental system of $\mathfrak{r}_i$.

For each $i, 1 \leqq i \leqq \bar{l}$, there is a unique involution $w_i \in W_i$ which satisfies

(7) $$w_i(\Delta_i)=-\Delta_i\,.$$

The involution w_i induces a natural automorphism ι_i of Δ_i, where ι_i is defined by the following equation:

(8) $$(\iota_i \circ w_i)(\alpha)=-\alpha \qquad \text{for all } \alpha \in \Delta_i\,.$$

The automorphism ι_i of Δ_i will be called the *opposition automorphism*[1] of Δ_i. Equation (8) is equivalent to $w_i\alpha=-\iota_i(\alpha)$ for all $\alpha\in\Delta_i$, thus $w_i(\Delta_0)=-\Delta_0$ if and only if the opposition automorphism of Δ_i leaves Δ_0 invariant.

LEMMA 2.2. *Let S be an admissible subtorus of T, and Δ an X_0-fundamental system of $\mathfrak{r}$. If the opposition automorphism of Δ_i leaves Δ_0 invariant, then $w_i(\Delta)$ is an X_0-fundamental system of $\mathfrak{r}$.*

PROOF. Since $w_i\in W_i$, it follows that $w_i\chi-\chi\in(\mathfrak{r}_i)_{\boldsymbol{Z}}$ for all $\chi\in X$ ([2]-exposé 16), and in particular, if $\alpha\in\Delta-\Delta_i$, then

$$w_i\alpha=\alpha+\sum_{\alpha_j\in\Delta_i}m_j\alpha_j,\ m_j\in\boldsymbol{Z}. \tag{9}$$

Since $w_i\alpha\in\mathfrak{r}$, equation (9) implies $m_j\geqq 0$ for all $\alpha_j\in\Delta_i$. Thus if $\alpha\in\Delta^k$, $k\neq i$, we have

$$\pi(w_i\alpha)=\gamma_k+m\gamma_i,\ m\in\boldsymbol{Z}_+. \tag{10}$$

For each $k\neq i$, $1\leqq k\leqq\bar{l}$, let $m_k=\max_{\alpha\in\Delta^k}\{m\mid\pi(w_i\alpha)=\gamma_k+m\gamma_i\}$, and let β_k be an element of Δ^k such that $\pi(w_i\beta_k)=\gamma_k+m_k\gamma_i$. Let β_i be any element of Δ^i; then $\pi(w_i\beta_i)=-\gamma_i$. Since S is admissible, the set $(\gamma_1,\cdots,\gamma_{\bar{l}})$ is a basis for $Y_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$, and hence the set $(\pi(w_i\beta_k),\ 1\leqq k\leqq\bar{l})$ is also a basis for $Y_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$. Let Y be ordered lexicographically with respect to this latter basis, and let X_0 be given a linear order such that the elements of $-\Delta_0$ are positive. Finally, denote by $>$ the unique X_0-linear order on X inducing these orders on Y and X_0, respectively, and denote by $\mathfrak{r}_+$ the positive elements of $\mathfrak{r}$ with respect to $>$. It is clear that $w_i(\Delta_0)=-\Delta_0\subset\mathfrak{r}_+$, and $w_i(\Delta^i)=-\Delta^i\subset\mathfrak{r}_+$ (since $\pi(w_i(\Delta^i))=-\gamma_i$). If $k\neq i$, and $\alpha\in\Delta^k$, then (10) implies $\pi(w_i\alpha)=\gamma_k+m\gamma_i=(\gamma_k+m_k\gamma_i)+(m_k-m)(-\gamma_i)$ and $m_k-m\in\boldsymbol{Z}_+$; hence $w_i\alpha>0$. Thus $w_i(\Delta)\subset\mathfrak{r}_+$, so $w_i(\Delta)$ is an X_0-fundamental system of $\mathfrak{r}$.

REMARK. Only the fact that the distinct elements of $\bar{\Delta}$ are linearly independent over $\boldsymbol{Q}$ was used in the proof, hence the argument applies to the example at the beginning of this section.

The following theorem indicates the importance of the opposition automorphism of Δ_i.

THEOREM 2.3. *Let G be a connected semi-simple algebraic group, S an admissible subtorus of T, and $\gamma_i\in\bar{\Delta}$, a restricted fundamental system of $\mathfrak{r}$ corresponding to the X_0-fundamental system Δ. The following conditions are equivalent:*

1) The definition is due to J. Tits, who saw the importance of this involution in connection with the classification of connected semi-simple algebraic groups over k. Condition (i) of our Theorem 2.6 was shown to be necessary in the case of k-roots, as well as other conditions on the opposition automorphism. See [1] and [8], and also items 38, 40, 42 in the bibliography of [8]. I am indebted to Professor Tits for suggesting this involution should be looked at in the general case studied here.

(i) $\overline{W}$ *contains the reflection* r_{γ_i}.

(ii) $Z(S)$ *is a proper subgroup of* $N(S)\cap Z(S_{\gamma_i})$.

(iii) $w_i \in W_0'$, (*and* $\overline{w}_i = r_{\gamma_i}$).

(iv) *the opposition automorphism of* Δ_i *leaves* Δ_0 *invariant.*

PROOF. We have shown (i)$\Leftrightarrow$(ii) in Proposition 1.8. Clearly (iii)$\Rightarrow$(i). Suppose (i) holds; then there is an element $s_i \in N(T)\cap N(S)$ such that $\overline{w}_{s_i} = r_{\gamma_i}$. Since $w_{s_i}(\Delta_0)$ and $-\Delta_0$ are both fundamental systems of $\mathfrak{r}_0$ (Proposition 1.3, 1.2), there exists $w \in W_0$ such that $ww_{s_i}(\Delta_0) = -\Delta_0$. Since $s_i \in N(T)\cap N(S)\cap Z(S_{\gamma_i})$ (Proposition 1.8), it follows that $ww_{s_i} \in W_i \cap W_0'$, and hence $ww_{s_i}(\Delta_i)$ is an X_0-fundamental system of $\mathfrak{r}_i$ (Proposition 1.3). Since $ww_{s_i}(\Delta_0) = -\Delta_0$ and $\overline{ww_{s_i}(\Delta_i)} = \{-\gamma_i\}$, and $-\Delta_i$ is also an X_0-fundamental system of $\mathfrak{r}_i$ satisfying $(-\Delta_i)_0 = -\Delta_0$ and $\overline{-\Delta}_i = \{-\gamma_i\}$, Proposition 1.2(c) implies that $ww_{s_i}(\Delta_i) = -\Delta_i$. Thus $ww_{s_i} = w_i$, and $w_i \in W_0'$ and $\overline{w}_i = \overline{ww_{s_i}} = r_{\gamma_i}$, which proves (iii). If (iii) holds, then $w_i(\Delta_0) \subset X_0 \cap (-\Delta_i) = -\Delta_0$, so $w_i(\Delta_0) = -\Delta_0$ which implies (iv). Finally, we show (iv)$\Rightarrow$(iii). Condition (iv) implies $w_i(\Delta_0) = -\Delta_0$, so that to show $w_i(X_0) = X_0$, it suffices to show that if $\alpha, \alpha' \in \Delta^k$, then $w_i\alpha - w_i\alpha' \in X_0$ (Proposition 2.1). By equation (10), we have $\pi(w_i\alpha) = \gamma_k + m\gamma_i$, $\pi(w_i\alpha') = \gamma_k + n\gamma_i$, with $m, n \in \boldsymbol{Z}_+$. Since $\pi(w_i(\Delta^i)) = \{-\gamma_i\}$, we see that $\{-\gamma_i, \gamma_k + m\gamma_i, \gamma_k + n\gamma_i\} \subset \overline{w_i(\Delta)}$. If $w_i\alpha - w_i\alpha' \notin X_0$, then $m \neq n$, and this implies $\overline{w_i(\Delta)}$ contains a linearly dependent set. But since S is admissible, and $w_i(\Delta)$ is an X_0-fundamental system of $\mathfrak{r}$ (Lemma 2.2), this cannot occur. Thus $w_i\alpha - w_i\alpha' \in X_0$, which completes the proof.

As a result of Theorem 2.3, we can determine necessary and sufficient conditions for the group $\overline{W}$ to be the Weyl group of an (abstract) root system. We recall the definition in [4].

Given a vector space M over $\boldsymbol{Q}$ with a non-degenerate symmetric bilinear form $(\,,)$, a finite subset $\Phi \subset M$ which generates M over $\boldsymbol{Q}$ is called a root system in M if the following four conditions hold:

$R(1)$ $0 \notin \Phi$, and $x \in \Phi$ implies $-x \in \Phi$.

$R(2)$ $x - \dfrac{2(x, y)}{(y, y)} y \in \Phi$ for all $x, y \in \Phi$.

$R(3)$ $\dfrac{2(x, y)}{(y, y)} \in \boldsymbol{Z}$ for all $x, y \in \Phi$.

$R(4)$ If $x \in \Phi$ and $cx \in \Phi$ with $c \in \boldsymbol{Q}$, then $c = \pm 1$.

If only conditions $R(1)$, $R(2)$, $R(3)$ are satisfied, then Φ is called a root system in a wider sense. The elements of Φ are called roots, and the set of positive simple roots of Φ with respect to a linear order on M is called a fundamental system of Φ (a positive root is simple if it is not the sum of two positive roots). The group generated by the automorphisms of M of the form $x \to x - \dfrac{2(x, y)}{(y, y)} y$ for $x \in M$, $y \in \Phi$ is called the Weyl group of Φ. It is easily shown that condition $R(3)$ implies that if $x \in \Phi$ and $cx \in \Phi$ for $c \in \boldsymbol{Q}$, then $|c| = \frac{1}{2}, 1, 2$.

Now denote by $\check{\mathfrak{r}}$ the set of "reduced" restricted roots, that is, the subset of elements of $\bar{\mathfrak{r}}$ which cannot be written in the form $c\gamma$ with $\gamma \in \bar{\mathfrak{r}}$, $c \in \boldsymbol{Q}$, $c > 1$. In Theorem 2.6, we will give necessary and sufficient conditions for $\check{\mathfrak{r}}$ to be a root system in $Y_{\boldsymbol{Q}}$ with Weyl group $\overline{W}$. It is clear from our definitions that both $\bar{\mathfrak{r}}$ and $\check{\mathfrak{r}}$ satisfy condition $R(1)$, and $\check{\mathfrak{r}}$ satisfies condition $R(4)$; thus conditions which guarantee conditions $R(2)$ and $R(3)$ are needed.

Some properties of $\check{\mathfrak{r}}$ and the reflections r_{γ_i} ($\gamma_i \in \bar{\Delta}$) which are needed in the proof of Theorem 2.6 are collected in the next lemma.

LEMMA 2.4. *Let S be an admissible subtorus of T, $\bar{\Delta} = \{\gamma_1, \cdots, \gamma_{\bar{l}}\}$ a restricted fundamental system of $\mathfrak{r}$, and $\check{\mathfrak{r}}_+$ the set of positive roots in $\check{\mathfrak{r}}$ with respect to $\bar{\Delta}$. If $w_i \in W_0'$ for all i, $1 \leqq i \leqq \bar{l}$, then*

(a) *If $\langle\,,\rangle$ is any $\overline{W}$-invariant non-degenerate symmetric bilinear form on $Y_{\boldsymbol{Q}}$, then*

$$\bar{w}_i \eta = \eta - \frac{2\langle \eta, \gamma_i \rangle}{\langle \gamma_i, \gamma_i \rangle} \gamma_i \qquad \text{for all } \eta \in Y_{\boldsymbol{Q}},\ 1 \leqq i \leqq \bar{l}\,.$$

(b) *If $\gamma \in \check{\mathfrak{r}}_+$, and $\gamma \neq \gamma_i$, then $\bar{w}_i \gamma \in \check{\mathfrak{r}}_+$.*

(c) *For each $\gamma \in \check{\mathfrak{r}}$, there exists an index j ($1 \leqq j \leqq \bar{l}$) and a subset $\{i(1), \cdots, i(\nu)\} \subset \{1, 2, \cdots, \bar{l}\}$ such that $\gamma = \bar{w}_{i(1)} \cdots \bar{w}_{i(\nu)} \gamma_j$.*

(d) *If $\gamma \in \bar{\mathfrak{r}}$, then $\gamma = m\gamma'$ for some $\gamma' \in \check{\mathfrak{r}}$, $m \in \boldsymbol{Z}$.*

We omit the proof of the lemma since the arguments are standard ones. We note that (a) follows since Theorem 2.3 implies that $\bar{w}_i$ is the reflection in $Y_{\boldsymbol{Q}}$ with respect to the hyperplane $H_{\gamma_i} = \{\eta \in Y_{\boldsymbol{Q}} \mid \langle \gamma_i, \eta \rangle = 0\}$; that (b) follows from (a) and proposition 1.2(b′), and the fact that $\overline{W}$ leaves $\check{\mathfrak{r}}$ invariant, that (c) follows from (b), and (d) follows from (c) and Proposition 1.2 (see, e. g. [**2**], exposé 14, or [**6**]).

In the course of the proof of Theorem 2.6, we use some standard arguments and hence need the following notion of "Weyl chamber." For each restricted fundamental system $\bar{\Delta} = \{\gamma_1, \cdots, \gamma_{\bar{l}}\}$ of $\mathfrak{r}$, define

$$C_{\bar{\Delta}} = \{\omega^* \in Y_{\boldsymbol{Q}}^* \mid \omega^*(\gamma_i) > 0,\ 1 \leqq i \leqq \bar{l}\}\,. \tag{11}$$

Since S is admissible, $\bar{\Delta}$ is a basis for $Y_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$, and so $C_{\bar{\Delta}} \neq \phi$. It is easily seen from Proposition 1.2(b′) that $C_{\bar{\Delta}}$ is a Weyl chamber of $Y_{\boldsymbol{Q}}^*$ in the usual sense, that is, if we choose $\omega_0^* \in C_{\bar{\Delta}}$ and define $\bar{\mathfrak{r}}_+ = \{\gamma \in \bar{\mathfrak{r}} \mid \omega_0^*(\gamma) > 0\}$, then $C_{\bar{\Delta}} = \bigcap_{\gamma \in \bar{\mathfrak{r}}_+} H_\gamma^+$, where $H_\gamma^+ = \{\omega^* \in Y_{\boldsymbol{Q}}^* \mid \omega^*(\gamma) > 0\}$. From (11) and (6) we see that for each $\bar{w} \in \overline{W}$ and restricted fundamental system $\bar{\Delta}$ of $\mathfrak{r}$, one has

$$\bar{w}(C_{\bar{\Delta}}) = C_{\bar{w}(\bar{\Delta})}\,, \tag{12}$$

thus $\overline{W}$ acts on the set of all $C_{\bar{\Delta}}$.

The following lemma is easily proved (see [4]) and states that the usual "useful" properties of Weyl chambers hold for the $C_{\bar{\Delta}}$.

LEMMA 2.5. *Let S be an admissible torus of G.*

(a) *There is a one-to-one correspondence between restricted fundamental systems $\bar{\Delta}$ of $\mathfrak{r}$ and Weyl chambers $C_{\bar{\Delta}}$.*

(b) $\bigcup_{\bar{\Delta}: r.f.s.} C_{\bar{\Delta}} = Y_{\boldsymbol{Q}}^* - \bigcup_{\gamma\in\bar{\mathfrak{r}}} H_\gamma$ *(the union on the left is taken over all restricted fundamental systems of $\mathfrak{r}$).*

THEOREM 2.6. *Let G be a connected semi-simple algebraic group, S an admissible subtorus of T, and $\bar{\Delta}=\{\gamma_1, \cdots, \gamma_{\bar{l}}\}$ a restricted fundamental system of $\mathfrak{r}$. The following conditions are equivalent:*

(i) *The opposition automorphism of Δ_i leaves Δ_0 invariant for all i, $1\leqq i\leqq \bar{l}$.*

(ii) *$\bar{W}$ contains r_γ for all $\gamma\in\bar{\mathfrak{r}}$*

(iii) *$\bar{W}$ is generated by $\{r_{\gamma_i}, 1\leqq i\leqq \bar{l}\}$*

(iv) *$\check{\mathfrak{r}}$ is a root system in $Y_{\boldsymbol{Q}}$ (with respect to a $\bar{W}$-invariant metric), with fundamental system $\bar{\Delta}$, and Weyl group $\bar{W}$.*

PROOF. Suppose (i) holds; then by Theorem 2.3, $w_i \in W_0'$ and $r_{\gamma_i}=\bar{w}_i\in\bar{W}$ for all i, $1\leqq i\leqq \bar{l}$. If $\gamma\in\bar{\mathfrak{r}}$, then Lemma 2.4(c) implies there is an index j and an element $\bar{w}\in\bar{W}$ satisfying $\gamma=\bar{w}\gamma_j$. If we define $\bar{w}_\gamma=\bar{w}\bar{w}_j\bar{w}^{-1}$, then $\bar{w}_\gamma\neq 1$, $\bar{w}_\gamma^2=1$, and $\bar{w}_\gamma$ leaves H_γ pointwise fixed, so $\bar{w}_\gamma=r_\gamma$, and $r_\gamma\in\bar{W}$. If $\gamma\in\bar{\mathfrak{r}}$ is of the form $m\gamma'$ for some $\gamma'\in\check{\mathfrak{r}}$, $m\in\boldsymbol{Z}$, (Lemma 2.4(d)) then $r_\gamma=r_{\gamma'}$, so that the subgroup $\bar{W}'$ of $\bar{W}$ generated by $\{r_{\gamma_i}, 1\leqq i\leqq\bar{l}\}$ contains r_γ for all $\gamma\in\bar{\mathfrak{r}}$. Now Lemma 2.5(b) implies that $Y_{\boldsymbol{Q}}^*=(\bigcup_{\bar{\Delta}: r.f.s.} C_{\bar{\Delta}})\cup(\bigcup_{\gamma\in\bar{\mathfrak{r}}} H_\gamma)$, hence given any two Weyl chambers $C_{\bar{\Delta}}$ and $C_{\bar{\Delta}'}$, there is an element $\bar{w}'\in\bar{W}'$ such that $\bar{w}'(C_{\bar{\Delta}})=C_{\bar{w}'(\bar{\Delta})}=C_{\bar{\Delta}'}$. By Lemma 2.5(a), $\bar{w}'(\bar{\Delta})=\bar{\Delta}'$, so $\bar{W}'$ is transitive on the set $\{\bar{\Delta}: r.f.s.\}$. But the action of $\bar{W}$ is simple on this set (Proposition 1.3), so $\bar{W}'=\bar{W}$. Thus (i)$\Rightarrow$(iii), and in the course of the argument, we've shown (iii)$\Rightarrow$(ii). Since (ii)$\Rightarrow$(i) (Theorem 2.3), and (iv) clearly implies (ii), we only need to show (ii)$\Rightarrow$(iv). From the construction of $\bar{w}_\gamma=r_\gamma$ above, and from Lemma 2.4(a), it follows that $\bar{w}_\gamma\eta=\eta-\dfrac{2\langle\eta,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma$ for all $\eta\in Y_{\boldsymbol{Q}}$, $\gamma\in\bar{\mathfrak{r}}$. Since $\bar{W}$ leaves $\check{\mathfrak{r}}$ invariant, condition $R(2)$ holds for $\check{\mathfrak{r}}$. For each $\gamma\in\check{\mathfrak{r}}$, define $\mathfrak{r}_\gamma=\{\alpha\in\mathfrak{r}\mid\pi(\alpha)=c\gamma,\ c\in\boldsymbol{Q}\}$. Condition (ii) implies that $\gamma=\bar{w}\gamma_j$ for some $\bar{w}\in\bar{W}$ and some j (Theorem 2.3, Lemma 2.4(c)), so that $\gamma\in\bar{w}(\bar{\Delta})$, which is a restricted fundamental system of $\mathfrak{r}$ (Proposition 1.3). Thus by the argument following Proposition 2.1, we see that $\mathfrak{r}_\gamma$ is the root system of the connected reductive group $Z(S_\gamma)$, and by Lemma 2.4(d), $\mathfrak{r}_\gamma=\{\alpha\in\mathfrak{r}\mid\pi(\alpha)=m\gamma,\ m\in\boldsymbol{Z}\}$. Condition (ii) also implies that there is an element $s\in N(T)\cap N(S)\cap Z(S_\gamma)$, $s\notin Z(S)$ such that $\bar{w}_s=r_\gamma$ (Proposition 1.8). Since w_s is in the Weyl group of $Z(S_\gamma)$, we have $w_s\chi-\chi\in(\mathfrak{r}_\gamma)_{\boldsymbol{Z}}$ for all $\chi\in X$, hence $\bar{w}_s\eta-\eta\in(\bar{\mathfrak{r}}_\gamma)_{\boldsymbol{Z}}$ for all $\eta\in Y$. In particular, $\bar{w}_s\gamma'-\gamma'$

$=\frac{2\langle\gamma,\gamma'\rangle}{\langle\gamma,\gamma\rangle}\gamma\in(\bar{\mathfrak{r}}_\gamma)_{\boldsymbol{Z}}$ for all $\gamma'\in\bar{\mathfrak{r}}$, and since $(\bar{\mathfrak{r}}_\gamma)_{\boldsymbol{Z}}=(\gamma)_{\boldsymbol{Z}}$, it follows that $\frac{2\langle\gamma,\gamma'\rangle}{\langle\gamma,\gamma\rangle}\in\boldsymbol{Z}$. Thus $R(3)$ holds for $\check{\mathfrak{r}}$; since $R(1)$ and $R(4)$ also hold and $\bar{\Delta}\subset\check{\mathfrak{r}}$ generates $Y_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$, $\check{\mathfrak{r}}$ is a root system in $Y_{\boldsymbol{Q}}$. Proposition 1.2(b$'$) shows $\bar{\Delta}$ is a fundamental system of $\check{\mathfrak{r}}$, and (iii) shows $\overline{W}$ is the Weyl group of $\check{\mathfrak{r}}$.

DEFINITION. A subtorus S of T will be said to be "of root system type" if S is admissible and one of the equivalent conditions (i)-(iv) of Theorem 2.6 is satisfied.

COROLLARY 2.7. *Let S be of root system type.*

(a) *$\overline{W}$ acts simply transitively on the set $\{\bar{\Delta}: r.f.s.\}$, and W_0' acts simply transitively on the set $\{\Delta: X_0$-fundamental system$\}$.*

(b) *$N(S)=Z(S)$ if and only if $S\subset$ center G.*

(c) *Let Δ be any X_0-fundamental system of $\mathfrak{r}$, and $\mathfrak{r}_+$ the positive roots of $\mathfrak{r}$ with respect to Δ. If $U_{\bar{\Delta}}$ is the group generated by $\{P_\alpha,\alpha\in\mathfrak{r}_+-\mathfrak{r}_0\}$, then G is generated by $U_{\bar{\Delta}}$ and $N(S)\cap N(T)$.*

PROOF. (a) The first part of this statement was proved in showing (i)$\Rightarrow$(iii) in Theorem 2.6; the second part then follows easily using Proposition 1.2(c).

(b) If $S\not\subset$ center G, then $\bar{\mathfrak{r}}\neq\phi$ (if $\bar{\mathfrak{r}}=\phi$, then $\mathfrak{r}_0=\mathfrak{r}$ implies $Z(S)=G$), so that there are at least two restricted fundamental systems, $\bar{\Delta}$, $-\bar{\Delta}$ of $\mathfrak{r}$. By (a), there is an element $\bar{w}_s\in\overline{W}$, $s\in N(S)$ such that $\bar{w}_s(\bar{\Delta})=-\bar{\Delta}$; since $\bar{w}_s\neq 1$, we see $s\notin Z(S)$ (Proposition 1.5).

(c) G is generated by T and $\{P_\alpha,\alpha\in\mathfrak{r}\}$ ([2], exposé 13), and $N(S)\supset T$, $N(S)\supset P_\alpha$ for all $\alpha\in\mathfrak{r}_0$. By definition, $U_{\bar{\Delta}}\supset P_\alpha$ for all $\alpha\in\mathfrak{r}_+-\mathfrak{r}_0$. By (a), there is an element $s\in N(T)\cap N(S)$ such that $w_s(\Delta)=-\Delta$; then $w_sP_\alpha=s^{-1}P_\alpha s=P_{w_s\alpha}$, so $sU_{\bar{\Delta}}s^{-1}\supset P_{-\alpha}$ for all $\alpha\in\mathfrak{r}_+-\mathfrak{r}_0$.

COROLLARY 2.8. *Let the assumptions be as in Theorem 2.6. The set $\bar{\mathfrak{r}}$ is a root system in a wider sense in $Y_{\boldsymbol{Q}}$ with fundamental system $\bar{\Delta}$ and Weyl group $\overline{W}$ if and only if $\check{\mathfrak{r}}$ is also, and $\frac{2\langle\gamma,\gamma'\rangle}{\langle\gamma,\gamma\rangle}\in\boldsymbol{Z}$ for all $\gamma\in\bar{\mathfrak{r}}-\check{\mathfrak{r}}$, $\gamma'\in\bar{\mathfrak{r}}$.*

Corollary 2.8 follows immediately from our proof of (ii)$\Rightarrow$(iv) in Theorem 2.6.

REMARK. There are numerous examples to illustrate that $\check{\mathfrak{r}}$ can be a root system and $\bar{\mathfrak{r}}$ not a root system in a wider sense. A simple case is: let G be a simple group of type A_3, with fundamental system $\Delta=\{\alpha_1,\alpha_2,\alpha_3\}$, and let S be the admissible torus whose annihilator is generated by $\alpha_2-\alpha_1$, $\alpha_3-\alpha_2$. Since $\Delta_0=\phi$, condition (i) of Theorem 2.6 is trivially satisfied, so $\check{\mathfrak{r}}$ is a root system. However, $\bar{\mathfrak{r}}=\{\pm\gamma,\pm2\gamma,\pm3\gamma\}$ where $\gamma=\pi(\alpha_i)$, $i=1,2,3$, so that $\bar{\mathfrak{r}}$ cannot satisfy condition $R(3)$ of root system.

The condition on the opposition automorphisms in (i) of Theorem 2.6 also guarantees that W_0' has a nice structure, and that $\overline{W}$ is isomorphic to the subgroup of W generated by the w_i, $1\leqq i\leqq \bar{l}$.

THEOREM 2.9. *Let the assumptions be as in Theorem* 2.6, *and let V be the subgroup of W generated by the set $\{w_i, 1 \leqq i \leqq \bar{l}\}$. Then S is of root system type if and only if $W_0' = V \cdot W_0$, a semidirect product.* (*Hence if S is of root system type, then V is isomorphic to $\overline{W}$ under the restriction to V of the canonical homomorphism $W_0' \to \overline{W}$, defined by* (4).)

PROOF. If $W_0' = V \cdot W_0$, then $V \subset W_0'$, so $w_i \in W_0'$ for all i, $1 \leqq i \leqq \bar{l}$. Thus by Theorem 2.3, S is of root system type. Conversely, if S is of root system type, then $w_i \in W_0'$ for all i, $1 \leqq i \leqq \bar{l}$ (Theorem 2.3), and $\bar{w}_i = r_{\gamma_i}$. Since $\overline{W}$ is generated by $\{\bar{w}_i, 1 \leqq i \leqq \bar{l}\}$ (Theorem 2.6, (iii)), it follows that $\bar{V} = \overline{W}$, where $\bar{V}$ is the canonical image of V in $\overline{W}$. This implies, by Corollary 1.4, that $W_0' = V \cdot W_0$. Thus we only need to show $V \cap W_0 = \{1\}$. If we put $w_0 = 1$, then any element $w \in V \cap W_0$ can be written $w = w_{i(1)} \cdots w_{i(p)}$, where $\{i(1), \cdots, i(p)\} \subset \{0, 1, \cdots, \bar{l}\}$. We use induction on p; clearly if $p = 1$, we must have $w = w_0 = 1$. Assume for all $k < p$ that if $w = w_{i(1)} \cdots w_{i(k)} \in W_0$, then $w = 1$. Suppose $w = w_{i(1)} \cdots w_{i(p)} \in W_0$; clearly we may assume $w_{i(p)} \neq 1$. Now $\bar{w}\gamma_{i(p)} = \bar{w}_{i(1)} \cdots \bar{w}_{i(p)}\gamma_{i(p)} = \gamma_{i(p)} > 0$, and since $\bar{w}_{i(p)}\gamma_{i(p)} = -\gamma_{i(p)}$, there exists an index k such that $\bar{w}_{i(m)} \cdots \bar{w}_{i(p)}\gamma_{i(p)} < 0$ for all m satisfying $k < m \leqq p$, and $\bar{w}_{i(k)} \cdots \bar{w}_{i(p)}\gamma_{i(p)} > 0$ (note $w_{i(k)} \neq 1$). If we put $\bar{w}' = \bar{w}_{i(k+1)} \cdots \bar{w}_{i(p-1)}$, then $\bar{w}'\gamma_{i(p)} \in \check{\mathfrak{r}}$, and $\bar{w}'\gamma_{i(p)} > 0$, and $\bar{w}_{i(k)}\bar{w}'\gamma_{i(p)} < 0$ hence $\bar{w}'\gamma_{i(p)} = \gamma_{i(k)}$ (Lemma 2.4(b)). This implies $\bar{w}'\bar{w}_{i(p)}\bar{w}'^{-1} = \bar{w}_{i(k)}$, so $\bar{w}'\bar{w}_{i(p)} = \bar{w}_{i(k)}\bar{w}'$. Multiplying this equation by $\bar{w}_{i(1)} \cdots \bar{w}_{i(k)}$, we have $1 = \bar{w} = \bar{w}_{i(1)} \cdots \bar{w}_{i(k-1)}\bar{w}_{i(k+1)} \cdots \bar{w}_{i(p-1)}$, so by Proposition 1.3 and the induction hypothesis, $w_{i(1)} \cdots w_{i(k-1)}w_{i(k+1)} \cdots w_{i(p-1)} = 1$. Thus we can write $w = (w_{i(1)} \cdots w_{i(k-1)})w_{i(k)}(w_{i(1)} \cdots w_{i(k-1)})^{-1}w_{i(p)}$. Since $w_i(\Delta_0) = -\Delta_0$ for all $i \neq 0$ (Theorem 2.6 (i)), $w_0(\Delta_0) = \Delta_0$, and $i(k)$, $i(p) \neq 0$, it follows that $w(\Delta_0) = \Delta_0$. But since $w \in W_0$, this implies $w = 1$. Corollary 1.4 implies the second assertion in the theorem.

Given any abstract group H which is generated by a set of involutions $R = \{r_i\}$, $i \in I$ (I an index set), the *length* of any element $h \in H$ is denoted $l(h)$, and defined as the least positive integer m such that h can be written as a product of m of the r_i. A product $r_{i(1)} \cdots r_{i(k)}$ is called *reduced* if $l(r_{i(1)} \cdots r_{i(k)}) = k$. The set R is called a "good system of involutive generators"[2] of H if the following condition is satisfied for any choice of indices $i(0)$, $i(1)$, $\cdots$, $i(m)$, and any positive integer m: (c) If $r_{i(1)} \cdots r_{i(m)}$ is reduced, and $r_{i(0)}r_{i(1)} \cdots r_{i(m)}$ is not reduced, then there exists an integer j ($1 \leqq j \leqq m$) such that $r_{i(0)}r_{i(1)} \cdots r_{i(j-1)} = r_{i(1)} \cdots r_{i(j)}$.

A classic example of such a group and set of generators is the Weyl group of a semi-simple algebraic group, and the set of fundamental reflections. It is also known that the Weyl group of an abstract root system Φ has a good system of involutive generators, namely, the reflections corresponding to a

2) The definition is due to H. Matsumoto, C. R. Acad. Sci. Paris, 258, p. 3419. Such systems have also been studied by J. Tits, N. Iwahori, and H. Hijikata.

fundamental system of Φ. (See, e. g. N. Iwahori,. "Discrete Reflection Groups in Euclidean Spaces," Berkeley, 1965.) Thus, when S is of root system type, $\overline{W}$ has a good system of involutive generators, $\{r_{\gamma_i}, \gamma_i \in \bar{\Delta}\}$. Theorem 2.9 then implies:

COROLLARY 2.10. *Let the assumptions be as in Theorem* 2.6. *If S is of root system type, then the set $\{w_i, 1 \leqq i \leqq \bar{l}\}$ is a good system of involutive generators for V.*

§ 3. The admissible torus T^{Γ}.

In this section, we examine a class of admissible tori which are a natural generalization of maximal k-trivial tori. We continue to assume G is a connected, semi-simple algebraic group.

Denote by $\mathrm{Aut}(G, T)$ the group of rational automorphisms of G which leave T invariant, and fix Γ, a non-trivial subgroup of $\mathrm{Aut}(G, T)$. We denote by T^{Γ} the identity component of the closed subgroup of T left pointwise fixed by Γ. We are going to show that T^{Γ} is an admissible subtorus of T.

Each element of Γ can be considered as an element of $\mathrm{Aut}(X, \mathfrak{r})$ (the Cartan group of T) in a natural manner, namely, for each $\chi \in X$, $\sigma \in \Gamma$, χ^{σ} is defined by the equation:

$$\chi^{\sigma}(t) = \chi(t^{\sigma^{-1}}) \qquad \text{for all } t \in T. \tag{13}$$

We will also use the symbol Γ to denote the subgroup of $\mathrm{Aut}(X, \mathfrak{r})$ formed by the automorphisms $\chi \to \chi^{\sigma}$, for $\sigma \in \Gamma$. Since $\mathrm{Aut}(X, \mathfrak{r})$ is finite, the subgroup Γ of $\mathrm{Aut}(X, \mathfrak{r})$ is also: let $d = [\Gamma : 1]$. We define submodules X_0 and X^{Γ} of X as follows:

$$X_0 = \{\chi \in X \mid \sum_{\sigma \in \Gamma} \chi^{\sigma} = 0\} \tag{14}$$

$$X^{\Gamma} = \{\chi \in X \mid \chi^{\sigma} = \chi \qquad \text{for all } \sigma \in \Gamma\}.$$

Since $X_{0\boldsymbol{Q}}$ and $X^{\Gamma}_{\boldsymbol{Q}}$ are the kernel and image, respectively, of the homomorphism of $X_{\boldsymbol{Q}} \to X_{\boldsymbol{Q}}$ given by $\chi \to \sum_{\sigma \in \Gamma} \chi^{\sigma}$, it follows that $X_{\boldsymbol{Q}} = X_{0\boldsymbol{Q}} + X^{\Gamma}_{\boldsymbol{Q}}$, a direct sum. If $\chi \in X$ and $\sigma \in \Gamma$, then $\chi - \chi^{\sigma} \in X_0$, and χ is written with respect to this direct sum as follows:

$$\chi = d^{-1} \sum_{\sigma \in \Gamma} (\chi - \chi^{\sigma}) + d^{-1} \sum_{\sigma \in \Gamma} \chi^{\sigma}. \tag{15}$$

In particular, (15) shows that elements of the form $\chi - \chi^{\sigma}$ where $\chi \in X$ and $\sigma \in \Gamma$ generate X_0 over $\boldsymbol{Q}$. In fact, since any fundamental system Δ of $\mathfrak{r}$ generates X over $\boldsymbol{Q}$, the set $\{\alpha - \alpha^{\sigma} : \alpha \in \Delta, \sigma \in \Gamma\}$ generates X_0 over $\boldsymbol{Q}$.

It is clear from (14) that X_0 and X^{Γ} are both Γ-invariant co-torsion free submodules of X, hence the annihilators of X_0 and X^Γ in T are Γ-invariant subtori of T. We show that the annihilator of X_0 in T is just T^Γ. If $\chi \in X_0$, then $\sum_{\sigma \in \Gamma} \chi^\sigma = 0$, so for each $t \in T^\Gamma$, we have $1 = \prod_{\sigma \in \Gamma} \chi^\sigma(t) = \prod_{\sigma \in \Gamma} \chi(t) = (\chi(t))^d$. Since $\chi(T^\Gamma)$ is a connected subgroup of G_m, it follows that $\chi(t) = 1$ for all $t \in T^\Gamma$. Conversely, if $t \in T$ annihilates X_0, then $(\chi - \chi^{\sigma^{-1}})(t) = 1$ for all $\chi \in X$, $\sigma \in \Gamma$, so $\chi(t) = \chi(t^\sigma)$ for $\chi \in X$, $\sigma \in \Gamma$, which implies $t = t^\sigma$ for all $\sigma \in \Gamma$, so $t \in T^\Gamma$.

Since $\chi^\sigma \equiv \chi \pmod{X_0}$ for all $\chi \in X$, $\sigma \in \Gamma$, it follows that a linear order $>$ on X is an X_0-linear order if and only if the following condition holds:

$$\text{(16)} \qquad \text{If } \chi \notin X_0, \text{ then } \chi > 0 \text{ implies } \chi^\sigma > 0 \qquad \text{for all } \sigma \in \Gamma\,.$$

A linear order on X satisfying (16) will be called a *Γ-linear order on X*, and a fundamental system of $\mathfrak{r}$ with respect to such an order will be called a *Γ-fundamental system of* $\mathfrak{r}$.

Since the action of Γ on X leaves $\mathfrak{r}$ and X_0 invariant, it follows that if Δ is a Γ-fundamental system of $\mathfrak{r}$, and $\sigma \in \Gamma$, then Δ^σ is another Γ-fundamental system of $\mathfrak{r}$ (of course, $\overline{\Delta^\sigma} = \bar{\Delta}$). The following lemma makes explicit how an element $\alpha \in \Delta - \Delta_0$ is related to $\alpha^\sigma \in \Delta^\sigma - \Delta_0^\sigma$.

LEMMA 3.1. *Let $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ be a Γ-fundamental system of $\mathfrak{r}$. Each $\sigma \in \Gamma$ defines a permutation of $\Delta - \Delta_0$ (we write $\alpha_i \to \alpha_{i(\sigma)}$) which satisfies: if $\alpha_i \in \Delta - \Delta_0$, then $\alpha_i^\sigma = \alpha_{i(\sigma)} + \sum_{\alpha_j \in \Delta_0} m_j \alpha_j$, where $m_j \in \mathbf{Z}_+$, and $\alpha_i \equiv \alpha_{i(\sigma)} \pmod{X_0}$.*

PROOF. For *any* $\alpha_i \in \Delta$ and $\sigma \in \Gamma$, (16) implies that we may write $\alpha_i^\sigma = \sum_{j=1}^{l} c_{ij}(\sigma) \alpha_j$, where $c_{ij}(\sigma) \in \mathbf{Z}_+$ if $\alpha_i \notin \Delta_0$, and $c_{ij}(\sigma) = 0$ if $\alpha_i \in \Delta_0$ and $\alpha_j \notin \Delta_0$ (Proposition 1.2). We may assume (by reordering if necessary) that $\Delta - \Delta_0 = \{\alpha_1, \cdots, \alpha_m\}$, $\Delta_0 = \{\alpha_{m+1}, \cdots, \alpha_l\}$. Then the integral matrices $(c_{ij}(\sigma))$, $(c_{ij}(\sigma^{-1}))$ are both of the form $\left(\begin{array}{c|c} \geqq 0 & \geqq 0 \\ \hline 0 & * \end{array}\right)$, and their product is the identity matrix. Thus the upper left submatrix is an $m \times m$ permutation matrix. For each i, $1 \leqq i \leqq m$, denote $i(\sigma) = k$ if the i, k^{th} entry is 1. Then if $\alpha \in \Delta - \Delta_0$, we have $\alpha_i^\sigma = \alpha_{i(\sigma)} + \sum_{\alpha_j \in \Delta_0} c_{ij}(\sigma) \alpha_j$, and since $\alpha_i^\sigma \equiv \alpha_i \pmod{X_0}$, it follows that $\alpha_i \equiv \alpha_{i(\sigma)} \pmod{X_0}$.

Using this lemma, it is now easy to show that T^Γ is an admissible subtorus of T[3].

PROPOSITION 3.2. *T^Γ is an admissible subtorus of T.*

PROOF. Let Δ be a Γ-fundamental system of $\mathfrak{r}$; then the set $\{\alpha_i^\sigma - \alpha_i\,;\ \alpha_i \in \Delta,\ \sigma \in \Gamma\}$ generates X_0 over $\mathbf{Q}$. If $\alpha_i \in \Delta_0$, then $\alpha_i^\sigma \in \mathfrak{r}_0$, so $\alpha_i^\sigma \in (\Delta_0)_{\mathbf{Z}}$ (Proposition 1.2). If $\alpha_i \notin \Delta_0$, then $\alpha_i^\sigma - \alpha_i = (\alpha_{i(\sigma)} - \alpha_i) + \sum_{\alpha_j \in \Delta_0} m_j \alpha_j$, $m_j \in \mathbf{Z}$, and

3) (An alternate proof which can be used without change is given in [4], Proposition 5(b). The proof using Lemma 3.1 is also due to Satake.)

$\alpha_{i(\sigma)} \equiv \alpha_i \pmod{X_0}$ (Lemma 3.1). Thus X_0 is generated over $\boldsymbol{Q}$ by Δ_0 and elements of the form $\alpha_k - \alpha_i$, where $\alpha_i \equiv \alpha_k \pmod{X_0}$, so T^{Γ} is admissible (Proposition 2.1).

REMARK. Although every subtorus of T of the form T^{Γ} for some $\Gamma \subset \mathrm{Aut}(G, T)$ is admissible, the strong condition (i) of Theorem 2.6 shows that many (in fact, most) of these are not of root system type. For instance, if Γ is a subgroup of W generated by a subset $\{w_{\alpha_{i(1)}}, \cdots, w_{\alpha_{i(k)}}\}$ of reflections, where $\alpha_{i(1)}, \cdots, \alpha_{i(k)}$ belong to a fundamental system Δ of $\mathfrak{r}$, then it is easily shown that X_0 is generated over $\boldsymbol{Q}$ by $\alpha_{i(1)}, \cdots, \alpha_{i(k)}$, and hence Δ is an X_0-fundamental system, and $\Delta_0 = \{\alpha_{i(1)}, \cdots, \alpha_{i(k)}\}$. In this case, for each $\gamma_j \in \bar{\Delta}$, $\Delta \cap \pi^{-1}(\gamma_j) = \Delta^j$ consists of just one root, and so unless the set Δ_0 is "well chosen", the opposition automorphism of $\Delta_j = \{\alpha\} \cup \Delta_0$ will not leave Δ_0 invariant.

For the remainder of this section, we fix a Γ-fundamental system Δ of $\mathfrak{r}$.

LEMMA 3.3. *For each $\sigma \in \Gamma$, there exists a unique element $w_\sigma \in W_0$ satisfying $w_\sigma \Delta = \Delta^\sigma$.*

PROOF. Since Δ^σ is a Γ-fundamental system of $\mathfrak{r}$, $\Delta_0^\sigma = \Delta^\sigma \cap \mathfrak{r}_0$ is a fundamental system of $\mathfrak{r}_0$ (Proposition 1.2), hence there is a unique element $w_\sigma \in W_0$ satisfying $w_\sigma \Delta_0 = \Delta_0^\sigma$. Since $\overline{w_\sigma \Delta} = \bar{\Delta} = \overline{\Delta^\sigma}$, it follows that $w_\sigma \Delta = \Delta^\sigma$ (Proposition 1.2).

This lemma enables us to define another action of Γ on X as follows:

$$\chi^{[\sigma]} = w_\sigma^{-1} \chi^\sigma \qquad \text{for each } \chi \in X,\ \sigma \in \Gamma. \tag{17}$$

Since $\sigma \in \Gamma$ and $w_\sigma \in W_0$ are automorphisms of X which leave $\mathfrak{r}$ and X_0 invariant, $[\sigma]$ is also such an automorphism. But the definition of w_σ in Lemma 3.3 implies that $[\sigma]$ also leaves Δ invariant, thus $[\sigma] \in \mathrm{Aut}(X, \mathfrak{r}, \Delta, \Delta_0)$. We will denote by $[\Gamma]$ the subgroup of $\mathrm{Aut}(X, \mathfrak{r}, \Delta, \Delta_0)$ defined by the set $\{[\sigma], \sigma \in \Gamma\}$.

It is clear from (17) that $\chi^{[\sigma]} \equiv \chi \pmod{X_0}$ for all $\chi \in X$, $\sigma \in \Gamma$, and hence the restriction of each $[\sigma] \in [\Gamma]$ to $\Delta - \Delta_0$ is a permutation satisfying $\alpha^{[\sigma]} \equiv \alpha \pmod{X_0}$ for all $\alpha \in \Delta - \Delta_0$. In fact, this permutation coincides with the one defined in Lemma 3.1.

LEMMA 3.4. *For each $\alpha_i \in \Delta - \Delta_0$, and $\sigma \in \Gamma$, one has $\alpha_i^{[\sigma]} = \alpha_{i(\sigma)}$.*

PROOF. Since $w_\sigma^{-1} \in W_0$, we have $w_\sigma^{-1} \alpha_i^\sigma = \alpha_i^\sigma + \chi_0$, where $\chi_0 \in (\Delta_0)_{\boldsymbol{Z}}$. Thus $\alpha_i^{[\sigma]} = w_\sigma^{-1} \alpha_i^\sigma = \alpha_i^\sigma + \chi_0 = \alpha_{i(\sigma)} + \chi_0'$, where $\chi_0' \in (\Delta_0)_{\boldsymbol{Z}}$ (Lemma 3.1). Since $\alpha_i^{[\sigma]} \in \Delta - \Delta_0$, it follows that $\alpha_i^{[\sigma]} = \alpha_{i(\sigma)}$.

We can reformulate the condition $\alpha^{[\sigma]} \equiv \alpha \pmod{X_0}$ for $\alpha \in \Delta - \Delta_0$ in the following manner: if $\alpha \in \Delta - \Delta_0$, and $\pi(\alpha) = \gamma$, then $\alpha^{[\sigma]} \in \Delta \cap \pi^{-1}(\gamma)$ for all $\sigma \in \Gamma$. The following proposition states that, in fact, every element of $\Delta \cap \pi^{-1}(\gamma)$ is of the form $\alpha^{[\sigma]}$ for some $\sigma \in \Gamma$. For each $\chi \in X$, we call the set $\{\chi^{[\sigma]} : [\sigma] \in [\Gamma]\}$ the $[\Gamma]$-orbit of χ.

PROPOSITION 3.5. *For $\gamma \in \bar{\Delta}$, $\Delta \cap \pi^{-1}(\gamma)$ is a $[\Gamma]$-orbit.*

PROOF. Let $\alpha_i, \alpha_j \in \Delta \cap \pi^{-1}(\gamma)$. By Lemma 3.4, it suffices to show that there exists a $\sigma \in \Gamma$ such that $\alpha_{i(\sigma)} = \alpha_j$. Since $\alpha_i - \alpha_j \in X_0$, we have by (14), $\sum_{\sigma \in \Gamma} (\alpha_i - \alpha_j)^\sigma = 0$, which implies

$$\sum_{\sigma \in \Gamma} \alpha_{i(\sigma)} + \chi_0 = \sum_{\sigma \in \Gamma} \alpha_{j(\sigma)} + \chi_0',$$

where $\chi_0, \chi_0' \in (\Delta_0)_{\boldsymbol{Z}_+}$ (Lemma 3.1). Since these are equal linear combinations of fundamental roots (with non-negative coefficients), every term on the right also appears on the left. But α_j is a term on the right (note: $j(id) = j$), and $\alpha_j \notin \Delta_0$, hence $\alpha_j = \alpha_{i(\sigma)}$ for some $\sigma \in \Gamma$.

Using our notation in § 2, Proposition 3.5 shows that when $S = T^\Gamma$, the disjoint union $\Delta - \Delta_0 = \Delta^1 \cup \cdots \cup \Delta^{\bar{l}}$ (where $\Delta^i = \Delta \cap \pi^{-1}(\gamma_i)$) is just the decomposition of $\Delta - \Delta_0$ into orbits under the action of $[\Gamma]$.

COROLLARY 3.6. *X_0 (defined in (14)) is generated over $\boldsymbol{Q}$ by Δ_0 and the set $\{\alpha^{[\sigma]} - \alpha : \alpha \in \Delta - \Delta_0, \sigma \in \Gamma\}$.*

PROOF. This is an immediate consequence of Proposition 2.1, Proposition 3.2, and Proposition 3.5.

REMARK. The group $\mathrm{Aut}(X, \mathfrak{r}, \Delta)$ is well known for G a simple group, so the fact that Δ^i is a $[\Gamma]$-orbit, where $[\Gamma] \subset \mathrm{Aut}(X, \mathfrak{r}, \Delta)$ means that we can determine for this case the maximum number of elements in $\Delta - \Delta_0$ which have the same restriction $\gamma_i \in \bar{\Delta}$. Except for D_4, Δ^i can have at most two elements, and for $G = D_4$, Δ^i can have at most three elements. This observation shows that there are admissible tori (even of root system type) which are not of the form T^Γ. The subtorus of G, where G is of type A_3, noted in the remark after Corollary 2.8 provides a simple example. (The example is easily generalized to G of type A_l, $\Delta = \{\alpha_1, \cdots, \alpha_l\}$, and S the subtorus of T whose annihilator is generated by $(\alpha_i - \alpha_j, i \neq j)$.)

§ 4. Γ as an automorphism group of W and subgroups of fixed points.

We continue to assume that Γ is a fixed subgroup of $\mathrm{Aut}(G, T)$, and examine two distinct actions of Γ on the Weyl group W which correspond in a natural manner to the actions of Γ on $(X, \mathfrak{r})$ defined by (13) and (17). Our notations and assumptions in § 3 continue.

For each $w \in W$ and $\sigma \in \Gamma$, the element $w^\sigma \in W$ is defined by the following equation:

$$w^\sigma \chi^\sigma = (w\chi)^\sigma \qquad \text{for all } \chi \in X. \tag{18}$$

Using (18), each element $\sigma \in \Gamma$ determines an element $(w \to w^\sigma)$ in $\mathrm{Aut}(W)$; we will also denote by Γ the subgroup of $\mathrm{Aut}(W)$ formed by these elements.

It is clear that Γ leaves W_0' invariant. Let $s\in N(T)$, and $\sigma\in\Gamma$; then for any $\chi\in X$ and $t\in T$, we have $(w_s\chi)^\sigma(t)=w_s\chi(t^{\sigma^{-1}})=\chi(s^{-1}t^{\sigma^{-1}}s)$, and also $w_{s^\sigma}\chi^\sigma(t)=\chi^\sigma(s^{-\sigma}ts^\sigma)=\chi(s^{-1}t^{\sigma^{-1}}s)$. This proves

$$w_s^\sigma=w_{s^\sigma} \qquad \text{for all } s\in N(T),\ \sigma\in\Gamma\,. \tag{19}$$

Since $\mathrm{Aut}\,(X,\mathfrak{r})$ is finite there is a non-degenerate symmetric bilinear form $\langle,\rangle$ on X_Q which is invariant under $\mathrm{Aut}\,(X,\mathfrak{r})$. Thus for any $\alpha\in\mathfrak{r}$, $\sigma\in\Gamma$, and $\chi\in X$, we have $(w_\alpha\chi)^\sigma=\left(\chi-\frac{\langle\chi,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha\right)^\sigma=\chi^\sigma-\frac{\langle\chi^\sigma,\alpha^\sigma\rangle}{\langle\alpha^\sigma,\alpha^\sigma\rangle}\alpha^\sigma=w_{\alpha^\sigma}\chi^\sigma$. This implies:

$$w_\alpha^\sigma=w_{\alpha^\sigma} \qquad \text{for all } \alpha\in\mathfrak{r},\ \sigma\in\Gamma\,. \tag{20}$$

Since Γ leaves $\mathfrak{r}_0$ invariant, (20) implies that Γ leaves W_0 invariant.

In the case $S=T^\Gamma$ which we are now considering, we will denote W_0' by W_Γ; thus by definition, $W_\Gamma=\{w\in W\,|\,w(X_0)=X_0\}$, where X_0 is defined in (14). Then Γ leaves W_Γ invariant.

If Δ is a Γ-fundamental system of $\mathfrak{r}$, then the set $\{w_\sigma,\ \sigma\in\Gamma\}$ defined in Lemma 3.3 satisfies the relation:

$$w_\sigma^\tau w_\tau=w_{\sigma\tau}\,, \qquad \text{for all } \sigma,\tau\in\Gamma\,. \tag{21}$$

Using Lemma 3.3, one can also show:

$$w^\sigma\equiv w \pmod{W_0} \qquad \text{for all } w\in W_\Gamma,\ \sigma\in\Gamma\,. \tag{22}$$

More precisely, if $w\in W_\Gamma$, and $\sigma\in\Gamma$, then Lemma 3.3 implies that $w^\sigma=w_\sigma'ww_\sigma^{-1}$, where $w_\sigma\Delta=\Delta^\sigma$, and $w_\sigma'(w\Delta)=(w\Delta)^\sigma$. Since W_0 is normal in W_Γ, (22) results.

Now denote by W^Γ the subgroup of W left pointwise fixed by Γ. Equation (18) implies that W^Γ is just the centralizer of Γ in W (where Γ and W are both considered as subgroups of $\mathrm{Aut}\,(X,\mathfrak{r})$). W^Γ is a subgroup of W_Γ, since if $w\in W^\Gamma$ and $\chi\in X_0$, we have $\sum_{\sigma\in\Gamma}(w\chi)^\sigma=\sum_{\sigma\in\Gamma}w\chi^\sigma=w\sum_{\sigma\in\Gamma}\chi^\sigma=0$. Equation (18) implies that W^Γ also leaves X^Γ invariant.

It would be interesting to know the structure of the group W^Γ; so far, we have not been able to solve this in general. We can, however, observe several facts. It is clear from (20) that W^Γ contains the subgroup of W generated by the reflections w_α, where $\alpha^\sigma=\pm\alpha$ for all $\sigma\in\Gamma$, and these are the only reflections in W^Γ (with respect to roots $\alpha\in\mathfrak{r}$). If $W_0=\{1\}$, then (22) implies that $W_\Gamma=W^\Gamma$; however, $W_0=\{1\}$ is not a necessary condition for $W_\Gamma=W^\Gamma$ to occur, as the example at the end of this section illustrates.

Questions of structure can be answered with respect to a different action of Γ on W, which corresponds to the action of $[\Gamma]$ on X in (17). We fix a Γ-fundamental system Δ of $\mathfrak{r}$ for the remainder of this section. For each $\sigma\in\Gamma$ and $w\in W$, the element $w^{[\sigma]}\in W$ is defined by the following equation:

(23) $$w^{[\sigma]}\chi^{[\sigma]}=(w\chi)^{[\sigma]} \qquad \text{for all } \chi\in X\,.$$

The set $\{[\sigma]:\sigma\in\Gamma\}$ forms a subgroup of $\mathrm{Aut}(W)$ which we denote by $[\Gamma]$. It is clear from (23) that $[\Gamma]$ leaves W_Γ invariant. An alternate way of stating (23) is that the automorphism $w^{[\sigma]}$ of $(X, \mathfrak{r})$ is just a composition of automorphisms of $(X, \mathfrak{r})$, namely:

(24) $$w^{[\sigma]}=[\sigma]\circ w\circ[\sigma]^{-1} \qquad \text{for all } w\in W,\ \sigma\in\Gamma\,.$$

Since $w^{[\sigma]}\chi^{[\sigma]}=w^{[\sigma]}w_\sigma^{-1}\chi^\sigma$, and $(w\chi)^{[\sigma]}=w_\sigma^{-1}(w\chi)^\sigma=w_\sigma^{-1}w^\sigma\chi^\sigma$ for all $\chi\in X$, it follows that $w^{[\sigma]}w_\sigma^{-1}=w_\sigma^{-1}w^\sigma$, or

(25) $$w^{[\sigma]}=w_\sigma^{-1}w^\sigma w_\sigma \qquad \text{for all } w\in W,\ \sigma\in\Gamma\,.$$

In particular, if we apply (25) to w_α, $\alpha\in\mathfrak{r}$, then (20) implies:

(26) $$w_\alpha^{[\sigma]}=w_{\alpha^{[\sigma]}} \qquad \text{for all } \alpha\in\mathfrak{r},\ \sigma\in\Gamma\,.$$

From (26), we see that not only does $[\Gamma]$ leave W_0 invariant, but also the sets of reflections $\{w_\alpha, \alpha\in\mathfrak{r}\}$, $\{w_\alpha, \alpha\in\Delta\}$, $\{w_\alpha, \alpha\in\Delta_0\}$. Since W_0 is normal in W_Γ, (22) and (25) imply

(27) $$w^{[\sigma]}\equiv w \pmod{W_0}\,, \qquad \text{for all } w\in W_\Gamma,\ \sigma\in\Gamma\,.$$

In addition, using (21) and (25), one can show:

(28) $$w_\sigma^{[\tau]}=w_\tau^{-1}w_{\sigma\tau} \qquad \text{for all } \sigma,\ \tau\in\Gamma\,.$$

Now denote by $W^{[\Gamma]}$ the subgroup of W left pointwise fixed by $[\Gamma]$. In general, $W^{[\Gamma]}$ is not a subgroup of W_Γ, but when T^Γ is of root system type, $W^{[\Gamma]}$ contains the subgroup V (Theorem 2.9), as we shall prove. We first generalize some results of R. Steinberg [7].

If Δ' is any subset of Δ, we call the subgroup of W generated by the reflections w_α, $\alpha\in\Delta'$ the Weyl group of Δ'.

LEMMA 4.1. *If Δ' is a $[\Gamma]$-invariant subset of Δ, and W' is the Weyl group of Δ', then*

(a) *W' is invariant under $[\Gamma]$ and W' is a normal subgroup of the group generated by W' and $[\Gamma]$ in* $\mathrm{Aut}(X, \mathfrak{r})$.

(b) *If w' is the unique element of W' satisfying $w'(\Delta')=-\Delta'$, then $w'\in W^{[\Gamma]}$.*

PROOF. (a) The first statement follows from (26), and then the second follows from (24).

(b) Since Δ' is $[\Gamma]$-invariant, we have $w'^{[\sigma]}(\Delta')=(w'(\Delta'^{[\sigma]^{-1}}))^{[\sigma]}=-\Delta'$ for all $\sigma\in\Gamma$. Since $w'^{[\sigma]}\in W'$ (by (a)), we must have $w'^{[\sigma]}=w'$ for all $\sigma\in\Gamma$.

We have shown (Proposition 3.5) that if $\bar{\Delta}=\{\gamma_1,\cdots,\gamma_{\bar{l}}\}$, then the subset $\Delta^i=\Delta\cap\pi^{-1}(\gamma_i)$ is a $[\Gamma]$-orbit; since Δ_0 is also left fixed by $[\Gamma]$, $\Delta_i=\Delta^i\cup\Delta_0$ is a $[\Gamma]$-invariant subset of Δ. Since w_i is the unique element of the Weyl group W_i of Δ_i which satisfies $w_i(\Delta_i) = -\Delta_i$, we have:

COROLLARY 4.2. *If $\bar{\Delta} = \{\gamma_1, \cdots, \gamma_{\bar{l}}\}$, then $w_i \in W^{[\Gamma]}$ for $1 \leqq i \leqq \bar{l}$.*

If we combine Corollary 4.2 with Theorem 2.9, we obtain:

THEOREM 4.3. *If T^Γ is of root system type, then there exists a subgroup $V \subset W^{[\Gamma]}$ having a "good system of involutive generators" such that $W_\Gamma = V \cdot W_0$ is a semi-direct product, and $\overline{W}$ is isomorphic to V under the canonical homomorphism $W_\Gamma \to \overline{W}$.*

When T^Γ is of root system type, we can combine Corollary 1.4 with Theorem 4.3 (and use the second isomorphism theorem), to obtain the following lattice of subgroups of W, where each of the "vertical" quotients is isomorphic to $\overline{W}$.

$$\begin{tikzcd}[row sep=small] W_\Gamma \arrow[dr, dash] \arrow[ddd, dash] & & \\ & W_\Gamma^{[\Gamma]} \arrow[dr, dash] \arrow[ddd, dash] & \\ & & V \arrow[ddd, dash] \\ W_0 \arrow[dr, dash] & & \\ & W_0^{[\Gamma]} \arrow[dr, dash] & \\ & & \{1\} \end{tikzcd} \qquad \cong \overline{W}$$

If we apply Lemma 4.1 to the set of $[\Gamma]$-orbits Δ' of Δ, part (b) yields a corresponding set of elements $w' \in W^{[\Gamma]}$. This set is, in fact, a good system of involutive generators for the group $W^{[\Gamma]}$. This result is obtained by applying to our case Theorems 2 and 3 of [3] (and is true whether or not T^Γ is of root system type).

PROPOSITION 4.4 (Hijikata). *Let Δ be a Γ-fundamental system of $\mathfrak{r}$, and let $\Delta = \Delta_{(1)} \cup \cdots \cup \Delta_{(k)}$ be the decomposition of Δ into $[\Gamma]$-orbits. If v_j is the involution in the Weyl group of $\Delta_{(j)}$ satisfying $v_j(\Delta_{(j)}) = -\Delta_{(j)}$, then the set $\{v_j, 1 \leqq j \leqq k\}$ is a good system of involutive generators of $W^{[\Gamma]}$.*

Note that if $\bar{\Delta} = \{\gamma_1, \cdots, \gamma_{\bar{l}}\}$, then $\bar{l}$ of the orbits $\Delta_{(j)}$ in Corollary 4.5 are of the form Δ^j, and the rest are $[\Gamma]$-orbits of elements in Δ_0. If $\Delta_{(j)} = \Delta^j$, then the involution w_j is a product of the involution v_j with some of the involutions v_n, where $\Delta_{(n)} \subset \Delta_0$. Thus Proposition 4.4 also implies Corollary 4.2.

We close this section with an example which illustrates some applications of our theorems, and shows that even if W_0 is a non-trivial proper subgroup of W_Γ, that one can have $W_\Gamma = W^\Gamma$.

EXAMPLE. We first remark that if $\sigma \in \mathrm{Aut}(X, \mathfrak{r})$, then there is an element

$\varphi_\sigma \in \mathrm{Aut}(G, T)$ such that ${}^t\varphi_\sigma^{-1} = \sigma$ (and moreover, φ_σ is unique up to inner automorphism by an element of T), ([2], exposé 23). Thus by choosing such a $\varphi_\sigma \in \mathrm{Aut}(G, T)$, we can identify the group generated by σ in $\mathrm{Aut}(X, \mathfrak{r})$ with the group generated by φ_σ in $\mathrm{Aut}(G, T)$, and this identification agrees with (13).

Now let G be a simple group of type A_3, with fundamental system $\varDelta = \{\alpha_1, \alpha_2, \alpha_3\}$, and let $\varGamma = \{1, \sigma\}$ where $\sigma = \mathrm{Aut}(X, \mathfrak{r})$ satisfies:

$$\alpha_1^\sigma = \alpha_3 + \alpha_2,\ \alpha_2^\sigma = -\alpha_2,\ \alpha_3^\sigma = \alpha_1 + \alpha_2\,.$$

(Since $\varDelta^\sigma = w_{\alpha_2}(\varDelta)$, σ is an automorphism, and it is clear that $\sigma^2 = 1$.) Since X_0 is generated over Q by $\{\alpha_i^\sigma - \alpha_i, i = 1, 2, 3\}$, we see that X_0 is generated over Q by $\{\alpha_2, \alpha_3 - \alpha_1\}$. By (16), we see that $\varDelta$ is an X_0-fundamental system, and $\varDelta_0 = \{\alpha_2\}$, $\bar{\varDelta} = \{\gamma_1\}$. Lemma 3.3 implies $w_{\alpha_2} = w_\sigma$, so by (17), we have $\alpha_1^{[\sigma]} = \alpha_3$, $\alpha_2^{[\sigma]} = \alpha_2$, $\alpha_3^{[\sigma]} = \alpha_1$, thus $[\sigma]$ is the opposition automorphism of $\varDelta$. Since $\varDelta^1 = \{\alpha_1, \alpha_3\}$, we have $\varDelta_1 = \varDelta$, and so Theorem 2.6 (i) implies $T^\varGamma$ is of root system type. If $w \in W_1 = W$ is the involution satisfying $w(\varDelta) = -\varDelta$, then $V = \{1, w\}$ and since $W_0 = \{1, w_{\alpha_2}\}$, Theorem 2.9 implies that $W_\varGamma = V \cdot W_0$ contains four elements. Clearly (20) implies $w_{\alpha_2} \in W^\varGamma$, and it is easily verified that $w \in W^\varGamma$, and hence $ww_{\alpha_2} \in W^\varGamma$. Since $W^\varGamma \subset W_\varGamma$, we must have $W_\varGamma = W^\varGamma = \{1, w_{\alpha_2}, ww_{\alpha_2}, w\}$.

§ 5. k-roots and maximal k-trivial tori.

We wish to make a few comments about how our results relate to the case where G is a connected semi-simple (or reductive) algebraic group defined over a field k. In this case, we take T a maximal torus defined over k, and splitting over K, where K/k is finite Galois, and determine the group $\varGamma$ by $\mathrm{Gal}(K/k)$ as follows. Each $\sigma \in \mathrm{Gal}(K/k)$ determines an automorphism $\chi \to \chi^\sigma$ of $(X, \mathfrak{r})$ and the transposed inverse φ_σ defined by $\chi^\sigma(t) = \chi(\varphi_\sigma^{-1}(t))$, for $t \in T$, $\chi \in X$, is a rational automorphism of T. Thus $\varGamma$ is taken as the group $\{\varphi_\sigma : \sigma \in \mathrm{Gal}(K/k)\}$. ($\varGamma$ is a subgroup of $\mathrm{Aut}(T)$ rather than $\mathrm{Aut}(G, T)$, but with the exception of (19), we have only used the fact that $\varGamma \subset \mathrm{Aut}(T)$. Even (19) holds true if φ_σ is extended to a rational automorphism of (G, T) since by [2], exposé 23, an element $\varPsi_\sigma \in \mathrm{Aut}(G, T)$ satisfying $\varPsi_\sigma | T = \varphi_\sigma$ is unique up to inner automorphism by elements of T).

It is known ([4]) for an arbitrary field k, that the module X_0 defined in (14) is the annihilator of a maximal k-trivial torus of T. If T is chosen so as to contain a maximal k-trivial torus of G, we see that a maximal k-trivial torus of G is just $T^\varGamma$.

Corollary 2.8 shows that to prove that the set $\bar{\mathfrak{r}}$ (called k-roots) is a root system in a wider sense with Weyl group $\overline{W}$, it suffices to verify one of the conditions of Theorem 2.6 (or Theorem 2.3, for $1 \leqq i \leqq \bar{l}$), since $R(3)$ can then be shown for $\mathfrak{r}$ using a reduction to the case of a simple reduced root (see [4], p. 225–226). The important fact that can be used to prove any of these conditions is the conjugacy (by k-rational elements of G) of maximal k-trivial tori of G, and (for k perfect) k-Borel subgroups of G.

The main interest, of course, in the study of k-roots of G is a result of the classification problem; that is, to describe (relative to k) the structure of G, and make a complete classification in terms of certain invariants, of all possible G defined over a given field k (up to k-isogeny). One of the invariants that can be used to describe G is the $[\Gamma]$-diagram (or k-index) of G; that is, the Dynkin diagram of a Γ-fundamental system, indicating which vertices are in Δ_0, and which are in the same $[\Gamma]$-orbit.

Condition (i) of Theorem 2.6 makes it possible to list, for each simple group, all possible $[\Gamma]$-diagrams which can occur. Although this hardly solves the classification problem (the existence of groups G defined over k that "fit" the diagrams must be proved), it helps cut it down to size. By a reduction to the case of a $[\Gamma]$-diagram of a single restricted fundamental root (i. e., the Δ_i of § 2), the problem can be attacked in its simplest form.

For an excellent overall view of the classification problem and techniques used in its solution, see [8]. A general exposition of the problem for k a perfect field, and the solution to the problem for k a $\mathfrak{p}$-adic field appears in [5]. (M. Kneser's work is of key importance in the $\mathfrak{p}$-adic case; see "Galois-Kohomologie halbeinfacher algebraisher Gruppen über $\mathfrak{p}$-adischen Körpern," I, II, Math. Zeit., **88** (1965), 40–47, **89** (1965) 250–272). For details of the solution when k is the field of real numbers, see S. Araki, "On root systems and an infinitesimal classification of irreducible symmetric spaces", J. Math. Osaka City U., Vol. 13, 1–34.

Finally, a special case should be mentioned. When $\Delta_0=\phi$, the group G is said to be of "Steinberg type", that is, G contains a Borel group defined over k. (If k is a finite field, for instance, this is the case.) In this case, since $W_0=\{1\}$, the automorphisms $\sigma\in\Gamma$ and $[\sigma]\in[\Gamma]$ in Aut $(X,\mathfrak{r})$ (and in Aut (W)) coincide, and $W_\Gamma=W^\Gamma=W^{[\Gamma]}$ (by (22)). The set $\{w_i, 1\leqq i\leqq \bar{l}\}$ is just a set of fundamental reflections relative to the k-roots, and is a good system of involutive generators for the Weyl group $\overline{W}=W_\Gamma$ of $\bar{\mathfrak{r}}$ (Theorem 4.3). (Also, see [7].)

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