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A remark on the cohomology group and the dimension of product spaces

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1. In § 4 of the paper [4] several theorems concerning the dimension of product spaces were given. Proofs of all theorems except Theorem 5 depend heavily on Künneth formula which was proved by R. C. O'Neil [7]. However this formula is false. It is known by a counter example given by G. Bredon (see **2**). The purpose of this paper is devoted to correct some of theorems in § 4 of [4] and to prove the related results. However it is not known whether Theorems 6–9 hold or not though they are proved partly in this paper.

Throughout this paper all spaces are Hausdorff and have finite covering dimension and we mean by H^* the unrestricted Čech cohomology group.

2. R. C. O'Neil [7] gave the following theorems.

A. Let G be an abelian group. If $X \times Y$ is paracompact, then

$$H^n(X \times Y : G) \cong \sum_{q=0}^{n} H^q(X : H^{n-q}(Y : G)).$$

B. Let L be a principal ideal domain. If X is compact and Y is paracompact, then there is an exact sequence

$$0 \to \sum_{q=0}^{n} H^q(X : L) \underset{L}{\otimes} H^{n-q}(Y : L) \to H^n(X \times Y : L)$$

$$\to \sum_{q=0}^{n} H^{q+1}(X : L) *_L H^{n-q}(Y : L) \to 0.$$

The following example was given by G. Bredon. Let X be a solenoid, so X is a 1-dimensional compact metric space and $H^1(X) \cong R$ (=the group of all rational numbers). For $n = 2, 3, \cdots$, let Y_n be a 2-dimensional finite simplicial polytope such that $H^2(Y_n) \cong Z_n$ (=the cyclic group of order n). Let Y be a disjoint union of Y_n, $n = 2, 3, \cdots$. Then Y is a 2-dimensional locally finite polytope and $H^2(Y) \cong \prod_{n=2}^{\infty} Z_n$. Since X is compact, by Peterson [8: Appendix], we have $H^3(X \times Y) \cong \prod_{n=2}^{\infty} H^3(X \times Y_n) \cong \prod_{n=2}^{\infty} H^1(X) \otimes H^2(Y_n) = 0$ and $H^1(X : H^2(Y)) \cong H^1(X) \otimes H^2(Y) \neq 0$. Thus, both theorems A and B are false.

The following theorem is easily proved by using sheaf theory.

THEOREM 1. *Let X and Y be paracompact spaces such that* $\dim X = p$ *and* $\dim Y = q$. *Suppose that either* (1) *X is compact, or* (2) *$X \times Y$ is hereditarily paracompact and Y is locally contractible. Then, for any abelian group G,* $H^{p+q}(X \times Y : G) \cong H^q(Y : H^p(X : G))$.

PROOF. Let f be the projection of $X \times Y$ onto Y. Let $\mathcal{H}^m(X : G)$ be the sheaf over Y generated by the presheaf

$$U \to H^m(f^{-1}(U) : G)$$

where U is an open set of Y. Then, by [**3**: Théorème 4.17.1], there is a spectral sequence such that $E_2^{nm} = H^n(Y : \mathcal{H}^m(X : G))$ and the term E_∞ is bigraded associated with filtrations on $H^{m+n}(X \times Y : G)$. Under the hypothesis of the theorem it is known that $\mathcal{H}^m(X : G)$ is a constant sheaf $H^m(X : G)$ over Y. In the case (1) it follows from [**3**: Théorème 4.11.1] and in the case (2) it follows from the paracompactness of $f^{-1}(U)$ and the local contractibity of Y. Thus we have $E_2^{nm} = H^n(Y : H^m(X : G))$. Since $\dim X = p$ and $\dim Y = q$, $E_2^{nm} = 0$ if $m > p$ or $n > q$. Thus $E_2^{qp} \cong H^{p+q}(X \times Y : G)$. This completes the proof.

REMARK 1. By Bredon's example mentioned above, we can not exchange X and Y in the conclusion of Theorem 1, that is, $H^{p+q}(X \times Y : G) \cong H^p(X : H^q(Y : G))$ does not generally hold.

REMARK 2. The following generalization to Theorem 1 is true.

(1) Theorem 1 holds if we replace "$\dim X = p$" by "$\mathrm{Max}\,\{m : H^m(X : G) \neq 0\} = p$".

(2) In the case (1) of Theorem 1, if $q > 1$, then, for each closed set B of Y, we have $H^{p+q}(X \times Y, X \times B : G) \cong H^q(Y, B : H^p(X : G))$.

3. Let G be an abelian group. The cohomological dimension $D(X : G)$ of a space X with respect to G is the largest integer n such that $H^n(X, A : G) \neq 0$ for some closed set A of X.

LEMMA 1. *Let X be paracompact. Let G and H be abelian groups. If there is an epimorphism $\alpha : G \to H$ and $D(X : H) = \dim X$, then $D(X : G) = \dim X$.*

PROOF. Take a closed subset A of X such that $H^n(X, A : H) \neq 0$, where $n = \dim X$. Since $0 \to \mathrm{Ker}\,\alpha \to G \to H \to 0$ is exact, the sequence $\cdots \to H^n(X, A : G) \to H^n(X, A : H) \to H^{n+1}(X, A : \mathrm{Ker}\,\alpha) = 0$ is exact. Thus $H^n(X, A : G) \neq 0$.

LEMMA 2. *Let X be a compact space with $D(X : G) = m$ and let Y be a paracompact space. Suppose that, for some closed G_δ set A of X, either* (1) *there is an epimorphism:* $H^m(X, A : G) \to G$ *and* $D(Y : G) = \dim Y$ *or* (2) $D(Y : H^m(X, A : G)) = \dim Y$. *Then* $D(X \times Y : G) \geqq D(X : G) + \dim Y$.

PROOF. There is a closed G_δ set B of Y such that either (1) $H^n(Y, B : G) \neq 0$ or (2) $H^n(Y, B : H^m(X, A : G)) \neq 0$, where $n = \dim Y$. Let $X_0 = X/A$ and $Y_0 = Y/B$, and let x_0 and y_0 be the points corresponding to A and B. Since

$D(X_0 : G) = m$ and $\dim Y_0 = n$, $H^{m+n}(X_0 \times Y_0 : G) \cong H^n(Y_0 : H^m(X_0 : G))$ by Theorem 1. Thus, in both cases (1) and (2), we can conclude $H^{m+n}(X_0 \times Y_0 : G) \neq 0$. Hence $D(X_0 \times Y_0 : G) \geqq m+n$. Since $X_0 \times Y_0 - \{x_0\} \times Y_0 \cup X_0 \times \{y_0\}$ is a union of a countable number of closed sets of $X \times Y$, sum theorem [**4**, p. 348] means $D(X \times Y : G) \geqq m+n$.

A compact space X is clc^∞ if, for each point x of X and a closed neighborhood U of x, there is a closed neighborhood V of x such that $V \subset U$ and the induced homomorphism: $\tilde{H}^i(U) \to \tilde{H}^i(V)$ is trivial for $i = 0, 1, 2, \cdots$, where $\tilde{H}^i$ is the reduced group.

The following is a generalization to Morita [**6**: Theorem 6] and Dyer [**1**: Corollary 7].

THEOREM 2. *Let X be a compact clc^∞ space and let Y be a paracompact space with $D(Y : G) = \dim Y$. If $D(X : R) = \dim X$, then $D(X \times Y : G) = \dim (X \times Y) = \dim X + \dim Y$.*

PROOF. Since X is a compact clc^∞ space and $D(X : R) = \dim X$, there are closed sets A, B and N of X such that (1) A is G_δ and $A \supset B$, (2) N is a neighborhood of B, (3) $H^m(N, B)$ contains an element γ of infinite order, where $m = \dim X$, and (4) the image of the induced homomorphism: $H^m(X, A) \to H^m(N, B)$ is finitely generated and contains the element γ. The proof is found in Dyer [**2**: p. 157]. This shows that there is an epimorphism of $H^m(X, A)$ onto Z. Hence there is an epimorphism of $H^m(X, A : G) = H^m(X, A) \otimes G$ onto G. By Lemma 2 we can conclude that $D(X \times Y : G) = \dim (X \times Y) = \dim X + \dim Y$.

COROLLARY 1. *Let X be a compact clc^∞ space. In order that $\dim (X \times Y) = \dim X + \dim Y$ for every paracompact space Y, it is necessary and sufficient that $D(X : R) = \dim X$.*

LEMMA 3. *Let X be a paracompact space with finite large inductive dimension and let G be an abelian group. Then, for each k, $0 \leqq k \leqq D(X : G)$, there is a closed subset X_k of X such that $D(X_k : G) = k$. If G is finitely generated, then the lemma holds for a normal space X with finite large inductive dimension. Consequently, if X is normal and $\operatorname{Ind} X < \infty$, then, for each k, $0 \leqq k \leqq \dim X$, there is a closed set X_k such that $\dim X_k = k$.*

PROOF. If $\operatorname{Ind} X = 0$, then it is obvious that the lemma is true. Suppose that the lemma holds in case $\operatorname{Ind} X \leqq n-1$. Let $\operatorname{Ind} X = n$ and $D(X : G) = m$. For each pair (F, U), F closed, U open and $F \subset U$, choose an open set $V_{(F,U)}$ such that $F \subset V_{(F,U)} \subset \bar{V}_{(F,U)} \subset U$ and $\operatorname{Ind}(\bar{V}_{(F,U)} - V_{(F,U)}) \leqq n-1$. If $D(\bar{V}_{(F,U)} - V_{(F,U)} : G) \leqq m-2$ for each pair (F, U), then $D(X : G) \leqq m-1$ by [**5**: Theorem 5]. Thus, there is some pair (F, U) such that $D(\bar{V} - V : G) = m-1$ or $= m$, where $V = V_{(F,U)}$. Since $\operatorname{Ind}(\bar{V} - V) \leqq n-1$, there is a closed subset X_k of $\bar{V} - V$ for each k, $0 \leqq k \leqq m-1$, such that $D(X_k : G) = k$. Then the closed sets X_k, $k = 0$, 1, $\cdots$, $m-1$, and $X_m=X$ satisfy the conclusion of the lemma.

THEOREM 3. *Let X be a compact space and let Y be a paracompact space with finite large inductive dimension. If* $\dim Y>0$, *then* $D(X\times Y:G)\geqq D(X:G)+1$ *for any abelian group G.*

PROOF. Since Y has finite large inductive dimension and $\dim Y>0$, by the previous lemma, there is a closed set Y' of Y such that $\dim Y'=1$. Take a closed G_δ set A of X such that $H^n(X, A:G)\neq 0$, where $n=D(X:G)$. Since $\dim Y'=1$, it follows from [4: Corollary 1] that $D(Y':H^n(X, A:G))=\dim Y'$. Hence, by Lemma 2, $D(X\times Y:G)\geqq D(X\times Y':G)\geqq D(X:G)+1$.

THEOREM 4. *Let G be one of the groups R, Q_p (=the p-primary part of the group of rationals modulo one) and Z_p, where p is prime. Let X be compact and let Y be paracompact. If* $D(X:G)=\dim X$ *and* $D(Y:G)=\dim Y$, *then* $D(X\times Y:G)=\dim(X\times Y)=\dim X+\dim Y$.

PROOF. Take a closed G_δ set A of X such that $H^m(X, A:G)\neq 0$, where $m=\dim X$. Since X is compact, $H^m(X, A:G)\cong H^m(X, A)\otimes G$. Thus, if G is one of the groups R, Q_p and Z_p, then G is a direct summand of $H^m(X, A:G)$. Hence, there is an epimorphism from $H^m(X, A:G)$ onto G. The theorem follows from Lemma 2.

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